Calculus 1 Midterm 1 Solutions

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Name:

UNI:

Rules:

1. You may reference any handwritten notes you have at any point during this exam.

2. No electronic devices.

3. Explain and justify everything that you're doing. Words are very important.

4. If you need more paper, ask. Don't forget to write your name and UNI on every extra sheet.

1	2	3	4	Total

1.
1.1) Give a precise definition of what it means for a function f to be continuous at a point a.

Solution: f is continuous at a if and only if

$$\lim_{x\to a} f(x) = f(a).$$

1.2) Give an example of a function g such that:
- 0 is in the domain of g
- g is discontinuous at 0

Solution: One such function is

$$g(x) = \begin{cases} 0, & x = 0 \\ 1, & x \in \mathbb{R}, x \neq 0 \end{cases}$$

1.3) Is the function $f(x) = \tan\left(\frac{e^x+x^2}{1-\sqrt{\log(x)}}\right)$ continuous on its domain? Why or why not?

Solution: Yes, because it's made out of simple functions.

1.4) Evaluate

$$\lim_{x\to 3} \tan\left(\frac{e^x + x^2}{1-\sqrt{\log(x)}}\right)$$

and explain how you did it.

Solution: From 1.3 we know that the function in the limit is continuous on its domain. By inspection, 3 is in the domain of this function (i.e. we can evaluate this function at $x = 3$). By the definition given in 1.1, with $a = 3$, we have

$$\lim_{x\to 3} \tan\left(\frac{e^x + x^2}{1-\sqrt{\log(x)}}\right) = \tan\left(\frac{e^3 + 3^2}{1-\sqrt{\log(3)}}\right).$$

2.

2.1) Suppose that f is a function with the following properties:

- The domain of f is $[0, 1]$
- $f(0) = 1$
- $f(1) = -1$

Is this alone enough to guarantee the existence of a number c such that $f(c) = 0$?
If so, explain why.
If not, give an example of a function which has these three properties, but for which such a c does not exist.

Solution: No, it isn't. A counterexample is the function

$$f(x) = \begin{cases} 1, & x = 0 \\ -1, & 0 < x \leq 1. \end{cases}$$

2.2) Prove that there is a real number x such that $\cos(x) = x$.

Solution: Define $f(x) := \cos(x) - x$. Note that x is a solution to the equation $\cos(x) = x$ if and only if $f(x) = 0$. Notice that f is continuous on $\mathbb{R}$, because it's made of simple functions. Moreover, we have $f(0) = 1$ and $f(2) = \cos 2 - 2$. Thus, by the Intermediate Value Theorem, for all y satisfying $\cos 2 - 2 \leq y \leq 1$, there exists a c in the interval $[0, 2]$ such that $f(c) = y$. By taking $y = 0$, we see that there exists a c such that $f(c) = 0$, which proves the assertion.

3.

3.1) Either explain why the statement below is true, or give a counterexample to the statement below.

"If $\lim_{x\to 1} g(x) = 0$, *then* $\lim_{x\to 1} \frac{f(x)}{g(x)}$ *can never exist, regardless of what the functions* f *and* g *are."*

The statement is false. A counterexample is $f(x) = 0$ and $g(x) = x - 1$. In this case the limit is 0.

3.2) Evaluate the following limits. Use limit laws for 3.2.1, and use the squeeze theorem for 3.2.2.

3.2.1) $\lim_{x\to\infty} \frac{x^2 - x - 6}{2x^2 - 8}$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2 - x - 6}{2x^2 - 8} &= \lim_{x\to\infty} \frac{1 - x^{-1} - 6x^{-2}}{2 - 8x^{-2}} \\
&= \frac{\lim_{x\to\infty} 1 - \lim_{x\to\infty} x^{-1} - \lim_{x\to\infty} 6x^{-2}}{\lim_{x\to\infty} 2 - \lim_{x\to\infty} 8x^{-2}} \\
&= \frac{1 - 0 - 0}{2 - 0} \\
&= \frac{1}{2}
\end{aligned}$$

In the first line we divided the top and bottom of the fraction by x^2, which we can do as long as $x \neq 0$ (and we're looking at $x \to \infty$, so we don't have to worry about this). In the second line we used limit laws, which we can only do if everything exists. Everything does exist, because we evaluate it in the third line.

3.2.2) $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Solution: Define $g(x) = x^2$ and $h(x) = 0$. Notice that for all $x \in \mathbb{R}$ we have $h(x) \leq f(x) \leq g(x)$. Moreover, $\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = 0$. The squeeze theorem then allows us to conclude that $\lim_{x\to 0} f(x) = 0$.

3.2.3) $\lim_{x\to 0} e^{-\frac{1}{x}}$

Solution: If x is small and positive, then $\frac{1}{x}$ is large and positive, so $-\frac{1}{x}$ is large and negative, so $e^{-\frac{1}{x}}$ is close to 0 when x is small and positive. Thus

$$\lim_{x\to 0^+} e^{-\frac{1}{x}} = 0.$$

If x is small and negative, then $-\frac{1}{x}$ is large and positive, so $e^{-\frac{1}{x}}$ is large and positive when x is small and negative. Thus

$$\lim_{x\to 0^-} e^{-\frac{1}{x}} = \infty.$$

Since the two one-sided limits aren't equal, the limit in the question does not exist.

4. An object attached to a spring is released at the time $t = 0$. The position of the object is given by $q(t) = \sin(t)$. Find the speed of the object when $t = 2\pi$. Only use techniques discussed in class, in the assignments, and in the first two chapters of Stewart.

Solution: For small Δt, the distance the object has travelled when starting at $t = 2\pi$ and ending at $t = 2\pi + \Delta t$ is

$$q(2\pi + \Delta t) - q(2\pi).$$

Using $q(t) = \sin t$, we have $q(2\pi) = 0$ and $q(2\pi + \Delta t) = \sin(\Delta t)$ (using the fact that sin is periodic with period 2π). The time elapsed between $t = 2\pi$ and $t = 2\pi + \Delta t$ is

$$(2\pi + \Delta t) - (2\pi) = \Delta t.$$

The speed of an object is approximately the distance travelled divided by the time it took to travel that distance, which in this case is

$$\frac{\sin(\Delta t)}{\Delta t}.$$

As we take Δt to be smaller and smaller, we look at the object in a smaller and smaller time interval, and this interval always includes $t = 2\pi$. As we take the limit as Δt goes to 0, we get the exact speed of the object at $t = 2\pi$, by definition. This speed is

$$\lim_{\Delta t \to 0} \frac{\sin(\Delta t)}{\Delta t} = 1.$$