

Calculus 1 Midterm 2

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Name:

UNI:

Rules:

1. Don't open this exam booklet until instructed.
2. You may reference any handwritten notes you have at any point during this exam.
3. No electronic devices.
4. Explain and justify everything that you're doing. Words are very important.
5. If you need more paper, ask. Don't forget to write your name and UNI on every extra sheet.

1	2	3	4	5	Total

1. 1b, 2c, 3a.

There are a number of ways to see that this is the correct pairing. For example, 1 is the only function of the top three that is decreasing on both sides of $x = 0$, so it must match to b, which is the only function that is negative both immediately to the left and immediately to the right of $x = 0$. The function 3 is the only top function whose derivative is bounded around $x = 0$ (as opposed to getting arbitrarily large), so it must match with a.

2.

2.1) By definition

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}.$$

If we take $h = -0.07$, then hopefully the resulting ratio will be close to the actual value of $f'(1)$. Doing this makes the ratio above equal to

$$\frac{3.02 - 3}{-0.07} = -\frac{2}{7}.$$

2.2) The slope between the points $(0.95, 5.99)$ and $(1, 6)$ is $\frac{1}{5}$, and the slope between the points $(1, 6)$ and $(1.05, 6.01)$ is also $\frac{1}{5}$. The derivative is the slope of the tangent line, and this is well approximated by these slopes we just computed. Because these slopes aren't changing, we should expect that the rate of change of the slopes is 0, i.e. that $g''(1) = 0$.

3.

3.1) Linear approximation says that $f(x) \approx f(a) + f'(a)(x - a)$. Taking $f(x) = \log x$, $a = 100$, and $x = 98$, this gives

$$\begin{aligned}\log 98 &\approx \log 100 + \frac{1}{100}(98 - 100) \\ &\approx 4.60517 - 0.02 \\ &\approx 4.58517\end{aligned}$$

3.2) Linear approximation is the practice of approximating a function with its tangent line. Either by computing that the second derivative of $\log x$ is $-x^{-2}$, or just from inspection of the graph, we can conclude that the function $\log x$ is concave down everywhere, which means that the value of $\log x$ is below the value of any of its tangent lines at x . Thus the estimate in 3.1 is an overestimate.

4. Differentiating with respect to y :

$$\begin{aligned}\frac{d}{dy} \left(x^2 y + \frac{9}{16} y^4 \right) &= \frac{d}{dy} x^3 \\ x^2 \frac{d}{dy} y + y \frac{d}{dy} x^2 + \frac{9}{16} \frac{d}{dy} y^4 &= \frac{d}{dy} x^3 \\ x^2 + 2xy \frac{dx}{dy} + \frac{9}{4} y^3 &= 3x^2 \frac{dx}{dy}\end{aligned}$$

In the second line we used the product rule, and in the third line, we used the chain rule (for example, $\frac{d}{dy} x^2 = \frac{dx}{dy} \frac{d}{dx} x^2$). Now we can solve for $\frac{dx}{dy}$:

$$\begin{aligned}x^2 + 2xy \frac{dx}{dy} + \frac{9}{4} y^3 &= 3x^2 \frac{dx}{dy} \\ x^2 + \frac{9}{4} y^3 &= (3x^2 - 2xy) \frac{dx}{dy} \\ \frac{dx}{dy} &= \frac{x^2 + \frac{9}{4} y^3}{3x^2 - 2xy}\end{aligned}$$

Substituting $x = 3$ and $y = 2$ then gives

$$\begin{aligned}\frac{dx}{dy} &= \frac{x^2 + \frac{9}{4} y^3}{3x^2 - 2xy} \\ &= \frac{3^2 + \frac{9}{4} 2^3}{3 \cdot 3^2 - 2 \cdot 3 \cdot 2} \\ &= \frac{9}{5}\end{aligned}$$

5. We know that, for any function f , if $f'(x)$ exists and is nonzero, then $(x, f(x))$ is not an extremum of any kind. We compute that

$$\frac{d}{dx}x \log x = \log x + 1,$$

so $x \log x$ can only have extrema when $\log x = -1$ or $x = \frac{9}{10}$ (since functions are not differentiable at endpoints; see assignment 7 problem 2).

We have $\log x = -1$ when $x = e^{-1}$. When $x < e^{-1}$, the derivative $\log x + 1$ will be negative, and when $x > e^{-1}$ the derivative $\log x + 1$ will be positive, so $(e^{-1}, -e^{-1})$ is a local and global minimum.

Because $x \log x$ is increasing when $x > e^{-1}$, we know that $(\frac{9}{10}, \frac{9}{10} \log \frac{9}{10})$ is a local maximum. It remains to determine whether or not it's a global maximum.

We have

$$\begin{aligned} \lim_{x \rightarrow 0^+} x \log x &= \lim_{x \rightarrow 0^+} \frac{\log x}{x^{-1}} \\ &= \lim_{x \rightarrow 0^+} \frac{x^{-1}}{-x^{-2}} \\ &= \lim_{x \rightarrow 0^+} -x \\ &= 0 \end{aligned}$$

using L'Hospital's rule. Thus, the function $x \log x$ gets arbitrarily close to 0 as x approaches 0, and in particular it gets larger than $\frac{9}{10} \log \frac{9}{10}$, which is negative.