Calculus 1 Practice Problems

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1.* The number of prime numbers less than x is well approximated by

$$\int_2^x \frac{dt}{\log t}.$$

How many prime numbers would you expect to find between 10^9 and $10^9 + 10^7$?

2. Define

$$f(x) := \int_{-100}^{x} e^{\frac{\cos t}{100000}} (t^2 - 4)\, dt.$$

For what x does f attain a local minimum?

3. Write down a Riemann sum which estimates the value of

$$\int_{-10}^{2} e^{-x^2}\, dx$$

reasonably well.

4.*
a) A certain random number generator generates

- the number $\frac{1}{3}$ with probability $\frac{\left(\frac{1}{3}\right)^2}{\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+1^2}$
- the number $\frac{2}{3}$ with probability $\frac{\left(\frac{2}{3}\right)^2}{\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+1^2}$
- the number 1 with probability $\frac{1^2}{\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+1^2}$

If you generate 1000 numbers from this random number generator and add them up, what number do you expect to get?

b) A certain random number generator generates

- the number 0.25 with probability $\frac{0.25^2}{0.25^2+0.5^2+0.75^2+1^2}$
- the number 0.5 with probability $\frac{0.5^2}{0.25^2+0.5^2+0.75^2+1^2}$
- the number 0.75 with probability $\frac{0.75^2}{0.25^2+0.5^2+0.75^2+1^2}$
- the number 1 with probability $\frac{1^2}{0.25^2+0.5^2+0.75^2+1^2}$

If you generate 1000 numbers from this random number generator and add them up, what number do you expect to get?

c) For large n, A certain random number generator generates

- the number $\frac{1}{n}$ with probability $\frac{\left(\frac{1}{n}\right)^2}{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+1^2}$
- the number $\frac{2}{n}$ with probability $\frac{\left(\frac{2}{n}\right)^2}{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+1^2}$
- ...
- the number 1 with probability $\frac{1^2}{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+1^2}$

If you generate 1000 numbers from this random number generator and add them up, what number do you expect to get, approximately?

d) For large n, a certain random number generator generates

- the number $\frac{1}{n}$ with probability $\frac{f\left(\frac{1}{n}\right)}{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f(1)}$
- the number $\frac{2}{n}$ with probability $\frac{f\left(\frac{2}{n}\right)}{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f(1)}$
- ...
- the number 1 with probability $\frac{f(1)}{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f(1)}$

What is the average value of the output of this random number generator, approximately?

5.* What is the average value of the function $\sin x$ on the interval $[0, \pi]$?

6. Define

$$f_n(x) := (1+x)^{\frac{1}{n}}.$$

Estimate $f_n(-0.3)$. (Your answer will depend on n.)

7.* In this problem we will find a good estimate for $\sqrt{12345}$.
a) Estimate $\sqrt{12345}$ using linear approximation with $f(x) = \sqrt{x}$ and take $a_0 = 10000$ to be your base point. Write down your answer as a single decimal number which you think is correct to at least the 10s place.
b) Let y_0 be the answer you found in part a), truncated at the 10s place (meaning replace all digits after the 10s place with zeros). Define a_1 to be y_0^2. Compute a_1 and then approximate $\sqrt{12345}$ using linear approximation with $f(x) = \sqrt{x}$ and a_1 as your base point. Write your answer as a single decimal number which you think is correc to at least the 1s place.
c) Let y_1 be the answer you found in part b), truncated at the 1s place. Define a_2 to be y_1^2. Compute a_2 and then approximate $\sqrt{12345}$ using linear approximation with $f(x) = \sqrt{x}$ and a_2 as your base point.

8.* Estimate $\sqrt{2019}$.

9. Estimate $\frac{9}{10}\log\frac{9}{10}$.

10. Estimate arctan(100).

11.
Minimize the function $x^2 + y^2$ given the constraint $x^2 + xy = 2$.
Maximize the function $x^2 + y^2$ given the constraint $x^2 + xy = 2$.

12. Find all the local and global extrema of the following functions:

a) $\frac{\log x}{x}$ on the interval $(0, 5]$.

b) $\arctan 2x + \frac{1}{7x}$ for $x \in \mathbb{R}, x \neq 0$.

c) $\arctan(\log |x^2 - x - 1|)$ for $-2 < x \leq 4$.

d) $\begin{cases} |x|, & -2 < x < 2, x \neq 0 \\ 1, & x = 0 \end{cases}$.

13.
a) State the definition of a global maximum, a global minimum, a local maximum, and a local minimum.

b) Give an example of a function $f : [0, 1] \to [0, 1]$ which has no global extrema, and only has local extrema at $x = 0$ and $x = 1$.

c) Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which has at least one local max, at least one local min, and for which $f'(x) = 0$ has no solutions.

d) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is everywhere differentiable and has no extrema of any kind, but for which there exist distinct x_1 and x_2 such that $f'(x_1) = f'(x_2) = 0$.

14. Alice is running down the street. Her position is given by

$$s(t) := \frac{1}{t}.$$

a) What is Alice's average velocity between $t = 1$ and $t = 1.1$?

b) Explain why Alice's velocity at $t = 1$ is defined to be -1.

15. Suppose $f(3) = 7$ and $f(3.03) = 6.99$. Guess the equation of the line tangent to the curve $y = f(x)$ at $x = 3$.

16. Use the definition of the derivative to prove that $\frac{d}{dx}x^2 = 2x$.

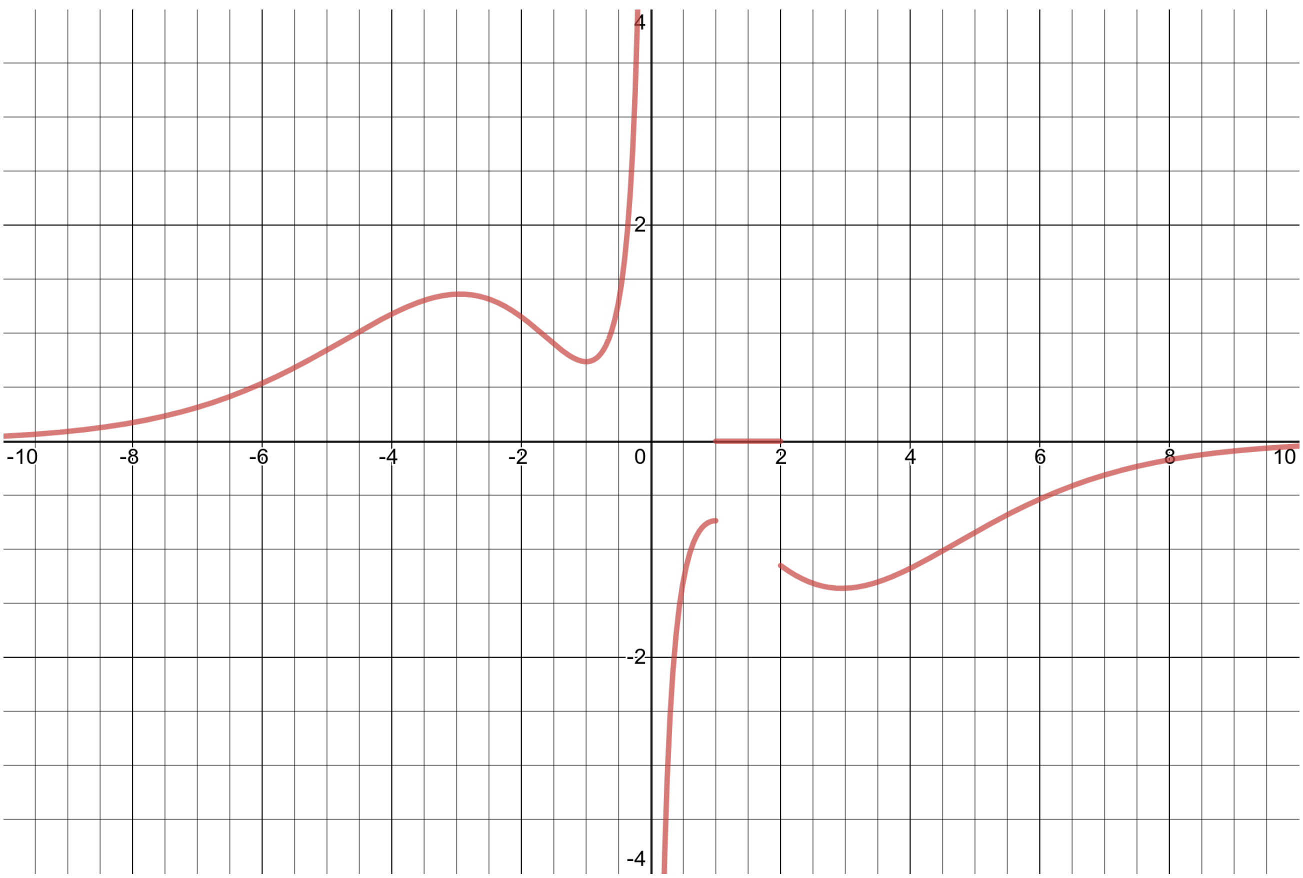
17. A 10-meter ladder is leaning against the wall of a building, and the base of the ladder is sliding away from the building at a rate of 3 meters per second. How fast is the top of the ladder sliding down the wall when the base of the ladder is 6 meters from the wall?

18. Suppose x and y are related via the equation $x^2 \cos y + 3^{xy} = \frac{2\sqrt{2}}{\pi^2} + \sqrt{3}$, and that $\frac{dy}{dt} = 2$. Find $\frac{dx}{dt}$ when $(x, y) = \left(\frac{2}{\pi}, \frac{\pi}{4}\right)$.

19. Suppose p and q are related via the equation $q \sin(p^2 q^2) = p$. At the point $(p, q) = \left(\frac{\pi^{\frac{1}{4}}}{2}, \pi^{\frac{1}{4}}\right)$ it is known that $\frac{dp}{dt} = 3$. Find $\frac{dq}{dt}$ at this point.

20. If x and y are related via the equation $x2^y + 2^{xy} = 80$, find $\frac{dx}{dy}$ at the point (2,3).

21. Let f be the function depicted below.
a) Give all values of x for which $f'(x) = 0$ (approximately).
b) Give all values of x for which $f''(x) = 0$ (approximately).



22. Sketch $\arctan(x^2)$.

23. Let $f(x(t))$ be a differentiable function. Suppose that $x(t) = e^t$ and that $\frac{df}{dx} = \frac{1}{1+x^4}$. What is $\frac{df}{dt}$?

24. True or false: $(2^9)' = 9 \cdot 2^8$.

25. True or false: $(x^x)' = x \cdot x^{x-1}$.

26. The Lambert W-function is the inverse function of the function $z \mapsto ze^z$. Prove that

$$\frac{dW}{dz} = \frac{1}{z + e^{W(z)}},$$

where W is the Lambert W-function.

27. Show that $\int_M^\infty \frac{dx}{x^2+0.0001 \log\log x} < \frac{1}{M}$.

28. True or false:
a) $\int_0^1 \sqrt{1+x^2}\, dx = \sqrt{2} - 1$.
b) $\int x^2 \cos x \, dx = \frac{1}{3}x^3 \sin x + C$.

29. True of false: There exists a function f and real numbers $a < b < c$ such that f has a vertical asymptote at b and $\int_a^c f(x)\, dx$ exists.

30.
a) Show that the function $x^5 - 2x^2 + x - 1$ has a root in the interval $[0, 2]$.

b) Explain why every 5th degree polynomial has at least one real root.

31. Prove that there's a real number x such that $2^x = 50x - 235$.

32. A Tibetan monk leaves the monastery at 7:00 am and takes his usual path to the top of the mountain, arriving at 7:00 pm. The following morning, he starts at 7:00 am at the top and takes the same path back, arriving at the monastery at 7:00 pm. Show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

33. Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose derivative is continuous everywhere.
a) Suppose there exist two points x_0 and x_1 with $x_0 < x_1$ and $f(x_0) > f(x_1)$. Prove that there exists an x^* such that $f'(x^*) < 0$.
b) Suppose that $f'(x) > 0$ for all x in the interval (a, b) Using the result from part a), prove that for all c_1 and c_2 in (a, b), if $c_1 < c_2$ then $f(c_1) < f(c_2)$.
c) Suppose that $f'(x) > 0$ for all x in the interval (a, b) Using the fundamental theorem of calculus, prove that for all c_1 and c_2 in (a, b), if $c_1 < c_2$ then $f(c_1) < f(c_2)$.

34. Let $f(x) = 2x + 1$ if $x < 1$ and $-x^2 + ax + b$ if $x \geq 1$. For what choice(s) of a and b will f be:
a) Continuous at $x = 1$?
b) Differentiable at $x = 1$?

35. Evaluate the following limits:

a) $\displaystyle\lim_{x\to-1} \frac{x^2-1}{x^2+3x+2}$

b) $\displaystyle\lim_{x\to\infty} \frac{3x^5+6x}{4x^5-3x^2+2}$

c) $\displaystyle\lim_{x\to\pi} \frac{\tan(x)}{x}$

d) $\displaystyle\lim_{x\to0} \frac{e^x-1-x}{\cos(x)-1}$

e) $\displaystyle\lim_{x\to0} \frac{\cos(5x)-1}{2^x-1-x\log 2}$

f) $\displaystyle\lim_{x\to0} \frac{x^2e^x}{e^{3x}-1-3x}$

g) $\displaystyle\lim_{x\to\infty} x^4e^{-x}$

h) $\displaystyle\lim_{n\to\infty} \begin{cases} 1+\frac{1}{n} & \text{if } n \text{ is even} \\ 1-\frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$

i) $\displaystyle\lim_{h\to0} \frac{1}{h}\int_2^{2+h} \tan\sqrt{t}\,dt$

36. Differentiate the following functions with respect to x:

a) $\frac{1}{x^{\frac{1}{3}}} + 17 \cdot 10^x + t$

b) $(\log(x^3 + 1))^4$

c) $x^2 e^x \sin x$

d) $\frac{x}{x^2 + 1}$

e) $(\log x + x^4 + 1)^{\frac{-3}{2}} + ((\arctan x)^2 - 2x + 3)^2 - x + 88$

f) $2^{x^2 \arcsin(3^x)}$

g) $\frac{4x^2}{\sqrt{e^{x^2} - 2}}$

37. Evaluate the following integrals:

a) $\displaystyle\int_{-2}^{2} 3x^5 + 2x^3 - x + 1\,dx$

b) $\displaystyle\int \frac{xe^{\arcsin r}}{\sqrt{1-r^2}}\,dr$

c) $\displaystyle\int_{3}^{4} 2^{2^g} 2^g\,dg$

d) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\sin(\sin u)\cos u\,du$

e) $\displaystyle\int \frac{\sqrt{\log t}}{t}\,dt$

f) $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{\infty} \frac{dx}{1+4x^2}$

g) $\displaystyle\int_{-\infty}^{\infty} \sin x\,dx$

h) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{e^{-x^2}}\,dx$

i) $\displaystyle\int_{0}^{1} \sqrt{s}\,ds$

j) $\displaystyle\int_{1}^{\infty} 7^{-t}\,dt$