1.1 $\quad x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x + \begin{pmatrix} e^t \\ t \end{pmatrix}$

$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ Step ① Diagonalize

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 3 = 0$$
$$= \lambda^2 - 4 + 3 = 0$$
$$\lambda = \pm 1$$

$\lambda_1 = 1$:
$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$v_1 = v_2$$
$$v_2 = 1 \qquad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -1$:
$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$3v_1 = v_2$$
$$v_2 = 3 \Rightarrow v_1 = 1 \qquad v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \qquad \lambda = 1 \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda = -1 \quad v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$
$$V^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad M^{-1} = \frac{1}{\det(M)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{quick formulas for } 2\times 2 \quad \det(M) = ad - bc}$$

$$A = V\Gamma V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

Step ② Transform

$$x' = Ax + \begin{pmatrix} e^t \\ t \end{pmatrix}$$
$$x' = V\Gamma V^{-1} x + \begin{pmatrix} e^t \\ t \end{pmatrix}$$
$$\underbrace{V^{-1}x'}_{y'} = \Gamma \underbrace{V^{-1}x}_{y} + V^{-1}\begin{pmatrix} e^t \\ t \end{pmatrix}$$

$$y' = \Gamma y + V^{-1}\begin{pmatrix} e^t \\ t \end{pmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^t \\ t \end{bmatrix}$$
$$y_1' = y_1 + \tfrac{1}{2}(3e^t - t) \Rightarrow y_1' - y_1 = \tfrac{1}{2}(3e^t - t)$$
$$y_2' = -y_2 + \tfrac{1}{2}(-e^t + t) \Rightarrow y_2' + y_2 = \tfrac{1}{2}(-e^t + t)$$

Step ③ Solve independent equs

$$\underline{y_1' = y_1 + \tfrac{1}{2}(3e^t - t)}$$
$$y_1 = c_1e^t + ate^t + (bt + c)$$
$$y_1' = c_1e^t + ate^t + ae^t + b$$
$$(c_1e^t + ate^t + ae^t + b) = (c_1e^t + ate^t + (bt + c)) + \tfrac{3}{2}e^t - \tfrac{t}{2})$$
$$c_1 + a = c_1 + \tfrac{3}{2} \Rightarrow a = \tfrac{3}{2} \qquad (e^t)$$
$$a = a \qquad (te^t)$$
$$0 = b - \tfrac{1}{2} \Rightarrow b = 1/2 \qquad (t)$$
$$b = c \Rightarrow c = 1/2 \qquad (\text{const.})$$

$$y_1 = c_1e^t + \tfrac{3}{2}te^t + \tfrac{1}{2}t + \tfrac{1}{2}$$
$$y_1' = c_1e^t + \tfrac{3}{2}e^t + \tfrac{3}{2}te^t + \tfrac{1}{2}$$
$$y_1' - y_1 = \tfrac{3}{2}e^t - \tfrac{1}{2}t \ \checkmark$$

$$\underline{y_2' = -y_2 + \tfrac{1}{2}(-e^t + t)}$$
$$y_2 = c_2e^{-t} + ae^t + (bt + c)$$
$$y_2' = -c_2e^{-t} + ae^t + b$$
$$(-c_2e^{-t} + ae^t + b) = -(c_2e^{-t} + ae^t + (bt + c)) + \tfrac{1}{2}(-e^t + t)$$
$$-c_2 = -c_2 \qquad (e^{-t})$$
$$a = -a - \tfrac{1}{2} \Rightarrow a = -\tfrac{1}{4} \qquad (e^t)$$
$$b = -c \Rightarrow c = -\tfrac{1}{2} \qquad (\text{const.})$$
$$0 = -b + \tfrac{1}{2} \Rightarrow b = \tfrac{1}{2} \qquad (t)$$

$$y_2 = c_2e^{-t} - \tfrac{1}{4}e^t + \tfrac{1}{2}t - \tfrac{1}{2}$$
$$y_2' = -c_2e^{-t} - \tfrac{1}{4}e^t + \tfrac{1}{2}$$
$$y_2 + y_2' = -\tfrac{1}{2}e^t + \tfrac{1}{2}t \ \checkmark$$

$$y = \begin{bmatrix} c_1e^t + \tfrac{3}{2}te^t + \tfrac{1}{2}t + \tfrac{1}{2} \\ c_2e^{-t} - \tfrac{1}{4}e^t + \tfrac{1}{2}t - \tfrac{1}{2} \end{bmatrix}$$

Step ④ Reverse Transform

$$x = Ry$$
$$x = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_1e^t + \tfrac{3}{2}te^t + \tfrac{1}{2}t + \tfrac{1}{2} \\ c_2e^{-t} - \tfrac{1}{4}e^t + \tfrac{1}{2}t - \tfrac{1}{2} \end{bmatrix}$$
$$x = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_1e^t \\ c_2e^{-t} \end{bmatrix} + \begin{bmatrix} \tfrac{3}{2}te^t - \tfrac{1}{4}e^t + t \\ \tfrac{3}{2}te^t - \tfrac{3}{4}e^t + 2t - 1 \end{bmatrix}$$

$$x' = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_1e^t \\ -c_2e^{-t} \end{bmatrix} + \begin{bmatrix} \tfrac{3}{2}te^t + \tfrac{5}{4}e^t + 1 \\ \tfrac{3}{2}te^t + \tfrac{3}{4}e^t + 2 \end{bmatrix}$$

$$x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

$$\begin{bmatrix} \tfrac{3}{2}te^t + \tfrac{5}{4}e^t + 1 \\ \tfrac{3}{2}te^t + \tfrac{3}{4}e^t + 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} \tfrac{3}{2}te^t - \tfrac{1}{4}e^t + t \\ \tfrac{3}{2}te^t - \tfrac{3}{4}e^t + 2t - 1 \end{bmatrix}$$
$$= \begin{bmatrix} \tfrac{3}{2}te^t + \tfrac{1}{4}e^t + 1 \\ \tfrac{3}{2}te^t + \tfrac{3}{4}e^t - t + 2 \end{bmatrix} + \begin{bmatrix} e^t \\ t \end{bmatrix} \ \checkmark$$