Math 218 — Assignment 1 — Solutions

Alex Cowan and Leonard Korreshi

1. Find all functions $f(x)$ such that

$$x^{0.2024}\frac{df}{dx} + \frac{1}{\cos(f(x))} = 0$$

for $0 < x < \frac{\pi}{2}$.

Solution. We want to treat f as an independent variable and then use separation of variables.

$$\begin{aligned} x^{0.2024}\frac{df}{dx} &= -\frac{1}{\cos(f)} \\ \cos(f)\,df &= -\frac{dx}{x^{0.2024}} \\ \int \cos(f)\,df &= -\int \frac{1}{x^{0.2024}}\,dx \\ \sin(f) + C_1 &= -\frac{x^{0.7976}}{0.7976} + C_2 \\ \sin(f) &= -\frac{x^{0.7976}}{0.7976} + C \qquad (1) \\ f(x) &= \arcsin\left(-\frac{x^{0.7976}}{0.7976} + C\right). \qquad (2) \end{aligned}$$

Giving (2) with no further comment was sufficient for full credit for this problem. However, it is not the case that (2) is valid for all $C \in \mathbb{R}$; the circumstances of the problem impose further restrictions on the admissible values of C, and it does not denote an arbitrary real number in this problem. It's not too important to understand these details so early into the semester. We expound below for students up for doing everything properly.

Recall that $\arcsin(x)$ defined[1] only for $-1 \le x \le 1$. This means that for (2) to be valid (if all quantities are required to be real), we must have

$$-1 \le -\frac{x^{0.7976}}{0.7976} + C \le 1. \qquad (3)$$

Alternatively, we could reason that it must be possible to satisfy (1) with $x, f \in \mathbb{R}$; this also leads to the requirement (3).

The problem statement requires that (2) work for all $0 < x < \frac{\pi}{2}$. This means that the free parameter C must be such that (3) holds for all $0 < x < \frac{\pi}{2}$. As the function $-\frac{x^{0.7976}}{0.7976}$ is monotonic[2] we need only consider the endpoints $x = 0$ and $x = \frac{\pi}{2}$:

$$-1 \le C \le 1 \qquad \text{and} \qquad -1 + \frac{(\frac{\pi}{2})^{0.7976}}{0.7976} \le C \le 1 + \frac{(\frac{\pi}{2})^{0.7976}}{0.7976}. \qquad (4)$$

As C must satisfy both the left and right conditions of (4) simultaneously, we end up requiring

$$-1 + \frac{(\frac{\pi}{2})^{0.7976}}{0.7976} \le C \le 1. \qquad (5)$$

[1] Alternatively, $\arcsin(x) \in \mathbb{R}$ iff ("if and only if") $-1 \le x \le 1$.

[2] The function f is *monotonic* if the function f is increasing (everywhere in its domain), or the function f is decreasing (everywhere in its domain).

All in all, the complete and fully precise solution to this problem would not be only (2), but

$$f(x) = \arcsin\left(-\frac{x^{0.7976}}{0.7976} + C\right) \qquad \text{for any } C \text{ such that} \quad -1 + \frac{(\frac{\pi}{2})^{0.7976}}{0.7976} \leq C \leq 1.$$

In general, when giving general solutions with free parameters, the proper way to do it is to specify exactly what values the free parameters may take.

2. Solve

$$x\frac{dy}{dx} - y = x^3$$

for $x > 0$. (This problem is in the course notes, which you can check if you want to. Your solution should be more detailed than what's written there.)

Solution. This problem requires the integrating factor technique. First, divide the left and right hand sides of the ODE $xy' - y = x^3$ by x to get it into a form where the coefficient of y' is 1. The ODE becomes

$$\frac{dy}{dx} - \frac{1}{x}y = x^2.$$

Then

$$p(x) = -\frac{1}{x} \implies I(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\log(x)+\tilde{C}} = \frac{k_0}{x},$$

where $k_0 := e^{\tilde{C}}$. Recall that throughout the course log will denote the natural logarithm, a.k.a. ln. Feel free to write ln if you prefer.

With our integrating factor $I(x) = \exp(\int p(x)\,dx)$ in hand, we calculate:

$$\begin{aligned}
\frac{dy}{dx} - \frac{1}{x}y &= x^2 \\
\frac{k_0}{x}\frac{dy}{dx} - \frac{k_0}{x^2}y &= k_0 x && \text{(multiplying both sides by } I(x)\text{)} \\
\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} &= x && \text{(dividing by } k_0\text{; we see that our choice of antiderivative of } p(x) \text{ didn't matter)} \\
\frac{d}{dx}\left(\frac{1}{x}y\right) &= x && \text{(recognizing the product rule; multiplying by } I(x) \text{ earlier always sets this up)} \\
\int \frac{d}{dx}\left(\frac{y}{x}\right) dx &= \int x\,dx \\
\frac{y}{x} &= \frac{x^2}{2} + C && \text{(using the fundamental theorem of calculus: } \int \tfrac{df}{dx}\,dx = f(x) + C\text{)} \\
\implies y &= \frac{x^3}{2} + Cx.
\end{aligned}$$

3. Let $y = y(t)$ be a function of time t, and let $\dot{y} := \frac{dy}{dt}$ denote its derivative. Draw a direction field for the differential equation

$$\dot{y} = (y+1)(4-y)$$

and describe y's behaviour as $t \to \infty$ in terms of the initial value $y(0)$. (The convention of denoting derivatives with respect to time with a dot is common in physics.)

Solution. One example of a direction field is as follows, though the axes have been scaled for readability. Slopes start at the point and move to the cross.

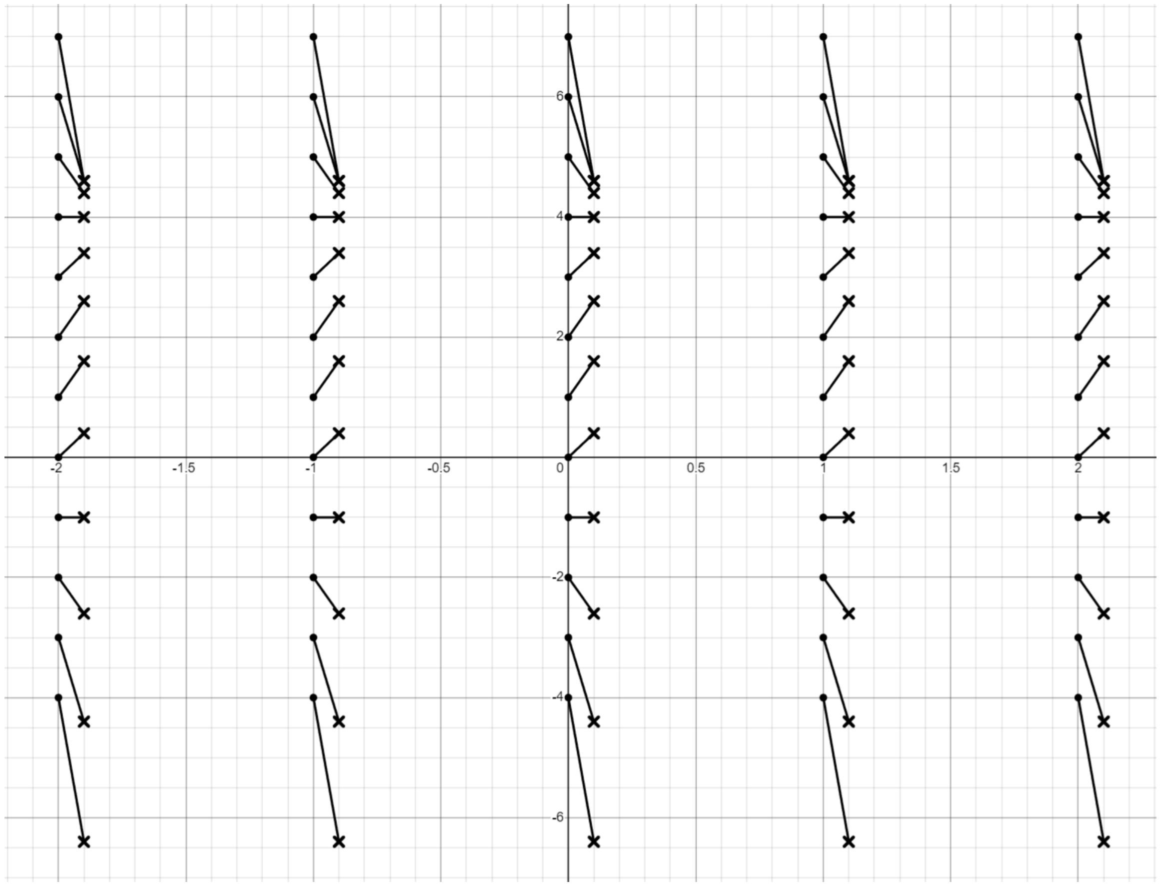

Figure 1: Slope field of $\dot{y} = (y+1)(4-y)$. The axes have been scaled for readability, and each slope starts at a dot and moves to the corresponding cross.

There are two asymptotes visible in the direction field, at $y = 4$ and $y = -1$. The first asymptote has slopes converging towards it, while the second has slopes diverging from it. We can describe the behaviour of the family of solutions to the ODE by describing the behaviour along these asymptotes, and the behaviour in each regin delimited by the asymptotes. Here this constitutes five regions/cases:

- $y < -1$
- $y = -1$
- $y > -1$ and $y < 4$
- $y = 4$
- $y > 4$.

Using Fig. 1 we observe the following behavior for y based on $y(0)$:

- When $y(0) < -1$, the function y is decreasing more and more quickly as y gets more and more negative. The rate of decrease $\dot{y}$ gets arbitrarily close to 0 as y nears -1, leading to a horizontal asymptote to the left of -1. The long term behaviour of a solution y, i.e. the limit of y as $t \to \infty$, is $-\infty$ for all solutions y in this region.
- When $y(0) = -1$, the function y is constant, i.e. the constant function $y = -1$ is a stable solution of the ODE. The limit of y as $t \to \infty$ is -1.
- When $y(0) > -1$ and $y(0) < 4$, the function y is increasing. It increases slowly for y near -1 or 4, and increases more quickly when y is farther from the two asymptotes, i.e. roughly half way between them. The limit of y as $t \to \infty$ is 4.

- When $y(0) = 4$, the function y is constant, i.e. the constant function $y = 4$ is a stable solution of the ODE. The limit of y as $t \to \infty$ is 4.
- When $y(0) > 4$, the function y is decreasing more and more quickly as y gets more and more positive. The rate of decrease $\dot{y}$ gets arbitrarily close to 0 as y nears 4, producing a horizontal asymptote to the right at $y = 4$. The limit of each solution y in this region as $t \to \infty$ is 4.