Math 218 — Assignment 2 solutions

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1. a) Give the general solution to the differential equation

$$y'' + 6y' + 13y = 0 \tag{1}$$

in terms of sin, cos, and exp with all coefficients real.

b) Give the general solution to (1) in terms of exp only and complex coefficients.

c) Express the free parameters of problem a) in terms of the free parameters of problem b).

d) Express the free parameters of problem b) in terms of the free parameters of problem a).

Solution. b) Pursuing the ansatz $y(x) = e^{rx}$,

$$(e^{rx})'' + 6(e^{rx})' + 13e^{rx} = (r^2 + 6r + 13)e^{rx}.$$

Because $e^t \neq 0$ for all $t \in \mathbb{C}$, if the above is to be 0 we must have

$$r^2 + 6r + 13 = 0 \quad \Longleftrightarrow \quad r = -3 \pm 2i$$

(using e.g. the quadratic formula to solve for r). In other words, the functions

$$y_1(x) \coloneqq e^{(-3+2i)x} \qquad \text{and} \qquad y_2(x) \coloneqq e^{(-3-2i)x}$$

both solve (1). They are linearly independent (meaning there are no $(\eta_1, \eta_2) \neq (0, 0)$ such that $\eta_1 y_1 + \eta_2 y_2 = 0$ for all x), and we expect 2 linearly independent solutions for an unconstrained second order ODE, so the general solution is

$$y_h(x) \coloneqq C_1 e^{(-3+2i)x} + C_2 e^{(-3-2i)x} \tag{2}$$

for any $C_1, C_2 \in \mathbb{C}$.

a) Let $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ be twice-differentiable, and set $z(x) \coloneqq u(x) + iv(x)$. Then

$$\frac{d}{dx} z(x) = \frac{d}{dx}(u(x) + iv(x)) = \frac{du}{dx} + i\frac{dv}{dx}.$$

Thus, for $z(x)$ of the form $u(x) + iv(x)$ with u, v as above, we see that

$$\frac{d}{dx}(\Re(z(x)) + i\Im(z(x))) = \Re\left(\frac{dz}{dx}\right) + i\Im\left(\frac{dz}{dx}\right).$$

Because the coefficients of (1) are real, if z is of the form we're assuming and is a solution to (1), i.e.

$$\frac{d^2 z}{dx^2} + 6\frac{dz}{dx} + 13z = 0,$$

then both $\Re(z(x))$ and $\Im(z(x))$ also solve (1). Let

$$z_h(x) \coloneqq e^{(-3+2i)x} = e^{-3x}(\cos(2x) + i\sin(2x)).$$

We see that z_h is of the form $u(x) + iv(x)$ for $u, v : \mathbb{R} \to \mathbb{R}$ and twice differentiable. Thus, because z_h is a solution to (1), as shown in b) above,

$$y_1(x) \coloneqq \Re(z_h(x)) = e^{-3x}\cos(2x) \qquad \text{and} \qquad y_2(x) \coloneqq \Im(z_h(x)) = e^{-3x}\sin(2x) \tag{3}$$

are solutions to (1) as well. Noting that y_1 and y_2 are linearly independent, a general solution to (1) with all coefficients real is

$$y_h(x) = D_1 e^{-3x} \cos(2x) + D_2 e^{-3x} \sin(2x) \tag{4}$$

for $D_1, D_2 \in \mathbb{R}$ arbitrary.

If these complex number manipulations are confusing, you can also just guess that y_1, y_2 as defined in (3) are solutions, and then plug them into (1) and verify that they are.

c) This question and d) are about expressing D_1, D_2 from (4) in terms of the C_1, C_2 from (2) and vice-versa. The reason to expect in the first place that this should be possible is because generically the number of free parameters in the general solution of an unconstrained 2^{nd} order ODE will be 2, so the four we have from (2) and (4) combined should be redundant.

We will need the following facts:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{5}$$
$$\cos(-\theta) = \cos\theta \tag{6}$$
$$\sin(-\theta) = -\sin\theta. \tag{7}$$

Substituting (5) into (2) yields

$$\begin{aligned} y_h(x) &= e^{-3x}\big(C_1(\cos 2x + i\sin 2x) + C_2(\cos(-2x) + i\sin(-2x))\big) \\ &= e^{-3x}\big(C_1(\cos 2x + i\sin 2x) + C_2(\cos 2x - i\sin 2x)\big) \qquad \text{(using (6) and (7))} \\ &= e^{-3x}\big((C_1+C_2)\cos 2x + (C_1 - C_2) i \sin 2x\big) \\ &= (C_1+C_2)e^{-3x}\cos 2x + i(C_1-C_2)e^{-3x}\sin 2x. \end{aligned} \tag{8}$$

If we want (8) to be equal to (4), then the coefficients of the $e^{-3x}\cos 2x$ and $e^{-3x}\sin 2x$ terms must match, i.e.

$$\begin{cases} D_1 = C_1 + C_2 \\ D_2 = i(C_1 - C_2). \end{cases} \tag{9}$$

This completes c).

d) We can solve for C_1 and C_2 in the linear system (9) with two equations and two unknowns. I'll show three ways to do this, and there are many other ways you could go about it. Feel free to do it however you like.

For the first way, write (9) as

$$\begin{cases} D_1 = C_1 + C_2 \\ -iD_2 = C_1 - C_2. \end{cases}$$

Adding up the two equations above and dividing by 2 gives $\frac{1}{2}(D_1 - iD_2) = C_1$, and taking their difference and dividing by 2 gives $\frac{1}{2}(D_1 + iD_2) = C_2$.

The second way I'll present uses the fact that

$$\Re(z) = \frac{z+\bar{z}}{2} \qquad \text{and} \qquad \Im(z) = \frac{z-\bar{z}}{2i}.$$

We also need to know that $\overline{e^{i\theta}} = e^{-i\bar{\theta}}$, and if $\theta \in \mathbb{R}$, then this is equal to $e^{-i\theta}$. From these observations we have that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

The two equalities above can be substituted into (4), and collecting terms then gives the result. This is very similar to how c) was solved above.

The third way, which you may consider much more complicated and which you are definitely completely free to ignore, starts by writing (5) as

$$\begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

where the second row is using (6) and (7). Taking $\theta = 2x$ and multiplying both sides by the scalar e^{-3x},

$$\begin{aligned} e^{-3x} \begin{pmatrix} e^{2ix} \\ e^{-2ix} \end{pmatrix} &= e^{-3x} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \cos 2x \\ \sin 2x \end{pmatrix} \\ \begin{pmatrix} e^{(-3+2i)x} \\ e^{(-3-2i)x} \end{pmatrix} &= \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} e^{-3x}\cos 2x \\ e^{-3x}\sin 2x \end{pmatrix}. \end{aligned}$$

We can now solve for the vector on the right by inverting the 2×2 matrix:

$$\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{-1} \begin{pmatrix} e^{(-3+2i)x} \\ e^{(-3-2i)x} \end{pmatrix} = \begin{pmatrix} e^{-3x}\cos 2x \\ e^{-3x}\sin 2x \end{pmatrix}.$$

To obtain (4) from the right hand side above, we can take the dot product with (D_1, D_2):

$$(D_1, D_2) \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{-1} \begin{pmatrix} e^{(-3+2i)x} \\ e^{(-3-2i)x} \end{pmatrix} = (D_1, D_2) \begin{pmatrix} e^{-3x}\cos 2x \\ e^{-3x}\sin 2x \end{pmatrix} = D_1 e^{-3x}\cos 2x + D_2 e^{-3x}\sin 2x.$$

You can invert the matrix however you like (and some ways amount to basically solving this problem in the first way presented above), but maybe you happen to know, or can guess/interpolate[1]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Hence

$$\begin{aligned} (D_1, D_2) \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{-1} \begin{pmatrix} e^{(-3+2i)x} \\ e^{(-3-2i)x} \end{pmatrix} &= (D_1, D_2) \cdot \frac{-1}{2i} \begin{pmatrix} -i & -i \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{(-3+2i)x} \\ e^{(-3-2i)x} \end{pmatrix} \\ &= (D_1, D_2) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & \frac{-1}{2i} \end{pmatrix} \begin{pmatrix} e^{(-3+2i)x} \\ e^{(-3-2i)x} \end{pmatrix} \\ &= (D_1, D_2) \begin{pmatrix} \frac{1}{2} e^{(-3+2i)x} + \frac{1}{2} e^{(-3-2i)x} \\ \frac{1}{2i} e^{(-3+2i)x} - \frac{1}{2i} e^{(-3-2i)x} \end{pmatrix} \\ &= \tfrac{1}{2} D_1 (e^{(-3+2i)x} + e^{(-3-2i)x}) + \tfrac{1}{2i} D_2 (e^{(-3+2i)x} - e^{(-3-2i)x}) \\ &= \frac{D_1 - iD_2}{2} e^{(-3+2i)x} + \frac{D_1 + iD_2}{2} e^{(-3-2i)x}. \end{aligned}$$

[1] This is very much off the beaten path, but e.g. for the diagonal entries you could be led to a guess by thinking of diagonal matrices, which you can invert by inspection. For the top right entry you could think of the map $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ which is a "homomorphism" from the integers under addition to 2×2 matrices, i.e. it preserves structure, turning addition into matrix multiplication: $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m+n \\ 0 & 1 \end{pmatrix}$. From this fact it's automatic that negating the top right entry in this case gives the matrix inverse, from which you can extrapolate. Alternatively, you could think of the linear transformation $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ which has the effect of rotating vectors by an angle θ; the inverse is then clearly the same matrix but with $\theta \mapsto -\theta$, and the top right/bottom left negations follow from (6) and (7). The scale $(ad - bc)^{-1}$ comes from $\det(A^{-1}) = (\det A)^{-1}$, i.e. det is a homomorphism. Anyway...

2. Give the general solution to

$$y'' + by' + cy = 0$$

with the initial condition $y(0) = y_0$.

Solution.

Step 1

Similarly to question 1 above, we start by implementing a trial function:

$$y = e^{mx}$$
$$\implies y' = me^{mx} \;\text{ and }\; y'' = m^2 e^{mx}.$$

Step 2

Next, we plug y, y', and y'' into the ODE and solve the characteristic equation:

$$(m^2 + bm + c)\, e^{mx} = 0.$$
$$\implies m_{1,2} = \frac{1}{2}\Big[- b \pm \sqrt{b^2 - 4c}\Big].$$

The nature of the solution y depends on the values of b and c. We consider 3 cases.

<u>**Case 1**</u>: $b^2 > 4c$ (distinct roots $\in \mathbb{R}$)

$$\implies y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

<u>**Case 2**</u>: $b^2 < 4c$ (distinct roots $\in \mathbb{C} \setminus \mathbb{R}$)

$$\implies y = (c_1 \cos \beta x + c_2 \sin \beta x)\, e^{\alpha x} \text{ , where } m_{1,2} = \alpha \pm i\beta.$$

<u>**Case 3**</u>: $b^2 = 4c$ (equal roots $\in \mathbb{R}$)

$$\implies y = (c_1 + c_2 x)\, e^{mx} \text{ , and } m = m_1 = m_2.$$

(Alternatively, cases 1 and 2 can be combined by allowing m_1 and m_2 in case 1 to be distinct complex numbers. The following manipulations do not assume that $m_{1,2} \in \mathbb{R}$; if you approached the problem using complex exponentials instead of trig functions, you can follow along with cases 1 and 3 only.

Moreover, the problem statement didn't specify $b, c \in \mathbb{R}$, though it would be reasonable to assume that. If you took $b, c \in \mathbb{C}$ arbitrary (good for you!), then do case 1 with $m_{1,2} \in \mathbb{C}$ and allow m to be complex in case 3.)

Step 3

Next, we implement the initial conditions (ICs):

$$y(0) = y_0.$$

<u>**Case 1**</u>: $y(0) = c_1 \cancelto{1}{e^{m_1 \cdot 0}} + c_2 \cancelto{1}{e^{m_2 \cdot 0}} = y_0,$

$$\implies c_1 + c_2 = y_0 \implies y(x) = c_1 e^{m_1 x} + (y_0 - c_1) e^{m_2 x} \text{ constitutes a general solution.}$$

<u>**Case 2**</u>: $y(0) = (c_1 \cancelto{1}{\cos(\beta \cdot 0)} + c_2 \cancelto{0}{\sin(\beta \cdot 0)})\, \cancelto{1}{e^{\alpha \cdot 0}} = y_0$, where $m_{1,2} = \alpha \pm i\beta$.

$$\implies c_1 = y_0 \implies y(x) = e^{\alpha x}[y_0 \cos(\beta x) + c_2 \sin(\beta x)] \text{ constitutes a general solution.}$$

<u>**Case 3**</u>: $y = (c_1 + \cancelto{0}{c_2(0)})\, \cancelto{1}{e^{m \cdot 0}} = y_0$, and $m = m_1 = m_2$.

$$\implies c_1 = y_0 \implies y(x) = y_0 e^{mx} + c_2 x e^{mx} \text{ constitutes a general solution.}$$

3. a) Give the general solution to

$$y'' + 6y' + 13y = e^{\alpha x}$$

for $\alpha \in \mathbb{C}$ such that $\alpha^2 + 6\alpha + 13 \neq 0$.

Solution.

Recall that a general solution for a non-homogeneous ODE is a combination of the solution to the corresponding homogeneous equation, and a particular solution to the non-homogeneous case, as follows

$$y(x) = y_h(x) + y_p(x).$$

Step 1 - Finding y_h.

$$y'' + 6y' + 13y = 0 \quad \leftarrow \text{homogeneous ODE}$$

Recalling the steps in question 1. a)....

1. Trial function: $y = e^{mx} \implies y' = me^{mx}$ and $y'' = m^2 e^{mx}$.
2. Characteristic: $(m^2 + 6m + 13)\, e^{mx} = 0 \implies m_{1,2} = -3 \pm 2i$
3. Homogeneous ODE solution: $y_h(x) = e^{-3x}[c_1 \cos(2x) + c_2 \sin(2x)]$.

Step 2 - Finding y_p.

$$y'' + 6y' + 13y = e^{\alpha x} \quad \leftarrow \text{inhomogeneous ODE}$$

1. Trial function: $y = Ae^{\alpha x} \implies y' = \alpha A e^{\alpha x}$ and $y'' = \alpha^2 A e^{\alpha x}$.
2. Characteristic: $A(\alpha^2 + 6\alpha + 13)\, e^{\alpha x} = e^{\alpha x} \iff A = \frac{1}{\alpha^2+6\alpha+13}$ $\leftarrow$ for this to hold, the denominator cannot equal zero, which is satisfied by the condition provided in the statement of the problem.
3. Inhomogeneous ODE particular solution: $y_p(x) = Ae^{\alpha x} = \frac{1}{\alpha^2+6\alpha+13} e^{\alpha x}$.

Step 3 - Combining the general solution.

Finally, the general solution to the inhomogeneous ODE provided in the problem is:

$$y(x) = y_h(x) + y_p(x) = e^{-3x}[c_1 \cos(2x) + c_2 \sin(2x)] + \frac{1}{\alpha^2 + 6\alpha + 13} e^{\alpha x},$$

for c_1, c_2 arbitrary constants, $\alpha \in \mathbb{C}$, and such that $\alpha^2 + 6\alpha + 13 \neq 0$.

3. b) Give the general solution to

$$y'' + 6y' + 13y = \sin(ax)$$

for $a \in \mathbb{R}$.

Solution 1.

Step 1 - Finding y_h.

$$y'' + 6y' + 13y = 0 \quad \leftarrow \text{homogeneous ODE}$$

Recalling the steps in question 1. a)....this is step the same as **3.a)** above.

1. Trial function: $y = e^{mx} \implies y' = me^{mx}$ and $y'' = m^2 e^{mx}$.
2. Characteristic: $(m^2 + 6m + 13)\, e^{mx} = 0 \implies m_{1,2} = -3 \pm 2i$
3. Homogeneous ODE solution: $y_h(x) = e^{-3x}[c_1 \cos(2x) + c_2 \sin(2x)]$.

Step 2 - Finding y_p.

$$y'' + 6y' + 13y = \sin(ax) \quad \leftarrow \text{inhomogeneous ODE}$$

1. Trial function: $y = A\cos(ax) + B\sin(ax) \quad \leftarrow$ we attempt to find A and B.
   $\implies y' = -aA\sin(ax) + aB\cos(ax)$ and $y'' = -a^2(A\cos(ax) + B\sin(ax))$.
2. Solving for A & B by subbing in the trial function, and its derivatives into the ODE, and then rearranging: $B = \frac{a^2+13}{a^4-10a^2+156}$, and $A = \frac{6a}{a^4-10a^2+156}$
3. Inhomogeneous ODE particular solution:

$$y_p(x) = A\cos(ax) + B\sin(ax) = \frac{6a}{a^4 - 10a^2 + 156}\cos(ax) + \frac{a^2 + 13}{a^4 - 10a^2 + 156}\sin(ax).$$

Step 3 - Combining the general solution.

Finally, the general solution to the inhomogeneous ODE provided in the problem is:

$$y(x) = y_h(x) + y_p(x) = e^{-3x}[c_1 \cos(2x) + c_2 \sin(2x)] + \frac{6a}{a^4 - 10a^2 + 156}\cos(ax) + \frac{a^2 + 13}{a^4 - 10a^2 + 156}\sin(ax),$$

for $a \in \mathbb{R}$, and c_1, c_2 arbitrary constants.

Solution 2.

Alternatively, we can solve this problem using the work we did in 3a). First, note that

1. if y_f is a solution to $y'' + 6y' + 13y = f(x)$ and y_g is a solution to $y'' + 6y' + 13y = g(x)$, then $y_f + y_g$ is a solution to $y'' + 6y' + 13y = f(x) + g(x)$, and
2. if y_h is a solution to $y'' + 6y' + 13y = h(x)$, then λy_h is a solution to $y'' + 6y' + 13y = \lambda h(x)$ for all $\lambda \in \mathbb{C}$.

These follow from the fact that "differentiation is a linear operator", i.e. $\frac{d}{dx}\big(af(x) + bg(x)\big) = a\frac{df}{dx} + b\frac{dg}{dx}$.

With these linearity observations, and the fact that $\sin ax = \frac{1}{2i}(e^{iax} - e^{-iax})$, and the solution of 3a), we can solve 3b) as follows.

From 3a) with $\alpha = ia$ (to which 3a applies: by assumption $a \in \mathbb{R}$ and therefore $\alpha := ia$ satisfies the hypothesis that $\alpha^2 + 6\alpha + 13 \neq 0$), we find that

$$y_+(x) := \frac{1}{-a^2 + 6ia + 13} e^{iax} \qquad \text{and} \qquad y_-(x) := \frac{1}{-a^2 - 6ia + 13} e^{-iax}$$

are solutions to

$$y'' + 6y' + 13y = e^{iax} \qquad \text{and} \qquad y'' + 6y' + 13y = e^{-iax}$$

respectively.

With the linearity observations listed above, we then deduce that

$$\frac{y_+ - y_-}{2i} = \frac{1}{-2ia^2 - 12a + 26i} e^{iax} - \frac{1}{-2ia^2 + 12a + 26i} e^{-iax}$$

is a particular solution to $y'' + 6y' + 13y = \frac{1}{2i}(e^{iax} - e^{-iax}) = \sin ax$. Combining with the homogeneous solution from problem 1 gives a general solution of

$$y = \frac{1}{-2ia^2 - 12a + 26i} e^{iax} - \frac{1}{-2ia^2 + 12a + 26i} e^{-iax} + C_1 e^{(-3+2i)x} + C_2 e^{(-3-2i)x}$$

for arbitrary $C_1, C_2 \in \mathbb{C}$.