Math 218 — Assignment 3

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Due 2024/09/27

1. a) Find the general solution of the DE

$$y'' + \omega_0^2 y = \alpha\cos(\omega t) \tag{1}$$

where ω_0, α, and ω are positive real constants. Which values(s) of ω have to be treated as special cases?

b) Find the unique solution of (1) with $\omega^2 \neq \omega_0^2$ subject to the initial conditions $y(0) = 0$ and $y'(0) = 0$.

c) Show that the solution in b) can be written

$$y(t) = A(t)\sin\big(\tfrac{1}{2}(\omega_0 + \omega)t\big)$$

where

$$A(t) := \frac{2\alpha}{\omega_0^2 - \omega^2}\sin\big(\tfrac{1}{2}(\omega_0 - \omega)t\big).$$

Give a qualitative sketch of the graph of $y(t)$ in the case where $\omega_0 - \omega$ is small compared to ω_0.

d) Show that the unique solution of (1) with $\omega^2 = \omega_0^2$ and initial conditions $y(0) = y'(0) = 0$ is

$$y(t) = \frac{\alpha}{2\omega_0} t\sin\omega_0 t.$$

Give a qualitative sketch of the graph.

2. Consider a mass whose position $y(t)$ satisfies the DE

$$y'' + 2\lambda y' + \omega^2 y = 0$$

with initial conditions $y(0) = y_0$ and $y'(0) = v_0$. Suppose that $\lambda \geq \omega > 0$, so that oscillations do not occur.

a) One expects that if v_0 is sufficiently large then the mass will pass through the equilibrium position $y = 0$ before coming to rest. Find the restriction on v_0 that will ensure that this happens in the case of critical damping $\lambda = \omega$. Also find the corresponding time t_0 for which $y(t_0) = 0$.

b) Find the time $t_{\min}$ at which y attains its minimum value $y_{\min}$. Show that $t_{\min}$ satisfies

$$t_{\min} = t_0 + \frac{1}{\omega},$$

and that

$$y_{\min} = -\left(\frac{v_0}{\omega} - y_0\right)e^{-\omega t_{\min}}.$$

c) Sketch the graphs of the displacement $y(t)$ and the velocity $y'(t)$, and label t_0, $t_{\min}$, v_0, and $y_{\min}$.

d) Show that if $\lambda > \omega$ then the mass will pass through the equilibrium position $y = 0$ if

$$\frac{v_0}{y_0} > \lambda + \sqrt{\lambda^2 - \omega^2},$$

and will not if

$$\frac{v_0}{y_0} < \lambda + \sqrt{\lambda^2 - \omega^2}.$$

What happens if

$$\frac{v_0}{y_0} = \lambda + \sqrt{\lambda^2 - \omega^2}\,?$$