Math 218 — Assignment 3 Solutions

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1. (a) Find the general solution of the DE

$$y'' + \omega_0^2\, y = \alpha \cos \omega t,$$

where ω_0, α and ω are constants. Which values(s) of ω have to be treated as special cases?

(b) Find the unique solution of the DE in (a) with $\omega^2 \neq \omega_0^2$, subject to the initial condition $y(0) = 0$, $y'(0) = 0$,

(c) Show that the solution in (b) can be written in the form

$$y = A(t) \sin\left[\tfrac{1}{2}(\omega_0 + \omega)t\right],$$

where

$$A(t) = \tfrac{2\alpha}{\omega_0^2 - \omega^2} \sin\left[\frac{1}{2}(\omega_0 - \omega)t\right].$$

Give a qualitative sketch of the graph of $y(t)$ in the case where $\omega_0 - \omega$ is small compared to ω_0. If you want to use specific values, use $\omega_0 = 11, \omega = 9$, but don't try to sketch the curve exactly.

(d) Show that the unique solution of the DE in (a) with $\omega^2 = \omega_0^2$, subject to the initial condition $y(0) = 0,\ y'(0) = 0$ is

$$y = \frac{\alpha}{2\omega_0} t \sin \omega_0 t.$$

Give a qualitative sketch of the graph.

Solution

(a) We first observe that the general solution of the homogeneous DE $y'' + \omega_0^2 y = 0$ is

$$y_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

To find y_p we must consider the two cases $\omega = \omega_0$ and $\omega \neq \omega_0$.
If $\omega \neq \omega_0$ then we use the trial function

$$y_p = a_1 \cos \omega t + a_2 \sin \omega t.$$

Then we have

$$\begin{aligned} y_p' &= -\omega a_1 \sin \omega t + \omega a_2 \cos \omega t \\ y_p'' &= -\omega^2 a_1 \cos \omega t - \omega^2 a_2 \sin \omega t. \end{aligned}$$

Putting these into the DE we get

$$-\omega^2 a_1 \cos \omega t - \omega^2 a_2 \sin \omega t + \omega_0^2 (a_1 \cos \omega t + a_2 \sin \omega t) = \alpha \cos \omega t.$$

Hence

$$-\omega^2 a_1 + \omega_0^2 a_1 = \alpha, \qquad -\omega^2 a_2 + \omega_0^2 a_2 = 0.$$

Thus

$$a_2 = 0, \qquad a_1 = \frac{\alpha}{\omega_0^2 - \omega^2}.$$

Therefore the solution is

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{\alpha}{\omega_0^2 - \omega^2} \cos \omega t.$$

If $\omega = \omega_0$ then we use the trial function

$$y_p = a_1 t \cos \omega_0 t + a_2 t \sin \omega_0 t.$$

Then we have

$$\begin{aligned} y_p' &= \omega_0 a_2 t \cos \omega_0 t - \omega_0 a_1 t \sin \omega_0 t + a_1 \cos \omega_0 t + a_2 \sin \omega_0 t \\ y_p'' &= -\omega_0^2 a_1 t \cos \omega_0 t - \omega_0^2 a_2 t \sin \omega_0 t 2\omega_0 a_2 \cos \omega_0 t - 2\omega_0 a_1 \sin \omega_0 t. \end{aligned}$$

Therefore the DE becomes

$$-\omega_0^2 a_1 t \cos \omega_0 t - \omega_0^2 a_2 t \sin \omega_0 t 2\omega_0 a_2 \cos \omega_0 t - 2\omega_0 a_1 \sin \omega_0 t + \omega_0^2 (a_1 t \cos \omega_0 t + a_2 t \sin \omega_0 t) = \alpha \cos \omega_0 t.$$

Simplifying we get

$$2\omega_0 a_2 \cos \omega_0 t - 2\omega_0 a_1 \sin \omega_0 t = \alpha \cos \omega_0 t.$$

Hence

$$a_1 = 0, \qquad a_2 = \frac{\alpha}{2\omega_0}.$$

So the solution is

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{\alpha}{2\omega_0} t \sin \omega_0 t.$$

(b) If $\omega^2 \neq \omega_0^2$, then $\omega \neq \omega_0$ and from part (a) we have

$$y = c_1 \cos \omega_0 t + c_s \sin \omega_0 t + \frac{\alpha}{\omega_0^2 - \omega^2} \cos \omega t.$$

Thus $y(0) = 0 \Rightarrow c_1 = -\frac{\alpha}{\omega_0^2 - \omega^2}$, and $y'(0) = 0$ gives us $0 = \omega_0 c_2$, hence

$$y(t) = \frac{\alpha}{\omega_0^2 - \omega^2} (\cos \omega t - \cos \omega_0 t).$$

(c) Using

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\left(\frac{A-B}{2}\right)$$

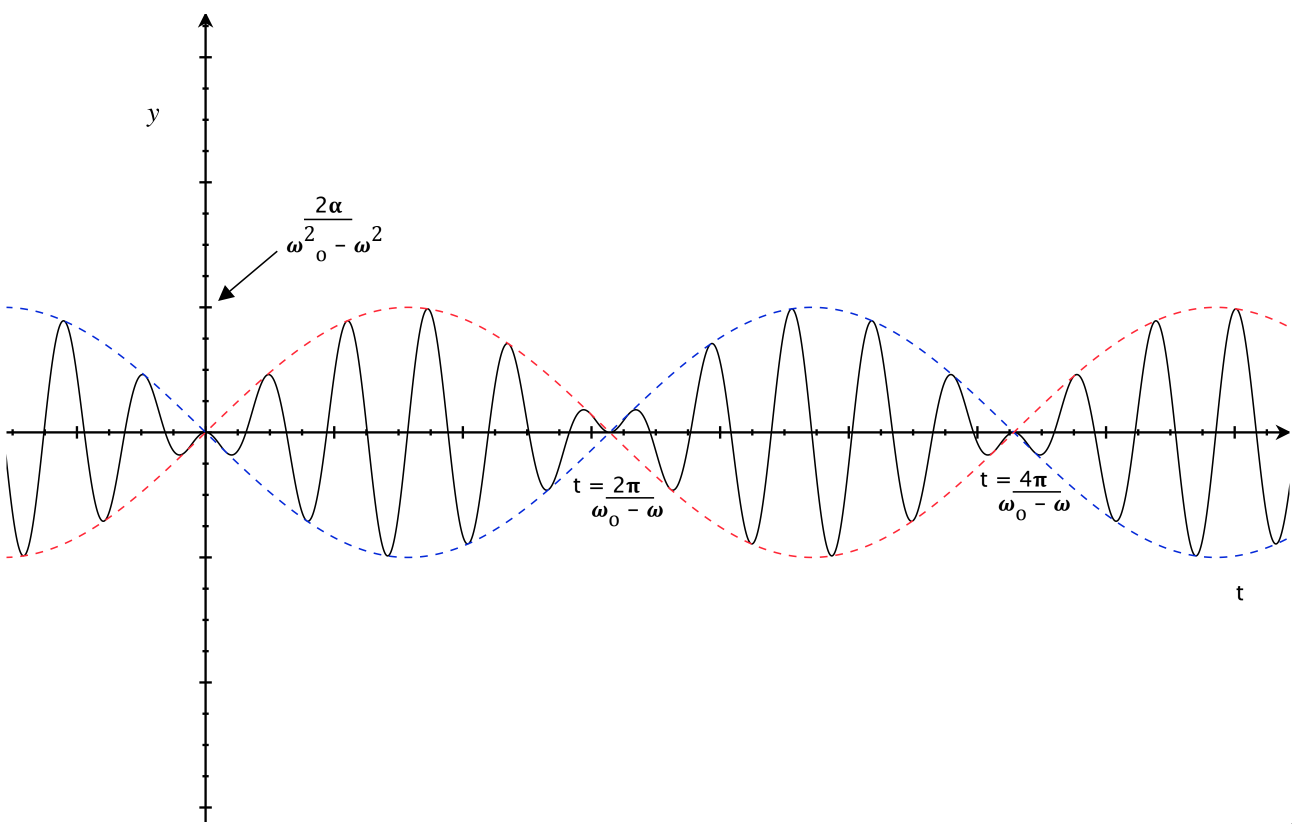
we have

$$\begin{aligned}\frac{\alpha}{\omega_0^2-\omega^2}(\cos\omega t - \cos\omega_0 t) &= -\frac{2\alpha}{\omega_0^2-\omega^2}\sin\left[\left(\frac{\omega-\omega_0}{2}\right)t\right]\sin\left[\left(\frac{\omega+\omega_0}{2}\right)t\right] \\ &= \frac{2\alpha}{\omega_0^2-\omega^2}\sin\left[\left(\frac{\omega_0-\omega}{2}\right)t\right]\sin\left[\left(\frac{\omega_0+\omega}{2}\right)t\right].\end{aligned}$$

Thus we can think of this as a sine function with frequency $\frac{1}{2}(\omega_0+\omega)$ with an amplitude function $A(t) = \frac{2\alpha}{\omega_0^2-\omega^2}\sin\left[\left(\frac{\omega_0-\omega}{2}\right)t\right]$. The amplitude function provides an envelope for the inner sine function to oscillate within.

Since $A(t)$ has frequency $\frac{\omega_0-\omega}{2}$ it will be much slower when $\omega_0 \approx \omega$ compared to the inner sine function (with frequency $\frac{1}{2}(\omega_0+\omega)$) which will oscillate quickly.

In the plot below the red and blue dashed curves represent the amplitude function $A(t)$ and $-A(t)$ respectively.



The phenomenon here is called beats. It's when you have two things oscillating at very close frequencies. It gives a sort of WAH-WAH-WAH effect. See https://www.youtube.com/watch?v=V8W4Djz6jnY

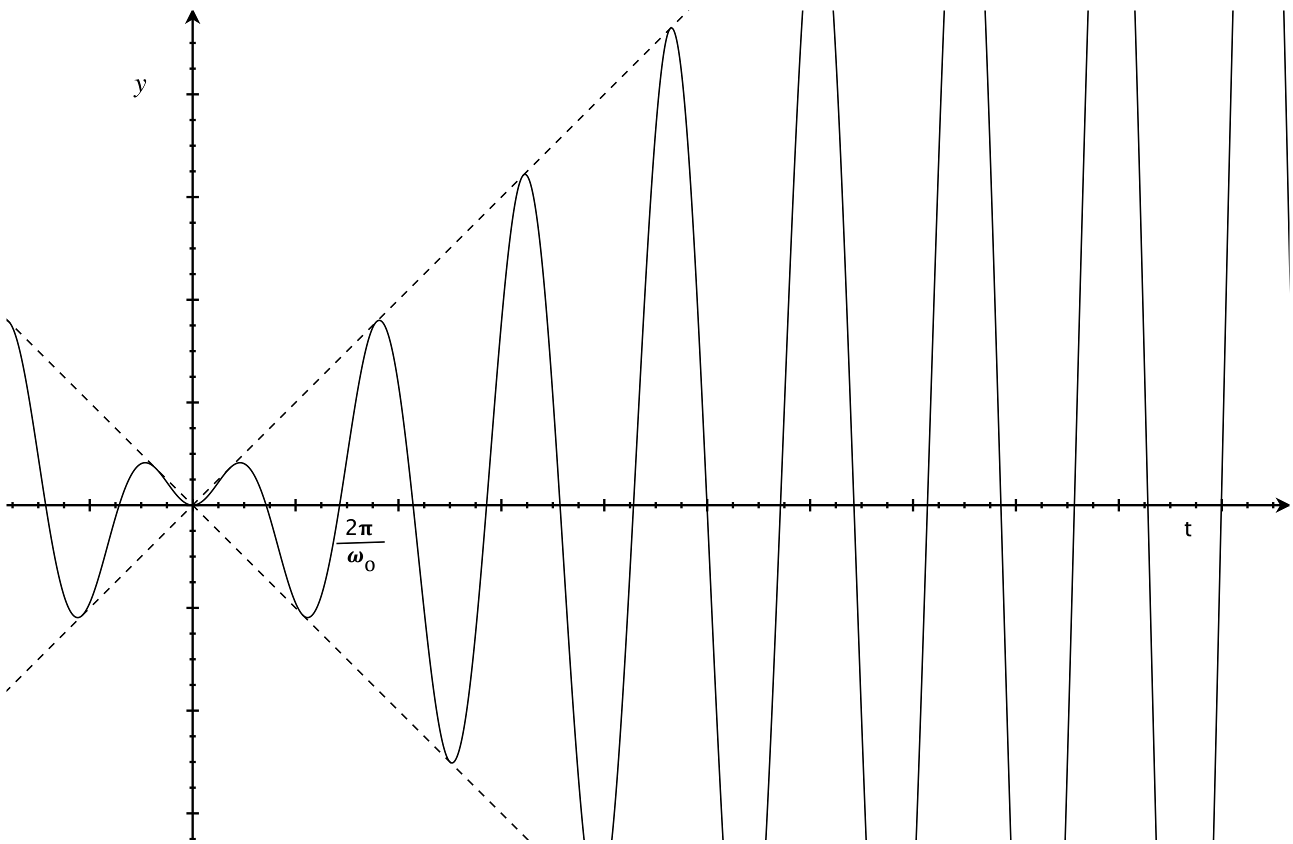
(d) If $\omega = \omega_0$ then

$$y = c_1 \cos \omega t + c_2 \sin \omega t + \frac{\alpha}{2\omega_0} t \sin \omega_0 t.$$

Hence $y(0) = 0$ gives $c_1 = 0$ and $y'(0) = 0$ gives $c_2 = 0$. Therefore

$$y = \frac{\alpha}{2\omega_0} t \sin \omega_0 t.$$

In the plot below the dashed lines are the curves $y = \frac{\alpha}{2\omega_0}$ and $y = -\frac{\alpha}{2\omega_0}$ again acting as an envelope for the inner sine function.



2. Consider a mass whose position $y(t)$ satisfies the DE

$$y'' + 2\lambda y' + \omega^2 y = 0 \tag{1}$$

with initial conditions $y(0) = y_0$ and $y'(0) = v_0$. Suppose that $\lambda \geq \omega > 0$, so that oscillations do not occur.

a) One expects that if v_0 is sufficiently large then the mass will pass through the equilibrium position $y = 0$ before coming to rest. Find the restriction on v_0 that will ensure that this happens in the case of critical damping $\lambda = \omega$. Also find the corresponding time t_0 for which $y(t_0) = 0$.

b) Find the time t_{min} at which y attains its minimum value y_{min}. Show that t_{min} satisfies

$$t_{min} = t_0 + \frac{1}{\omega},$$

and that

$$y_{min} = -\Big(\frac{v_0}{\omega} - y_0\Big)e^{-\omega t_{min}}.$$

c) Sketch the graphs of the displacement $y(t)$ and the velocity $y'(t)$, and label t_0, t_{min}, v_0, and y_{min}.

d) Show that if $\lambda > \omega$ then the mass will pass through the equilibrium position $y = 0$ if

$$\frac{v_0}{y_0} > \lambda + \sqrt{\lambda^2 - \omega^2},$$

and will not if

$$\frac{v_0}{y_0} < \lambda + \sqrt{\lambda^2 - \omega^2}.$$

What happens if

$$\frac{v_0}{y_0} = \lambda + \sqrt{\lambda^2 - \omega^2}?$$

First we solve the DE (1)

2. With the ansatz $y = e^{\alpha t}$,

$$\frac{d^2}{dt^2} e^{\alpha t} + 2\lambda \frac{d}{dt} e^{\alpha t} + \omega^2 e^{\alpha t} = 0$$

$$\Leftrightarrow (\alpha^2 + 2\lambda\alpha + \omega^2) e^{\alpha t} = 0$$

Because $e^z \neq 0$ for all $z \in \mathbb{C}$,

$$\Leftrightarrow \alpha^2 + 2\lambda\alpha + \omega^2 = 0$$

By assumption $\lambda = \omega > 0$, so

$$\alpha^2 + 2\omega\alpha + \omega^2 = 0$$
$$(\alpha + \omega)^2 = 0$$

$$\Leftrightarrow \alpha = -\omega$$

~~So the homogeneous soln before initial conditions is ___~~

This gives that $e^{-\omega t}$ is a soln to (1) (problem calls the eq (1))

We expect two solutions, but only got one bc $-\omega$ was a double root,

Trying $y(t) = te^{-\omega t}$:

$$\frac{dy}{dt} = e^{-\omega t} - \omega t e^{-\omega t},$$

$$\frac{d^2y}{dt^2} = -\omega e^{-\omega t} - \omega\left(te^{-\omega t}\right)$$

$$= -\omega e^{-\omega t} - \omega\left(e^{-\omega t} - \omega t e^{-\omega t}\right)$$

$$= -2\omega e^{-\omega t} + \omega^2 t e^{-\omega t}$$

So $y'' + 2\omega y' + \omega^2 y$

$$= \underbrace{-2\omega e^{-\omega t} + \omega^2 t e^{-\omega t}}_{y''} + 2\omega\underbrace{\left(e^{-\omega t} - \omega t e^{-\omega t}\right)}_{y'} + \omega^2 \underbrace{t e^{-\omega t}}_{y}$$

$$= -2\omega e^{-\omega t} + 2\omega e^{-\omega t} + \omega^2 t e^{-\omega t} - 2\omega^2 t e^{-\omega t} + \omega^2 t e^{-\omega t}$$

$= 0.$

The functions $y_1(t) := e^{-\omega t}$ and $y_2(t) := te^{-\omega t}$ are linearly indep (i.e. $\nexists\, c \in \mathbb{C}$ s.t. $y_1 + c y_2 = 0$),

so the $\Rightarrow$ $C_1 e^{-\omega t} + C_2 t e^{-\omega t} = y(t)$

Now we impose the initial conditions $y(0) = y_0$, $y'(0) = v_0$

$$y(0) = C_1 e^{-\omega \cdot 0} + C_2 \cdot 0 \cdot e^{-\omega \cdot 0} = C_1$$

$\Rightarrow C_1 = y_0$

(use bottom of prev page)

$$y'(0) = -C_1 \omega e^{-\omega t} + C_2\left(e^{-\omega t} - \omega t e^{-\omega t}\right)\Big|_{t=0}$$

$= -\omega C_1 + C_2 \Rightarrow v_0 = -\omega y_0 + C_2 \Rightarrow C_2 = v_0 + \omega y_0$ (sanity check units)

All in all, $y(t) = y_0 e^{-\omega t} + (v_0 + \omega y_0) t e^{-\omega t}$ (Sanity check units)

a) We want t_0 s.t. $y(t_0) = 0$. From context, we require $t_0 \geq 0$.

Let's set $y(t) = 0$ and solve for t.

$$0 = y_0 e^{-\omega t} + (v_0 + \omega y_0) t e^{-\omega t}$$

bc $e^{z} \neq 0$

$$\Leftrightarrow \quad 0 = y_0 + (v_0 + \omega y_0) t$$

Note: if $y_0 = 0$, then $t_0 = 0$ (matching physical intuition). We note this and henceforth assume we are in the case $y_0 \neq 0$

$$\Leftrightarrow \quad t_0 = -\frac{y_0}{v_0 + \omega y_0} = -\frac{1}{\frac{v_0}{y_0} + \omega}$$

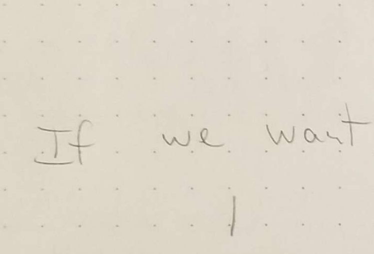
Also note that if $v_0 + \omega y_0 = 0$, there is a soln iff $y_0 = 0$, justifying $v_0 < -\omega y_0$ and not $v_0 \leq -\omega y_0$

If we want $t_0 > 0$, then

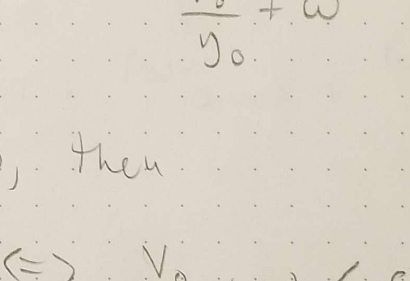
$$-\frac{1}{\frac{v_0}{y_0} + \omega} > 0 \quad \Leftrightarrow \quad \frac{v_0}{y_0} + \omega < 0$$

$$\Leftrightarrow \quad \frac{v_0}{y_0} < -\omega$$

Sanity check: sign makes sense:



vs.



So the condition on (y_0, t_0) is $y_0 = 0$ or $v_0 < -\omega y_0$

b) t_{min} will be such that $y'(t_{min}) = 0$

Let's set $y'(t) = 0$ and solve for t.

$$y'(t) = -\omega y_0 e^{-\omega t} + (v_0 + \omega y_0)\left(e^{-\omega t} - \omega t e^{-\omega t}\right)$$

$$= (-\omega y_0 + v_0 + \omega y_0) e^{-\omega t} - \omega (v_0 + \omega y_0) t e^{-\omega t}$$

$$= v_0 e^{-\omega t} - \omega (v_0 + \omega y_0) t e^{-\omega t}$$

Note: could have deduced this term for free

$y'(t) = 0 \iff \qquad = 0$

bc $e^{-} \neq 0$

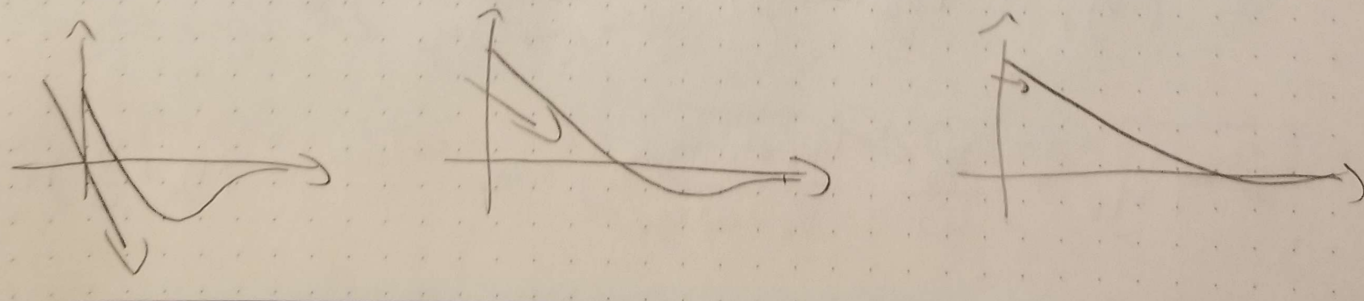
$\iff \quad v_0 - \omega (v_0 + \omega y_0) t = 0$ (sanity check units)

$$\iff t_{min} = \frac{v_0}{\omega (v_0 + \omega y_0)} = \frac{1}{\omega} \frac{1}{1 + \omega \frac{y_0}{v_0}}$$

Sanity check: $\lim_{v_0 \to -\omega y_0} \frac{1}{\omega} \frac{1}{1 + \omega \frac{y_0}{v_0}} = \infty$

(Recall: crosses axis iff $v_0 < -\omega y_0$) lim from the left only

This matches physical reasoning:



Question asks to show $t_{min} = t_0 + \frac{1}{\omega}$

$\rightsquigarrow$ Let's compute $t_{min} - t_0$. This should turn out to be $\frac{1}{\omega}$.

$$\frac{1}{\omega} \frac{1}{1 + \omega \frac{y_0}{v_0}} - \left(- \frac{1}{\frac{v_0}{y_0} + \omega} \right)$$

$$= \frac{1/\omega}{1 + \omega \frac{y_0}{v_0}} + \frac{1}{\omega + \frac{v_0}{y_0}}$$

Mult. top and bottom by $\frac{y_0}{v_0}$

$$= \frac{1/\omega}{1 + \omega \frac{y_0}{v_0}} + \frac{y_0 / v_0}{\omega \frac{y_0}{v_0} + 1}$$

$$= \frac{\frac{y_0}{v_0} + \frac{1}{\omega}}{1 + \omega \frac{y_0}{v_0}} = \frac{1}{\omega} \frac{\omega \frac{y_0}{v_0} + 1}{1 + \omega \frac{y_0}{v_0}} = \frac{1}{\omega}$$

Question also asks to find $y(t_{min})$ — Question accepts $e^{-\omega t_{min}}$ (there is a sign error in the question)

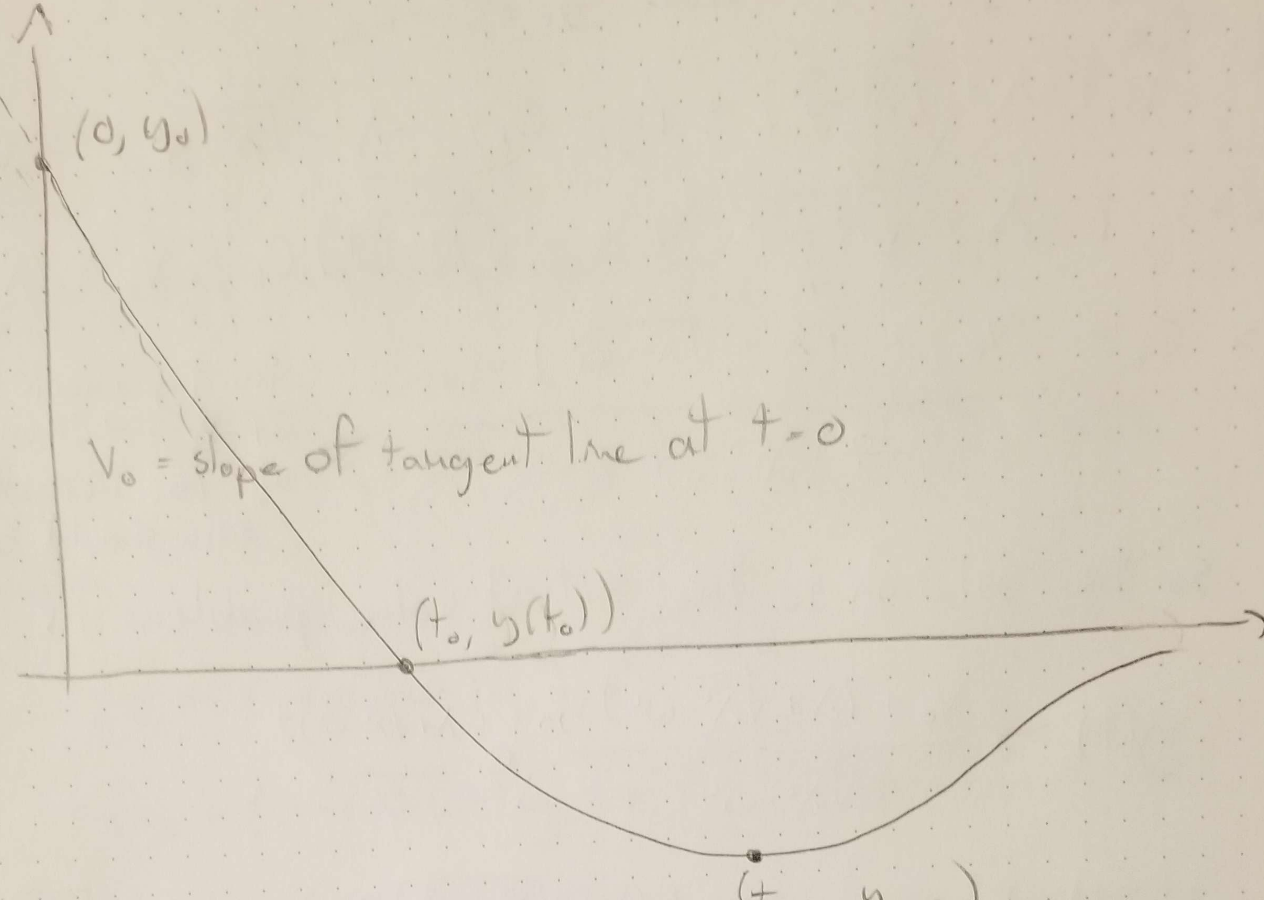
$$y(t_{min}) = y_0 e^{-\omega t_{min}} + (v_0 + \omega y_0) \frac{1}{\omega} \frac{1}{1 + \omega \frac{y_0}{v_0}} e^{-\omega t_{min}}$$

$$= \left(y_0 + (v_0 + \omega y_0) \frac{1}{\omega} \frac{1}{1 + \omega \frac{y_0}{v_0}} \right) e^{-\omega t_{min}}$$

$$= \left(y_0 + (v_0 + \omega y_0) \frac{1}{\omega} \frac{v_0}{v_0 + \omega y_0} \right) e^{-\omega t_{min}}$$

$$= \left(y_0 + \frac{v_0}{\omega} \frac{v_0 + \omega y_0}{v_0 + \omega y_0} \right) e^{-\omega t_{min}} = \left(y_0 + \frac{v_0}{\omega} \right) e^{-\omega t_{min}}$$

c)



d) We follow the same strategy/template that we just used for a) – c) in the case $\lambda = \omega$.

First, we solve the initial value problem

$$y'' + 2\lambda y' + \omega^2 y = 0, \quad y(0) = y_0, \quad y'(0) = V_0$$

Guessing $y = e^{\alpha t}$ gives the solution before of

$$y(t) = C_1 e^{(-\lambda + \sqrt{\lambda^2 - \omega^2})t} + C_2 e^{(-\lambda - \sqrt{\lambda^2 - \omega^2})t}$$

$y(0) = y_0 \Rightarrow C_1 + C_2 = y_0$

$y'(0) = V_0 \Rightarrow C_1(-\lambda + \sqrt{\lambda^2 - \omega^2}) + C_2(-\lambda - \sqrt{\lambda^2 - \omega^2}) = V_0$

$C_2 = y_0 - C_1$ from $y_0 = C_1 + C_2$

$$C_1\left(-\lambda + \sqrt{\lambda^2-\omega^2}\right) + (y_0 - C_1)\left(-\lambda - \sqrt{\lambda^2-\omega^2}\right) = V_0$$

$$\Leftrightarrow \left(-\lambda + \sqrt{\lambda^2-\omega^2} + \lambda + \sqrt{\lambda^2-\omega^2}\right) C_1 = V_0 + \left(\lambda + \sqrt{\lambda^2-\omega^2}\right) y_0$$

$$\Leftrightarrow C_1 = \frac{V_0 + \left(\lambda + \sqrt{\lambda^2-\omega^2}\right) y_0}{2\sqrt{\lambda^2-\omega^2}}$$

Sanity check:
If $\lambda \to \omega$, then this blows up (bc actually the soln should be $te^{-\omega t}$)

So the solution to the initial value problem is

$$y(t) = \frac{V_0 + \left(\lambda + \sqrt{\lambda^2-\omega^2}\right) y_0}{2\sqrt{\lambda^2-\omega^2}} e^{\left(-\lambda + \sqrt{\lambda^2-\omega^2}\right)t}$$

$$+ \left(y_0 - \frac{V_0 + \left(\lambda + \sqrt{\lambda^2-\omega^2}\right) y_0}{2\sqrt{\lambda^2-\omega^2}}\right) e^{\left(-\lambda - \sqrt{\lambda^2-\omega^2}\right)t}$$

$$= \frac{V_0 + \left(\lambda + \sqrt{\lambda^2-\omega^2}\right) y_0}{2\sqrt{\lambda^2-\omega^2}} e^{\left(-\lambda + \sqrt{\lambda^2-\omega^2}\right)t}$$

$$- \frac{V_0 + \left(\lambda - \sqrt{\lambda^2-\omega^2}\right) y_0}{2\sqrt{\lambda^2-\omega^2}} e^{\left(-\lambda - \sqrt{\lambda^2-\omega^2}\right)t}$$

Now we want to show that $\exists\, t_1 \geq 0$ s.t. $y(t_1) = 0$

if $-\frac{V_0}{y_0} > \lambda + \sqrt{\lambda^2-\omega^2}$

Sign error in question, sorry!

(factoring out)

$y(t) = 0$

$$\frac{e^{-\lambda t}}{2\sqrt{\lambda^2-\omega^2}}\Big((v_0 + (\lambda+\sqrt{\lambda^2-\omega^2})y_0)\, e^{\sqrt{\lambda^2-\omega^2}\,t}$$

$$- (v_0 + (\lambda-\sqrt{\lambda^2-\omega^2})y_0)\, e^{-\sqrt{\lambda^2-\omega^2}\,t}\Big) = 0$$

$\lambda > \omega$ by assumption

$\Leftrightarrow$ ($e^z \neq 0$)

$$(v_0 + (\lambda+\sqrt{\lambda^2-\omega^2})y_0)\, e^{\sqrt{\lambda^2-\omega^2}\,t}$$
$$-(v_0 + (\lambda-\sqrt{\lambda^2-\omega^2})y_0)\, e^{-\sqrt{\lambda^2-\omega^2}\,t} = 0$$

If $y_0 = 0$, then $t_0 = 0$ (The question is sloppy and implicitly assumes $y_0 \neq 0$.) We will assume $y_0 \neq 0$ henceforth.)

$\overset{y_0 \neq 0}{\Leftrightarrow}$

$$\left(\frac{v_0}{y_0} + \lambda + \sqrt{\lambda^2-\omega^2}\right) e^{\sqrt{\lambda^2-\omega^2}\,t}$$
$$-\left(\frac{v_0}{y_0} + \lambda - \sqrt{\lambda^2-\omega^2}\right) e^{-\sqrt{\lambda^2-\omega^2}\,t} = 0$$

By assumption, $\lambda > \omega$.

~~Because~~

That means $\sqrt{\lambda^2-\omega^2} \in \mathbb{R}$ and > 0.

① Let's consider the case $\frac{v_0}{y_0} + \lambda + \sqrt{\lambda^2-\omega^2} =: C > 0$

Writing $y(t) = C e^{\sqrt{\lambda^2-\omega^2}\,t} - (C - 2\sqrt{\lambda^2-\omega^2})\, e^{-\sqrt{\lambda^2-\omega^2}\,t}$

$= C(e^{\sqrt{\lambda^2-\omega^2}\,t} - e^{-\sqrt{\lambda^2-\omega^2}\,t}) + 2\sqrt{\lambda^2-\omega^2}\, e^{-\sqrt{\lambda^2-\omega^2}\,t}$

For $\theta > 0$, we have $e^{\theta t} - e^{-\theta t} > 0$ for all $t > 0$

and $e^{\theta t} - e^{-\theta t} = 0$ at $t=0$

(the growing exponential is bigger than the shrinking exponential, e.g. $2^1 > 2^{-1}$, $2^2 > 2^{-2}$, ...

Can also check that $e^{\theta t} + e^{-\theta t} = 0$ for $t=0$

and $\frac{d}{dt} e^{\theta t} - e^{-\theta t} = \theta(e^{\theta t} + e^{-\theta t}) > 0$)

Also, $2\theta e^{-\theta t} > 0$ for all $t \geq 0$ if $\theta > 0$

So $$\underbrace{\underbrace{C}_{\substack{>0 \\ \text{by assumption}}}\left(e^{\sqrt{\lambda^2-\omega^2}t} - e^{-\sqrt{\lambda^2-\omega^2}t}\right)}_{\geq 0} + \underbrace{2\sqrt{\lambda^2-\omega^2}\, e^{-\sqrt{\lambda^2-\omega^2}t}}_{>0} > 0$$ for all $t \geq 0$

This shows that if $\frac{v_0}{y_0} + \lambda + \sqrt{\lambda^2-\omega^2} > 0$, then

$y(t)$ does not cross $y=0$ for any $t \geq 0$

$\Leftrightarrow -\frac{v_0}{y_0} < \lambda + \sqrt{\lambda^2-\omega^2}$

$=: C$

② Now let's consider the case $\frac{v_0}{y_0} + \lambda + \sqrt{\lambda^2-\omega^2} < 0$

Again writing

$$y(t) = C\left(e^{\sqrt{\lambda^2-\omega^2}t} - e^{-\sqrt{\lambda^2-\omega^2}t}\right) + 2\sqrt{\lambda^2-\omega^2}\, e^{-\sqrt{\lambda^2-\omega^2}t}$$

At $t=0$, we have $y(0) = \underbrace{C(1-1)}_{=0} + \underbrace{2\sqrt{\lambda^2-\omega^2}}_{>0} > 0$

As $t \to \infty$, $e^{\sqrt{\lambda^2-\omega^2}t}$ gets arbitrarily large,

whereas $e^{-\sqrt{\lambda^2-\omega^2}t} \leq 1$. This implies that

for sufficiently large t, we will have $y(t) < 0$ (bc $C < 0$)

This implies (by the "intermediate value theorem", since $y(t)$ is clearly continuous on its domain) that

(the fact that $y(0) > 0$ and $y(\text{big}) < 0$)

there exists $0 < t_0 < \infty$ s.t. $y(t_0) = 0$.

③ Now we consider the case $\frac{v_0}{y_0} + \lambda + \sqrt{\lambda^2 - \omega^2} = 0$

then

$$y(t) = 0\, e^{\sqrt{\lambda^2-\omega^2}\, t} + 2\sqrt{\lambda^2-\omega^2}\, e^{-\sqrt{\lambda^2-\omega^2}\, t} > 0$$

and there is no t_0 s.t. $y(t_0) = 0$.

This conclusion could be arrived at through careful consideration of what the physical system is like: $t_0\left(\frac{v_0}{y_0}\right) \to \infty$ as $-\frac{v_0}{y_0} \to \lambda + \sqrt{\lambda^2 - \omega^2}$