Math 218 — Assignment 5

Alex Cowan

Due 2024/10/29

1. Give the general solution to each of the following differential equations.

1.1) $x' = \begin{pmatrix} 5i & 0 \\ 0 & -6 \end{pmatrix} x$ **1.2)** $x' = \begin{pmatrix} 3 & -2i \\ 0 & 1+i \end{pmatrix} x$ **1.3)** $x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x$

1.4) $x' = \begin{pmatrix} 0 & 1 \\ -11 & -3 \end{pmatrix} x$ **1.5)** $x' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x$ **1.6)** $x' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} x.$

2. Consider the initial value problem

$$x'(t) = Ax(t) \qquad \text{and} \qquad x(t_0) = c. \tag{1}$$

2.1) Write code which solves (1) numerically by modifying Euler's method to handle systems of linear first order differential equations with constant coefficients. Your code should take as input

- an $n \times n$ matrix A defined over $\mathbb{C}$,
- a starting time t_0,
- an ending time t_f,
- a list $c^T = (c_1, \ldots, c_n)$,
- a step size Δt,

and should output an approximation of $x(t_f)$.

2.2) In (1), take

$$A = \frac{1}{6445} \begin{pmatrix} -6408044856 + 2900994527i & -4456031192 + 10152071064i \\ -11196078372 + 3222022554i & 6408044856 + 22879134373i \end{pmatrix}$$

and $x(0) = (2, 1)^T$. Use your code to give a good numerical approximation of $x(10)$.

2.3) Determine the value of $x(10)$ analytically. Compare with your numerical approximation.

3. Fix $\lambda \in \mathbb{C}$ and let $J \coloneqq \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

3.1) What are the eigenvalues of J? For each eigenvalue, what is a basis of eigenvectors for the associated eigenspace (i.e. a basis for $\ker(J - \mu I)$, where μ is the eigenvalue)?
3.2) Find a simple expression for J^n, $n \in \mathbb{Z}_{\geq 0}$.[1]
3.3) Find a simple expression for e^{tJ}.
3.4) Solve the initial value problem $x' = Jx$ with $x(0) = c$ for $c \in \mathbb{C}^2$ arbitrary.
3.5) What is the general solution to the differential equation

$$x' = \begin{pmatrix} -14 & 9 \\ -16 & 10 \end{pmatrix} x?$$

[1] Hint (rot13): Pnyphyngr gur svefg srj cbjref ol unaq naq gura hfr vaqhpgvba.