1.1) Eigenvectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ with eigenvalues $5i$ and -6 by inspection

$x(t) = c_1 e^{5it}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{-6t}\begin{pmatrix}0\\1\end{pmatrix}$ is the general solution

1.2) By inspection, $\begin{pmatrix}3 & -2i\\ & 1+i\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 3\begin{pmatrix}1\\0\end{pmatrix}$

$\det\left(\lambda \underline{I} - \begin{pmatrix}3 & -2i\\ & 1+i\end{pmatrix}\right) = (\lambda-3)(\lambda-(1+i)) \Rightarrow$ other eigenvalue is $1+i$.

$\begin{pmatrix}3 & -2i\\ & 1+i\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} = (1+i)\begin{pmatrix}y_1\\y_2\end{pmatrix} \Leftrightarrow \begin{cases}3y_1 - 2i\,y_2 = (1+i)y_1\\ (1+i)y_2 = (1+i)y_2\end{cases}$

$\Rightarrow y_2 = \frac{2-i}{2i}y_1 \Rightarrow \begin{pmatrix}1\\ \frac{2-i}{2i}\end{pmatrix}$ is an eigenvector with eigenvalue $1+i$

General solution: $x(t) = c_1 e^{3t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{(1+i)t}\begin{pmatrix}1\\ \frac{2-i}{2i}\end{pmatrix}$

1.3) By inspection $\begin{pmatrix}3 & -2\\2 & -2\end{pmatrix} + \begin{pmatrix}1 & \\ & 1\end{pmatrix} = \begin{pmatrix}4 & -2\\2 & -1\end{pmatrix}$ has rank 1, and

$\begin{pmatrix}3 & -2\\2 & -2\end{pmatrix} - \begin{pmatrix}2 & \\ & 2\end{pmatrix} = \begin{pmatrix}1 & -2\\2 & -4\end{pmatrix}$ also has rank 1

$\Rightarrow$ Eigenvalues are -1 and 2

$\lambda=-1$: $\begin{pmatrix}4 & -2\\2 & -1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Leftrightarrow y_2 = 2y_1 \Rightarrow \begin{pmatrix}1\\2\end{pmatrix}$ is an eigenvector

Check: $\begin{pmatrix}3 & -2\\2 & -2\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\cdot1-2\cdot2\\2\cdot1-2\cdot2\end{pmatrix} = \begin{pmatrix}-1\\-2\end{pmatrix} = -\begin{pmatrix}1\\2\end{pmatrix}$

$\lambda=2$: $\begin{pmatrix}1 & -2\\2 & -4\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ by inspection

General Solution: $x(t) = c_1 e^{-t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}2\\1\end{pmatrix}$

1.4) $\det\left(\lambda \underline{I} - \begin{pmatrix} & 1\\-11 & -3\end{pmatrix}\right) = \lambda^2 + 3\lambda + 11$

$\lambda^2+3\lambda+11 = 0 \Leftrightarrow \lambda = \frac{-3 \pm \sqrt{35}\,i}{2}$ (as you found on the midterm)

Not just $\Rightarrow$!

$+$: $\begin{cases}\frac{-3+\sqrt{35}\,i}{2}y_1 - y_2 = 0\\ -11y_1 + \left(\frac{-3+\sqrt{35}\,i}{2} - 3\right)y_2 = 0\end{cases} \Leftrightarrow y_2 = \frac{-3+\sqrt{35}\,i}{2}y_1$

We don't have to make use of second eqn because $\det(\lambda\underline{I} - A) = 0$ means $\ker(\lambda\underline{I}-A)$ has $\dim \geq 1$, and e.g. $\begin{pmatrix}1\\1\end{pmatrix} \notin \ker$, so $\dim\ker = 1$!

More explicitly, it must be that second eqn is $\frac{-3-\sqrt{35}i}{2}$ times first one, because $(x-\lambda_1)(x-\lambda_2) = x - (\lambda_1+\lambda_2)x + \lambda_1\lambda_2$; we know from $\lambda^2+3\lambda+11=0$ that $\lambda_1\lambda_2 = 11$, so $-\frac{11}{\lambda_1} = -\lambda_2$

$\Rightarrow \quad y_2 = \dfrac{-3-\sqrt{35}i}{2}$ This is like complex conjugation, but for

$\sqrt{35}i \mapsto -\sqrt{35}i$, or for (one root of $\lambda^2+3\lambda+11$) $\leftrightarrow$ (other root). Because $\lambda^2+3\lambda+11$ doesn't explicitly involve any $\sqrt{-35}$'s, the solution must be symmetric under $\sqrt{-35} \mapsto -\sqrt{-35}$. Here there is an i, so you can also think of it as actual complex conjugation, but the idea also works for e.g. $w^2-2=0$: any algebraic relations must still hold under $\sqrt{2} \mapsto -\sqrt{2}$, because which root we decided to label "$\sqrt{2}$" was arbitrary. This is analogous to roots of polynomials with real coeff's coming in complex conjugate pairs. The study of this principle is called "Galois theory" and is a major branch of mathematics. Its namesake, Évariste Galois, was quite the character: one of the most influential mathematicians of all time, imprisoned for political activism as a teenager and killed in a duel over a woman at age 20.

General solution $x(t) = c_1 e^{\frac{-3+\sqrt{35}i}{2}t}\begin{pmatrix}\frac{-3+\sqrt{35}i}{2}\\ 1\end{pmatrix} + c_2 e^{\frac{-3-\sqrt{35}i}{2}t}\begin{pmatrix}\frac{-3-\sqrt{35}i}{2}\\ 1\end{pmatrix}$

1.5) $\det\left(\lambda I - \begin{pmatrix}1 & -1\\ 5 & -3\end{pmatrix}\right) = (\lambda-1)(\lambda+3)+5 = \lambda^2+2\lambda+2$

$= (\lambda+1)^2+1$ (completing the square by inspection).

$\Rightarrow \lambda = -1 \pm i$

$$\left(\lambda I - \begin{pmatrix}1 & -1\\ 5 & -3\end{pmatrix}\right)\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \iff \begin{cases}(\lambda-1)y_1 + y_2 = 0\\ 5y_1 + (3-\lambda)y_2 = 0\end{cases}$$

$\iff y_2 = (\lambda-1)y_1$ if $\lambda = -1 \pm i$ $\Rightarrow \begin{pmatrix}1 & -1\\ 5 & -3\end{pmatrix}\begin{pmatrix}1\\ 1\pm i\end{pmatrix} = (1\pm i)\begin{pmatrix}1\\ 1\pm i\end{pmatrix}$

As explained previously, if $\uparrow$, then the two eqns must be scalar multiples of eachother. This can be dangerous to use, because the second eqn serves as a good check you didn't make a mistake.

As explained while solving 1.4, the two eigenvals and eigenvecs must be complex conjugates (more generally and more precisely: Galois conjugates).

1.6) $\det\left(\lambda I - \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}\right) = (\lambda-\alpha)^2 + 1$

$(\lambda-\alpha)^2 + 1 = 0 \iff \lambda = \alpha \pm i$

$$\begin{cases} \big((\alpha \pm i) - \alpha\big) y_1 - y_2 = 0 \\ y_1 + \big((\alpha \pm i) - \alpha\big) y_2 = 0 \end{cases} \Rightarrow \left.\begin{cases} y_2 = \pm i y_1 \\ y_1 = \mp i y_2 \end{cases}\right\} \text{Same (multiply by } \mp i)$$

General solution: $x(t) = c_1 e^{(\alpha+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{(\alpha-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

Complex conjugates

There is a structure-preserving map
$\varphi: \mathbb{C} \to 2\times 2$ matrices with real coeffs given by
$\varphi(a+bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, i.e. $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is like the complex number $a+bi$.
See extra problems 2 problem 8 for more.

3.1) $\det(xI - J) = \det\begin{pmatrix} x-\lambda & -1 \\ & x-\lambda \end{pmatrix} = (x-\lambda)^2$

$\Rightarrow$ Both eigenvalues of J are λ.

3.2) Experimenting looking for a pattern:

$J^0 = \begin{pmatrix} 1 & 0 \\ & 1 \end{pmatrix}$

$J^1 = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$

$J^2 = J \cdot J = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ & \lambda^2 \end{pmatrix}$

$J^3 = \begin{pmatrix} \lambda^2 & 2\lambda \\ & \lambda^2 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ & \lambda^3 \end{pmatrix}$

Guess: $J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ & \lambda^n \end{pmatrix}$

Suppose $\nearrow$ is true for some particular n.

Then $J^{n+1} = J^n \cdot J = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ & \lambda^n \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$

$= \begin{pmatrix} \lambda^{n+1} & \lambda^n + n\lambda^n \\ & \lambda^{n+1} \end{pmatrix} = \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ & \lambda^{n+1} \end{pmatrix}$

So if e.g. $J^3 = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ & \lambda^3 \end{pmatrix}$, that implies $J^4 = \begin{pmatrix} \lambda^4 & 4\lambda^3 \\ & \lambda^4 \end{pmatrix}$,

which in turn implies $J^5 = \begin{pmatrix} \lambda^5 & 5\lambda^4 \\ & \lambda^5 \end{pmatrix}$, etc.

$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ & \lambda^n \end{pmatrix}$ holds for $n = 1$ e.g., so by (induction) it holds for all $n \geq 1$.

3.3) $\exp(tJ) = \sum_{n=0}^{\infty} \frac{t^n}{n!} J^n$ by def

$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ & \lambda^n \end{pmatrix}$

$= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{t^n \lambda^n}{n!} & \sum_{n=0}^{\infty} \frac{t^n n \lambda^{n-1}}{n!} \\ & \sum_{n=0}^{\infty} \frac{t^n \lambda^n}{n!} \end{pmatrix}$

$$= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(t\lambda)^n}{n!} & \sum_{n=0}^{\infty} nt \frac{(t\lambda)^{n-1}}{n(n-1)!} \\ & \sum_{n=0}^{\infty} \frac{(t\lambda)^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{t\lambda} & te^{t\lambda} \\ & e^{t\lambda} \end{pmatrix}$$

3.4) $x(t) = e^{tJ}c = \begin{pmatrix} e^{t\lambda} & te^{t\lambda} \\ & e^{t\lambda} \end{pmatrix}$

($e^{0J} = I$, so $x(0) = Ic = c$ as required)

3.5) $\det\left(\lambda I - \begin{pmatrix} -14 & 9 \\ -16 & 10 \end{pmatrix}\right)$

$= (\lambda+14)(\lambda-10) + 9 \cdot 16$

$= \lambda^2 + 4\lambda - 140 + 160 - 16$

$= \lambda^2 + 4\lambda + 4$

$= (\lambda+2)^2$

$\Rightarrow$ Both eigenvalues are -2

$-2I - \begin{pmatrix} -14 & 9 \\ -16 & 10 \end{pmatrix} = \begin{pmatrix} 12 & -9 \\ 16 & -12 \end{pmatrix}$

$\ker \begin{pmatrix} 12 & -9 \\ 16 & -12 \end{pmatrix}$ is spanned by $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ by inspection

It's a theorem that for any A over $\mathbb{C}$, $A = PJP^{-1}$ for some J in Jordan normal form (i.e. it is not true that every matrix is diagonalizable, but it is true that every matrix is Jordan-normal-form-able).

Both eigs of $\begin{pmatrix} -14 & 9 \\ -16 & 10 \end{pmatrix}$ are -2, so the only two possible J's are $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ and $\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$.

We only have one eigenvector (contrast with $\begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}$, which has eigenvectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$), so it must be that $J = \begin{pmatrix}-2 & 1\\ & -2\end{pmatrix}$.

P is the matrix that goes from "eigenvector coordinates" to "standard coordinates", i.e. it turns $\begin{pmatrix}1\\0\end{pmatrix}$ (interpreted as "one of the first eigenvector and zero of the second") into $\begin{pmatrix}3\\4\end{pmatrix}$, the eigenvector we found.

$\Rightarrow P = \begin{pmatrix}3 & p_1\\ 4 & p_2\end{pmatrix}$ for some p_1, p_2. Let's figure out what p_1, p_2 are.

$$A = PJP^{-1} \iff AP = PJ$$

$$\begin{pmatrix}-14 & 9\\ -16 & 10\end{pmatrix}\begin{pmatrix}3 & p_1\\ 4 & p_2\end{pmatrix} = \begin{pmatrix}3 & p_1\\ 4 & p_2\end{pmatrix}\begin{pmatrix}-2 & 1\\ 0 & -2\end{pmatrix}$$

$$\begin{pmatrix}-6 & -14p_1+9p_2\\ -8 & -16p_1+10p_2\end{pmatrix} = \begin{pmatrix}-6 & 3-2p_1\\ -8 & 4-2p_2\end{pmatrix}$$

$$\Rightarrow \begin{cases}-14p_1+9p_2 = 3-2p_1\\ -16p_1+10p_2 = 4-2p_2\end{cases}$$

$$\Rightarrow \begin{pmatrix}-12 & 9\\ -16 & 12\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix} \qquad (\tfrac{1}{4})$$

$$\Rightarrow -4p_1+3p_2 = 1$$

Any p_1, p_2 which solve this eqn will work, by construction, e.g. $(p_1, p_2) = (-1,-1)$ or $(-\frac{1}{4}, 0)$ or $(0, \frac{1}{3})$.

I'll pick ↗, giving $\begin{pmatrix}-14 & 9\\ -16 & 10\end{pmatrix} = \begin{pmatrix}3 & -1\\ 4 & -1\end{pmatrix}\begin{pmatrix}-2 & 1\\ & -2\end{pmatrix}\begin{pmatrix}3 & -1\\ 4 & -1\end{pmatrix}^{-1}$, which you can check by multiplying.

$$e^{tA} = e^{tPJP^{-1}} = Pe^{tJ}P^{-1}$$

$$\Big((PJP^{-1})^n = PJ\cancel{P^{-1}P}JP^{-1}\cdots = PJ^nP^{-1},$$

$$\text{so } \exp(tPJP^{-1}) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(PJP^{-1})^n = \sum_{n=0}^{\infty}\frac{t^n}{n!}PJ^nP^{-1}$$

$$= P\sum_{n=0}^{\infty}\frac{t^n}{n!}J^nP^{-1} = Pe^{tJ}P^{-1}\Big)$$

$$x' = Ax \Rightarrow x(t) = e^{tA}x(0) = Pe^{tJ}P^{-1}x(0),$$

So the general solution is

$$x(t) = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} e^{t\begin{pmatrix} -2 & 1 \\ & -2 \end{pmatrix}} \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}^{-1} x(0)$$

(Recall that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$)

$$= \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} e^{-2t} & te^{-2t} \\ & e^{-2t} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} x(0)$$

(Recall the notation $\lambda = -2$, $J = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$, $A = \begin{pmatrix} -14 & 9 \\ -16 & 10 \end{pmatrix}$, $P = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$

and set $v := \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $w := \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

Looking at the earlier calculations carefully, one can find that it is no surprise $(*)$ came out to be $(A-\lambda I)w = v$; this always happens, basically by tracing through the calculation we did, but generically.

Just as $v \in \ker(A-\lambda I)$, $w \in \ker\big((A-\lambda I)^2\big)$. The vector w is called a "generalized eigenvector".

The map $S \mapsto PSP^{-1}$ ("conjugation by P") is basically letting S act on the basis given by the columns p_1, p_2 of P. If $T := PSP^{-1}$, then Tv can be calculated as

$$\begin{tikzcd} v \arrow[r, "P^{-1}"] \arrow[d, "T"'] \arrow[dr, phantom, "\circlearrowleft"] & c \arrow[d, "S"] \\ Tv & Sc \arrow[l, "P"] \end{tikzcd}$$

where $c = (c_1, c_2)$ is the vector of the coefficients when expressing v in the basis (p_1, p_2), i.e. $v = c_1p_1 + c_2p_2$. The top arrow P^{-1} is changing coordinates to (c_1, c_2), then S is what it needs to be to ensure $T = PSP^{-1}$, and then P on the bottom transforms coords back

From this perspective, let's take coordinates where A acts as J. Note that $A - \lambda I = PJP^{-1} - \lambda PIP^{-1}$ $= P(J-\lambda I)P^{-1}$, and $\det(A-\lambda I) = \det(P)\det(J-\lambda I)\det(P^{-1})$ $= \det(J-\lambda I)$.

The matrix $J - \lambda I = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$ is "nilpotent": some power of it is the zero matrix: $(J-\lambda I)^2 = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The zero matrix is the only 2×2 matrix with a 2-dimensional kernel (because $B\begin{pmatrix}1\\0\end{pmatrix}$ is the first column of B, etc.), so if we want a basis for our space (the analogue of an eigenbasis), we need to build the zero matrix somehow.

This is why this Jordan normal form construction is useful:

- Always "Jordan-normal-form-able": $A = PJP^{-1}$ over $\mathbb{C}$
- Can give a basis of generalized eigenvectors by computing $\ker(A-\lambda I)^2$ (for 2D)

Moreover, $\ker(A-\lambda I)^2$ is pretty easy to compute, because $w \in \ker(A-\lambda I)^2 \iff (A-\lambda I)^2 w = 0 \iff (A-\lambda I)\left[(A-\lambda I)w\right] = 0$ $\iff (A-\lambda I)w \in \ker(A-\lambda I) \iff (A-\lambda I)w$ is proportional to v.

This is what the course notes do very explicitly, and why we could have predicted that $(*)$ would be $(A-\lambda I)w = v$.

If we had $J = \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$, then a basis of generalized eigenvectors would come from computing $\ker(A-\lambda I)^3$, because $J - \lambda I = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$, $(J-\lambda I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$, $(J-\lambda I)^3 =$ zero matrix, etc. for larger and larger Jordan blocks.