Math 218 — Assignment 6

Alex Cowan

Due 2024/11/05

1. Give the general solution to each of the following differential equations.

1.1) $x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x + \begin{pmatrix} e^t \\ t \end{pmatrix}$

1.2) $x' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} x + \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$

1.3) $x' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} x + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \ t > 0.$

2. Solve the following initial value problems.

2.1) $x' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} x, \ x(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

2.2) $x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}, \ x(\frac{\pi}{3}) = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ -1 \end{pmatrix}, \ 0 < t < \pi$

2.3) $x' = \begin{pmatrix} 2 & -3 & 5i \\ 0 & -7 & 11 \\ 0 & 0 & -13 \end{pmatrix} x + \begin{pmatrix} 17e^{2t} \\ -19e^t \\ 23e^{8t} \end{pmatrix}, \ x(0) = \begin{pmatrix} 109 \\ 1009 \\ 10009 \end{pmatrix}.$

3. This problem will walk you through the process of plotting trajectories in phase space.

Consider the system

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \mathbf{x}. \tag{1}$$

The general solution of (1) is

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t}.$$

Step 1: Draw an x_2 vs x_1 graph (i.e. x_1 is the horizontal axis and x_2 is the vertical axis).

Step 2: Solve each independent linear equation $x_1' = 0$ and $x_2' = 0$ based on the matrix equation. That is, set $x_1' = 0$ and solve for x_2 in terms of x_1. You should get a line similar in form to $y = mx$. Do the same thing for $x_2' = 0$ and draw these on your axes from Step 1.

Each of these represents a specific isocline. On the $x_1' = 0$ line you should draw a bunch of vertical tick marks. This is because when $x_1' = 0$ there is "no x_1 movement" so all movement happens vertically.

Similarly for the $x_2' = 0$ line you should draw a bunch of horizontal tick marks (these are horizontal for analogous reasons).

Step 3: Draw a line corresponding to the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and another line corresponding to the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. These should be "infinitely" long (like your lines in step 2) but follow the direction of each of these vectors.

Step 4: Each of the vector/lines in Step 3 represents the asymptotic behaviour of the solutions. That is, how the solutions behave as $t \to \infty$ and as t "started off" at $-\infty$.

To help draw a specific curve first ask yourself "as $t \to -\infty$ which of the two functions e^{-t} or e^{4t} will dominate?". Once you figure that out then the solution curves should start out "parallel" to the vector/line associated with that function and then curve toward and become "parallel" to the vector/line associated with the dominant exponential as $t \to +\infty$.

Finally, note that the sign of the coefficient of t in the exponentials (i.e. the eigenvalues) should help you determine if you are close to or away from the origin as $t \to \infty$ and as t "starts" at $-\infty$.

Step 5: Pick random (x_1, x_2) values and calculate the $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$ vector (i.e the $\mathbf{x}'$ vector). This will generate a vector field which the solutions should follow.

3.1) Use all the steps above to draw a collection of trajectories of (1) in phase space.

3.2) Do the same thing for the system

$$\mathbf{x}' = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \mathbf{x}.$$

You will need to adjust Step 3 and Step 4 by actually solving this system first.

4. Calculate the Laplace transform $\mathscr{L}\{y\}$ of the following functions.

4.1) $y(t) = e^{777t}$

4.2) $y(t) = te^{-2t}$

4.3) $y(t) = \begin{cases} 1 & 0 < a < t < b \\ 0 & \text{otherwise} \end{cases}$

4.4) $y(t) = \begin{cases} 0 & 0 \le t \le 1 \\ e^{3t} & t > 1 \end{cases}$

4.5) $y(t) = t \sin \pi t$

4.6) $y(t) = s(t),$ the sawtooth wave from extra problems 1 problem 2.6e).

5–8. (All bonus) Extra problems 2 problems 10–13. You will probably have to refer to earlier problems while doing these.