Math 218 — Assignment 6 solutions

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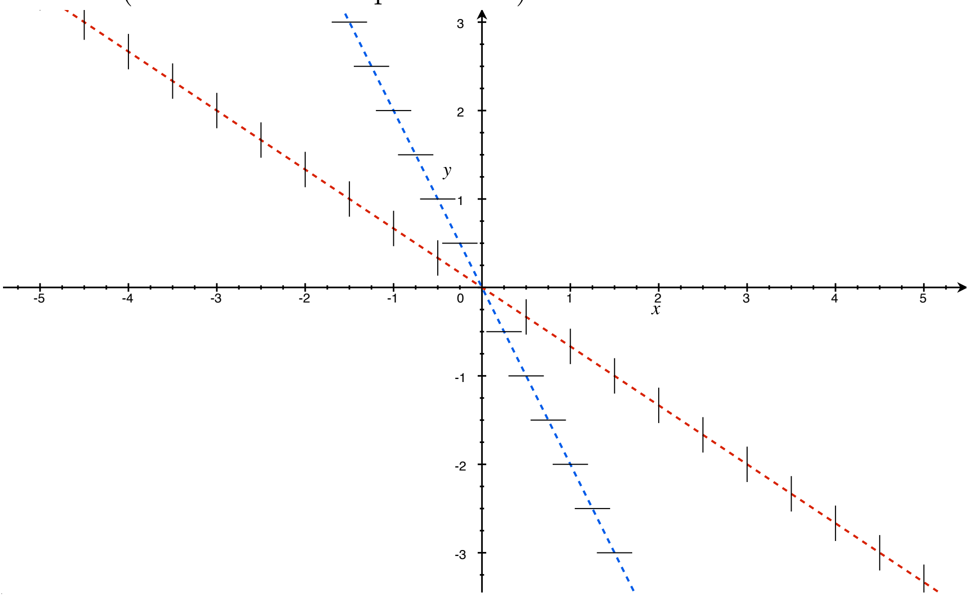
3.1)

Step 1 and **Step 2**

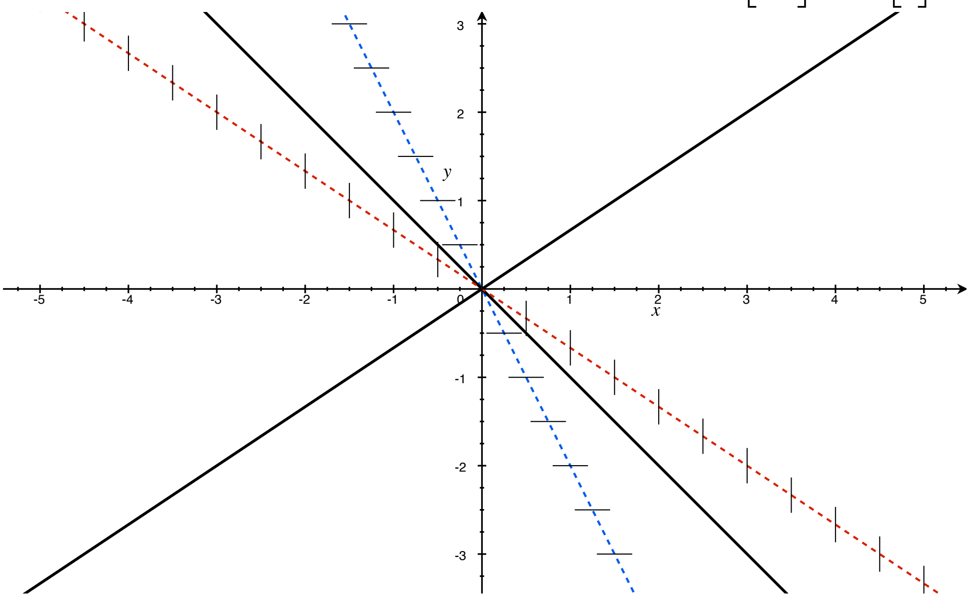
We have $x_1' = 2x_1 + 3x_2$ so setting $x_1' = 0$ yields $x_2 = -\frac{2}{3}x_1$. Since $x_1' = 0$ there is no horizontal movement so we get a bunch of vertical tick marks (red curve in the plot below).

Similarly we have $x_2' = 2x_1 + x_2$ so setting $x_2' = 0$ yields $x_2 = -2x_1$. This gives horizontal ticks marks (blue curve in the plot below).



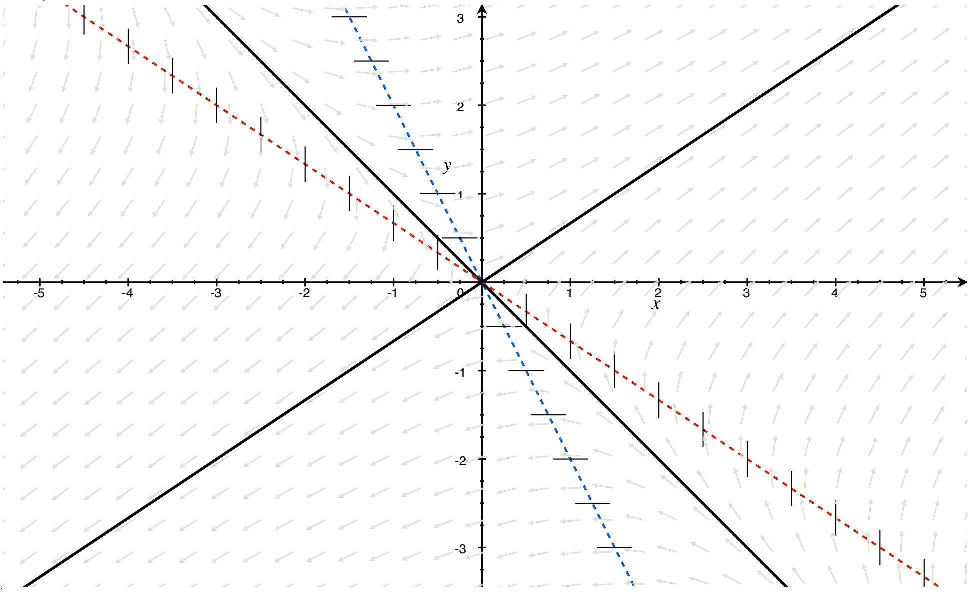
Step 3

Adding in the lines corresponding to the eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ yields:



Step 5

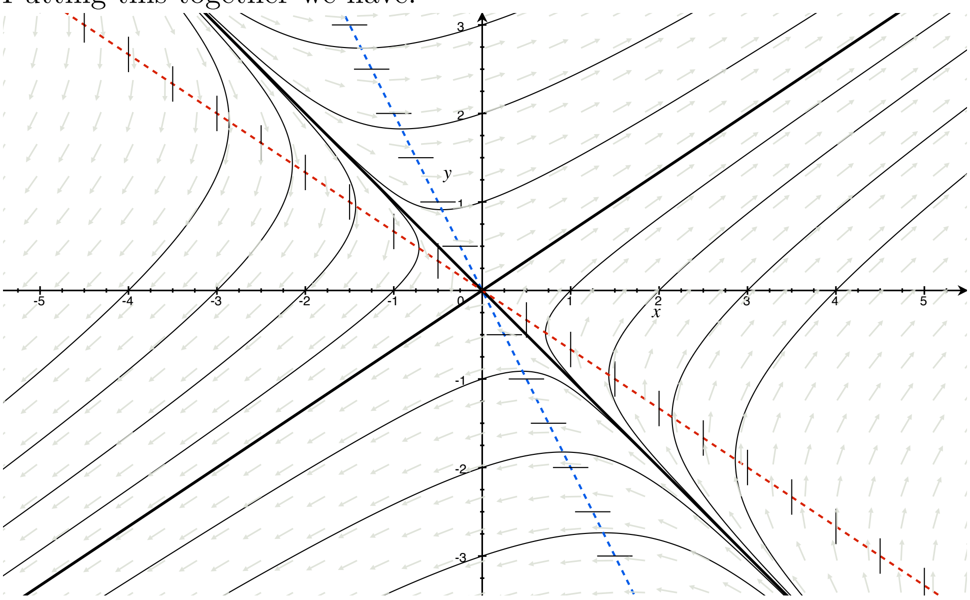
Each arrow just represents the vector that would be calculated by plugging in that specific (x_1, x_2) value into the RHS of the DE to generate $\mathbf{x}'$.



Step 4 As $t \to -\infty$ we have that e^{-t} will dominate e^{4t}. Also e^{-t} will be "blowing up" as $t \to -\infty$ so the solution curves will be far away from the origin. Since the coefficient of e^{-t} in the solution is the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then solutions will start off close to and "parallel" to this vector. Note that the specific position (above or below, top of the graph, bottom of the graph) would ultimately depend on the initial conditions.

Using a similar argument as $t \to \infty$ but where e^{4t} dominates and we move toward and "parallel" to $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ yields the rest of the curves.

Putting this together we have:



3.2)

For this system we first need to find the general solution. Using the eigenvalue/eigenvector method we should arrive at:

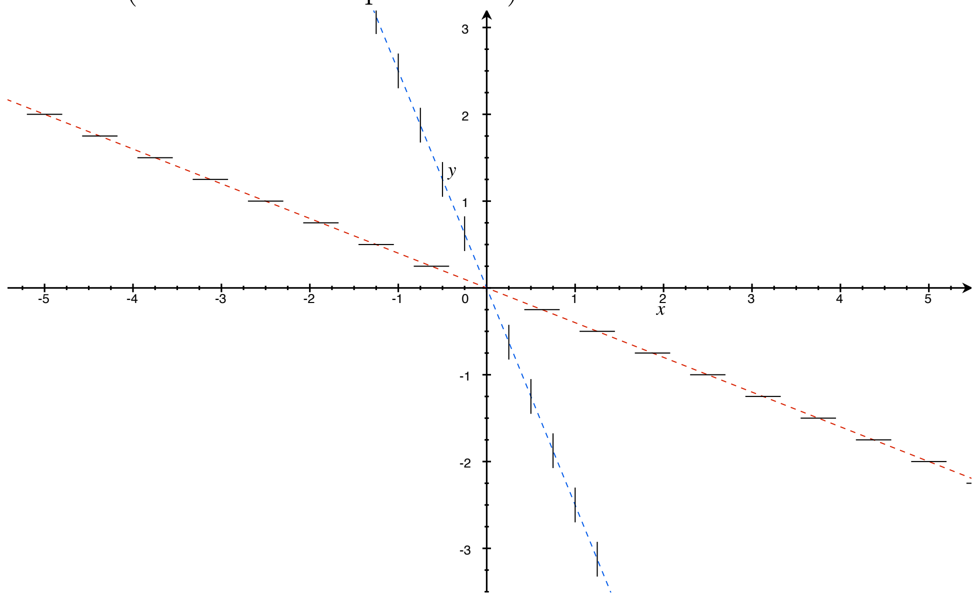
$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t}$$

The main difference here is that we now have two eigenvalues that are positive (3 and 7). So as $t \to -\infty$ the value of $\mathbf{x}$ approaches zero (i.e. the origin).

Step 1 and **Step 2**

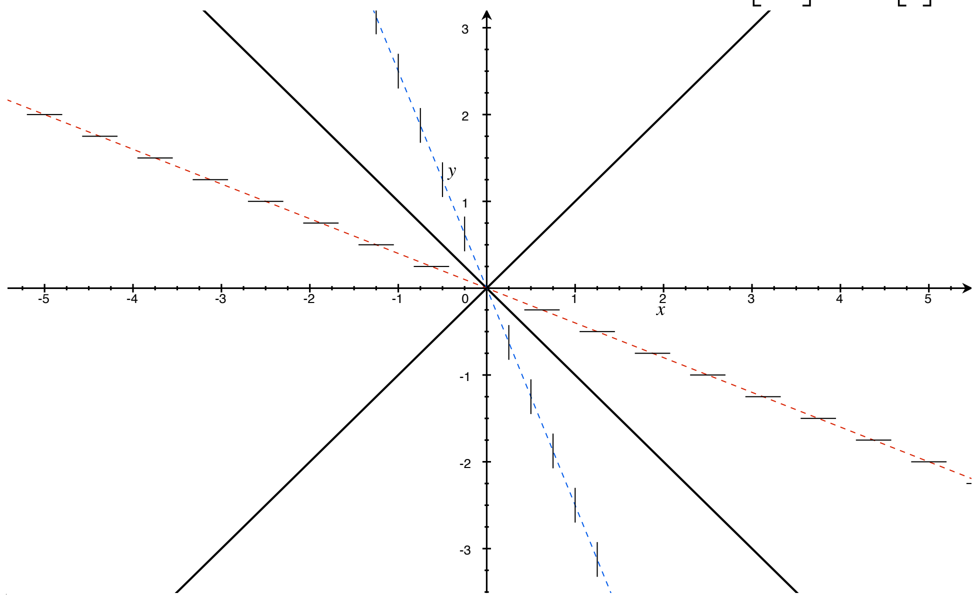
We have $x_1' = 5x_1 + 2x_2$ so setting $x_1' = 0$ yields $x_2 = -\frac{5}{2}x_1$. Since $x_1' = 0$ there is no horizontal movement so we get a bunch of vertical tick marks (blue curve in the plot below).

Similarly we have $x_2' = 2x_1 + 5x_2$ so setting $x_2' = 0$ yields $x_2 = -\frac{2}{5}x_1$. This gives horizontal ticks marks (red curve in the plot below).



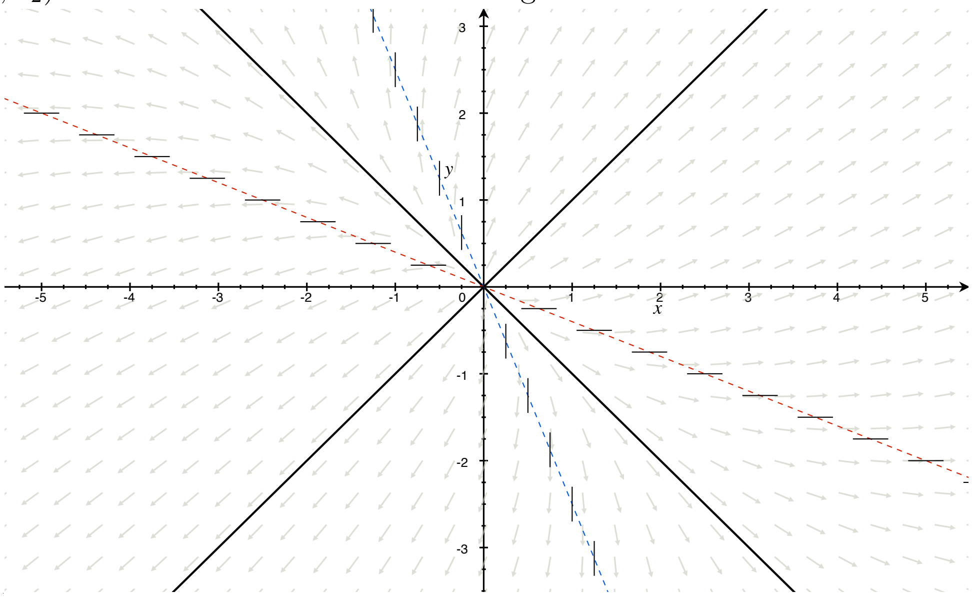
Step 3

Adding in the lines corresponding to the eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ yields:



Step 5

Each arrow just represents the vector that would be calculated by plugging in that specific (x_1, x_2) value into the RHS of the DE to generate $\mathbf{x}'$.



Step 4 As $t \to -\infty$ we have that e^{3t} will dominate e^{7t} (i.e. $e^{3t} > e^{7t}$). Another way to think about it is to factor out e^{3t} from the solution to get

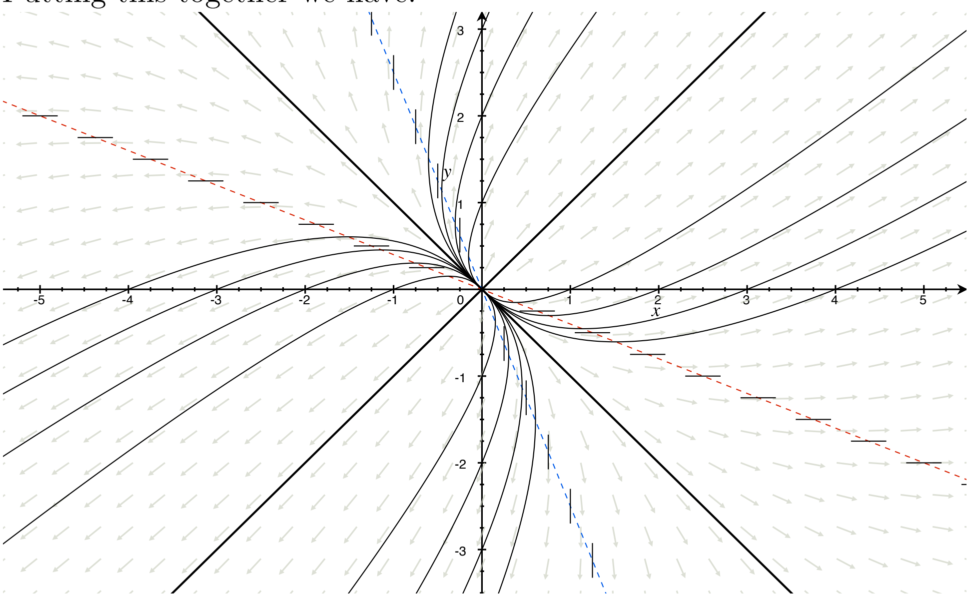
$$\mathbf{x} = e^{3t}\left(c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}\right)$$

which hopefully demonstrates that the second term in the brackets becomes less important as $t \to -\infty$.

Since e^{3t} will be shrinking as $t \to -\infty$ the solution curves will be close the origin. Also, since the coefficient of e^{3t} in the solution is the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then solutions will start off "parallel" to this vector (but still close to the origin which makes it tricky to see). Note that the specific position (above or below, quadrant 2 or quadrant 4) would ultimately depend on the initial conditions.

Using a similar argument as $t \to \infty$ but where e^{7t} dominates and we move toward and "parallel" to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ yields the rest of the curves. Note that as $t \to \infty$ both exponentials "blow up" and so we are moving away from the origin.

Putting this together we have:



1.) $\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda I\right) = \lambda^2 - (a+d)\lambda + ad - bc$

So λ is an eigenvalue of A if and only if $\lambda^2 - \operatorname{tr} A + \det A = 0$

1.1) Eigenvalues of $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ are ± 1 by inspection:

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} + \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \quad (\lambda = -1)$$

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \quad (\lambda = 1)$$

$\lambda = -1$: $3v_1 - v_2 = 0 \implies \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector

$\lambda = 1$: $v_1 - v_2 = 0 \implies \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

$$\implies \underset{A}{\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}} = \underset{P}{\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}} \underset{J}{\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}} \underset{P^{-1}}{\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}^{-1}}$$

$$\exp\left(t \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & \\ & e^{t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}^{-1}$$

Writing $x(t) = e^{tA} w(t) \rightsquigarrow x' = Ae^{tA} w + e^{tA} w' = Ax + e^{tA} w$

If $x' = Ax + f(t)$, then $f(t) = e^{tA} w'$

$\implies w = \int_0^t e^{-uA} f(u)\,du \implies x_p(t) = e^{tA} \int_0^t e^{-uA} f(u)\,du$

Here $f(u) = \begin{pmatrix} e^t \\ t \end{pmatrix}$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \implies P^{-1} = \frac{-1}{2} \begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix}$

$$e^{-uA} f(u) = \underbrace{\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} e^{u} & \\ & e^{-u} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}}_{\exp(-uA)} \begin{pmatrix} e^{u} \\ u \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -e^{u} & e^{u} \\ 3e^{-u} & -e^{-u} \end{pmatrix} \begin{pmatrix} e^{u} \\ u \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -e^{u} + 3e^{-u} & e^{u} - e^{-u} \\ -3e^{u} + 3e^{-u} & 3e^{u} - e^{-u} \end{pmatrix} \begin{pmatrix} e^{u} \\ u \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 3 - e^{2u} + ue^{u} - ue^{-u} \\ -3e^{2u} + 3 + 3ue^{u} - ue^{-u} \end{pmatrix}$$

$$\int_0^t \nearrow du = \frac{1}{2} \begin{pmatrix} 3u - \frac{1}{2} e^{2u} + ue^{u} - e^{u} + ue^{-u} + e^{-u} \\ -\frac{3}{2} e^{2u} + 3u + 3ue^{u} - 3e^{u} + ue^{-u} + e^{-u} \end{pmatrix} \Bigg|_0^t$$

$$= \frac{1}{2} \begin{pmatrix} 3t - \frac{1}{2} e^{2t} + \frac{1}{2} + te^{t} - e^{t} + 1 + te^{-t} + e^{-t} - 1 \\ -\frac{3}{2} e^{2t} + \frac{3}{2} + 3t + 3te^{t} - 3e^{t} + 3 + te^{-t} + e^{-t} - 1 \end{pmatrix}$$

$$\exp(tA) = \frac{1}{2}\begin{pmatrix} -e^{t}+3e^{-t} & -e^{t}+e^{-t} \\ 3e^{t}-3e^{-t} & -e^{t}+3e^{-t} \end{pmatrix}$$

$$\exp(tA)\int_0^t e^{-uA} f(u)\,du$$

$$= \frac{1}{4}\begin{pmatrix} -e^{t}+3e^{-t} & -e^{t}+e^{-t} \\ 3e^{t}-3e^{-t} & -e^{t}+3e^{-t} \end{pmatrix}$$

$$\cdot\begin{pmatrix} -\frac{1}{2}e^{2t}+te^{t}+te^{-t}-e^{t}+e^{-t}+3t+\frac{1}{2} \\ -\frac{3}{2}e^{2t}+3te^{t}+te^{-t}-3e^{t}+e^{-t}+3t+\frac{7}{2} \end{pmatrix}$$

General solution: $x(t) = \frac{1}{2}\begin{pmatrix} -e^{t}+3e^{-t} & -e^{t}+e^{-t} \\ 3e^{t}-3e^{-t} & -e^{t}+3e^{-t} \end{pmatrix} x(0) +$

Here is a way of computing $e^{tA}\int_0^t e^{-uA} f(u)du$ that's slightly less work, but a bit tricky:

$$e^{tA}\int_0^t e^{-uA} f(u)\,du = Pe^{tJ}P^{-1}\int_0^t Pe^{-uJ}P^{-1} f(u)\,du$$

$$= Pe^{tJ}\int_0^t e^{-uJ}P^{-1}f(u)\,du$$

$$= \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & \\ & e^{t} \end{pmatrix}\int_0^t \begin{pmatrix} e^{u} & \\ & e^{-u} \end{pmatrix}\frac{1}{2}\begin{pmatrix} -1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} e^{u} \\ u \end{pmatrix}du$$

$$= \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & \\ & e^{t} \end{pmatrix}\int_0^t \frac{1}{2}\begin{pmatrix} -e^{u} & e^{u} \\ 3e^{-u} & -e^{-u} \end{pmatrix}\begin{pmatrix} e^{u} \\ u \end{pmatrix}du$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & \\ & e^{t} \end{pmatrix}\int_0^t \begin{pmatrix} -e^{2u}+ue^{u} \\ 3-ue^{-u} \end{pmatrix}du$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & \\ & e^{t} \end{pmatrix}\begin{pmatrix} -\frac{1}{2}e^{2t}+te^{t}-e^{t}+\frac{3}{2} \\ 3t+te^{-t}+e^{-t}-1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2}e^{t}+t-1+\frac{3}{2}e^{-t} \\ 3te^{t}+t+1-e^{t} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 3te^{t}+2t-\frac{3}{2}e^{t}+\frac{3}{2}e^{-t} \\ 3te^{t}+4t-\frac{5}{2}e^{t}+\frac{9}{2}e^{-t} \end{pmatrix}$$

1.2) $\lambda^2 - \operatorname{tr} A\,\lambda + \det A = \lambda^2 + 5\lambda + 4 = 0$

$\Rightarrow \lambda = -1, -4$

$A + I = \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

$A + 4I = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}; \quad v_2 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$

$\Rightarrow A = \underbrace{\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} -1 & \\ & -4 \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}^{-1}}_{P^{-1}}$

$e^{tA} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & \\ & e^{-4t} \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$

$f(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$

$e^{tA} \int_0^t e^{-uA} f(u)\,du = P e^{tJ} \int_0^t e^{-uJ} P^{-1} f(u)\,du$

$= \frac{1}{3} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & \\ & e^{-4t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{u} & \\ & e^{4u} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} e^{-u} \\ -e^{-u} \end{pmatrix} du$

$= \frac{1}{3} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & \\ & e^{-4t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{u} & \\ & e^{4u} \end{pmatrix} \begin{pmatrix} (1-\sqrt{2})e^{-u} \\ (1+\sqrt{2})e^{-u} \end{pmatrix} du$

$= \frac{1}{3} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & \\ & e^{-4t} \end{pmatrix} \int_0^t \begin{pmatrix} 1-\sqrt{2} \\ (1+\sqrt{2})e^{3u} \end{pmatrix} du$

$= \frac{1}{3} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & \\ & e^{-4t} \end{pmatrix} \begin{pmatrix} (1-\sqrt{2})t \\ \frac{1}{3}(1+\sqrt{2})(e^{3t}-1) \end{pmatrix}$

$= \frac{1}{3} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} (1-\sqrt{2})te^{-t} \\ \frac{1}{3}(1+\sqrt{2})(e^{-t}-e^{-4t}) \end{pmatrix}$

General solution: $c_1 e^{-t} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} +$

1.3) $\lambda^2 - \operatorname{tr} A\lambda + \det \lambda = \lambda^2 - 0\lambda - 0 = 0$

$\Rightarrow$ both eigenvalues are 0.

$A = PJP^{-1}$, with J either $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

It can't be the former, because then $PJP^{-1} = 0 \neq A$,

so $A = P\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}P^{-1}$

Note that $A^2 = P\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 P^{-1} = P\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}P^{-1} = 0$

so $e^{tA} = I + tA = \begin{pmatrix} 4+t & -2 \\ 8 & -4+t \end{pmatrix}$

We can pick any starting pt $\rightarrow$ $\displaystyle\int_1^t \begin{pmatrix} 4-u & -2 \\ 8 & -4-u \end{pmatrix}\begin{pmatrix} u^{-3} \\ -u^{-2} \end{pmatrix} du$

$$= \int_1^t \begin{pmatrix} 4u^{-3} - u^{-2} + 2u^{-2} \\ 8u^{-3} + 4u^{-2} + u^{-1} \end{pmatrix} du$$

$$= \begin{pmatrix} -2u^{-2} - u^{-1} \\ -4u^{-2} - 4u^{-1} + \log|u| \end{pmatrix}\Bigg|_1^t$$

$$= \begin{pmatrix} -2t^{-2} - t^{-1} + 3 \\ -4t^{-2} - 4t^{-1} + \log t + 8 \end{pmatrix} \quad t>0$$

$$e^{tA}\int_1^t e^{-uA} f(u)\, du = \begin{pmatrix} 4+t & -2 \\ 8 & -4+t \end{pmatrix}\begin{pmatrix} -2t^{-2} - t^{-1} + 3 \\ -4t^{-2} - 4t^{-1} + \log t + 8 \end{pmatrix}$$

$$= \begin{pmatrix} -8t^{-2} - 4t^{-1} + 12 - 2t^{-1} - 1 + 3t + 8t^{-2} + 8t^{-1} - 2\log t - 16 \\ -16t^{-2} - 8t^{-1} + 24 + 16t^{-2} + 16t^{-1} - 4\log t - 32 - 4t^{-1} - 4 + t\log t + 8t \end{pmatrix}$$

$$= \begin{pmatrix} 2t^{-1} - 5 - 2\log t + 3t \\ 4t^{-1} - 12 - 4\log t + 8t + t\log t \end{pmatrix}$$

General solution: $\begin{pmatrix} 4+t & -2 \\ 8 & -4+t \end{pmatrix} x(0) +$

$\lambda^2 + 2\lambda + 5 = (\lambda+1)^2 + 4 \Rightarrow \lambda = -1 \pm 2i$

$\lambda_1 = -1+2i \Rightarrow A - \lambda I = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} + (1-2i)I$

$= \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}$ By inspection $v_1 := \begin{pmatrix} 2 \\ -i \end{pmatrix}$ is in $\ker(A - \lambda_1 I)$

$\lambda_2 = \overline{\lambda_1} \Rightarrow v_2 = \overline{v_1}$, since A is defined over $\mathbb{R}$.

$\Rightarrow A = \begin{pmatrix} 2 & 2 \\ -i & i \end{pmatrix} \begin{pmatrix} -1+2i & \\ & -1-2i \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -i & i \end{pmatrix}^{-1}$

The general solution (before imposing the initial condition) is

$x(t) = e^{tA} x(0) = \begin{pmatrix} 2 & 2 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{(-1+2i)t} & \\ & e^{(-1-2i)t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -i & i \end{pmatrix} x(0)$.

So the solution to the problem is $x(t) = e^{tA} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

2.2) $\lambda^2 - \mathrm{tr} A \lambda + \det A = \lambda^2 + 1 \Rightarrow$ eigenvalues are $\pm i$.

$\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} - iI$ has $v_1 := \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \in \ker(A - iI)$

$\lambda_1 = \overline{\lambda_2}$ and A defined over $\mathbb{R} \Rightarrow A = \begin{pmatrix} 2+i & 2-i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} i & \\ & -i \end{pmatrix} \begin{pmatrix} 2+i & 2-i \\ 1 & 1 \end{pmatrix}^{-1}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, so $\begin{pmatrix} 2+i & 2-i \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & -2+i \\ -1 & 2+i \end{pmatrix}$

Writing $x(t) =: e^{tA} w(t)$ (which we may do without loss of generality, since $(e^{tA})^{-1} = e^{-tA}$ always exists) implies, as explained in the solution to 1.1, that a particular solution is

$P e^{tJ} \int_0^t e^{-uJ} P^{-1} f(u)\,du$, which in this case is

$$\begin{pmatrix} 2+i & 2-i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{it} & \\ & e^{-it} \end{pmatrix} \int_0^t \begin{pmatrix} e^{-iu} & \\ & e^{iu} \end{pmatrix} \frac{1}{2i} \begin{pmatrix} 1 & -2+i \\ -1 & 2+i \end{pmatrix} \begin{pmatrix} 0 \\ \cos u \end{pmatrix} du$$

$$= \frac{1}{2i} \begin{pmatrix} 2+i & 2-i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{it} & \\ & e^{-it} \end{pmatrix} \int_0^t \begin{pmatrix} e^{-iu} & \\ & e^{iu} \end{pmatrix} \begin{pmatrix} (-2+i)\cos u \\ (2+i)\cos u \end{pmatrix} du$$

$$= \frac{1}{2i} \text{ --- } \int_0^t \begin{pmatrix} (-2+i) e^{-iu} \cos u \\ (2+i) e^{iu} \cos u \end{pmatrix} du$$

Exponentials are much easier to calculate with than trig fns, so

$$= \frac{1}{2i} \text{ --- } \int_0^t \frac{1}{2} \begin{pmatrix} (-2+i) e^{-iu} (e^{iu} + e^{-iu}) \\ (2+i) e^{iu} (e^{iu} + e^{-iu}) \end{pmatrix} du$$

$$= \frac{1}{4i}\begin{pmatrix} 2+i & 2-i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{it} & \\ & e^{-it} \end{pmatrix}\begin{pmatrix} (-2+i)\left(u - \frac{1}{2i}e^{-2iu}\right) \\ (2+i)\left(u + \frac{1}{2i}e^{2iu}\right) \end{pmatrix}\Bigg|_0^t$$

$$= \frac{1}{4i}\begin{pmatrix} 2+i & 2-i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{it} & \\ & e^{-it} \end{pmatrix}\begin{pmatrix} (-2+i)\left(t + \frac{1}{2i} - \frac{1}{2i}e^{-2it}\right) \\ (2+i)\left(t - \frac{1}{2i} + \frac{1}{2i}e^{2it}\right) \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2+i & 2-i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (1+2i)\left(te^{it} + \frac{e^{it}-e^{-it}}{2i}\right) \\ (1-2i)\left(te^{-it} + \frac{e^{it}-e^{-it}}{2i}\right) \end{pmatrix} \Big\rangle \text{ complex conjugates}$$

$$= \frac{1}{4}\begin{pmatrix} 2\,\mathrm{Re}\left((2+i)(1+2i)(te^{it} + \sin t)\right) \\ 2\,\mathrm{Re}\left((1+2i)(te^{it} + \sin t)\right) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \mathrm{Re}\left(5i(te^{it} + \sin t)\right) \\ \mathrm{Re}\left(te^{it} + 2ite^{it} + \sin t\right) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -5t\sin t \\ t\cos t - 2t\sin t + \sin t \end{pmatrix}$$

Checking first row: $(2, -5)\cdot(\,) = -5t\sin t - \frac{5}{2}t\cos t + 5t\sin t - \frac{5}{2}\sin t$

$= -\frac{5}{2}(t\cos t + \sin t) = -\frac{5}{2}(t\sin t)'$ ✓

General solution before initial conditions is

$$x(t) = c_1 e^{it}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} + c_2 e^{-it}\begin{pmatrix} 2-i \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -5t\sin t \\ t\cos t - 2t\sin t + \sin t \end{pmatrix}$$

$$x\left(\tfrac{\pi}{3}\right) = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ -1 \end{pmatrix} = \begin{pmatrix} (2+i)e^{\frac{\pi i}{3}} & (2-i)e^{-\frac{\pi i}{3}} \\ e^{\frac{\pi i}{3}} & e^{-\frac{\pi i}{3}} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{-5\pi}{2\sqrt{3}} \\ \frac{\pi}{6} - \frac{\pi}{\sqrt{3}} + \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (2+i)e^{\frac{\pi i}{3}} & (2-i)e^{-\frac{\pi i}{3}} \\ e^{\frac{\pi i}{3}} & e^{-\frac{\pi i}{3}} \end{pmatrix}^{-1}\begin{pmatrix} \frac{1+\sqrt{5}}{2} + \frac{5\pi}{4\sqrt{3}} \\ -1 - \frac{\pi}{12} + \frac{\pi}{2\sqrt{3}} - \frac{\sqrt{3}}{4} \end{pmatrix} \qquad e^{\pm\frac{\pi i}{3}} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

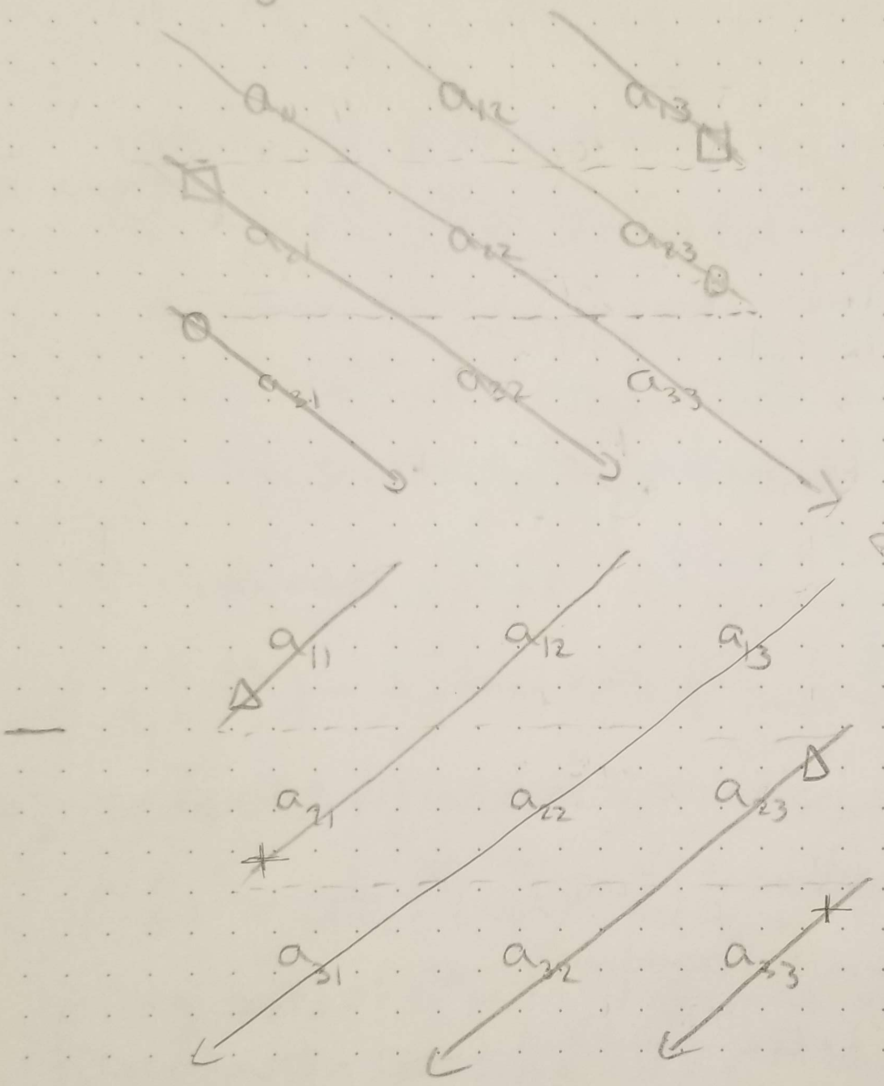
$$\det = (2+i) - (2-i) = 2i$$

$$(\;\cdots\;)^{-1} = \frac{1}{2i}\begin{pmatrix} e^{-\frac{\pi i}{3}} & (-2+i)e^{-\frac{\pi i}{3}} \\ -e^{\frac{\pi i}{3}} & (2+i)e^{\frac{\pi i}{3}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -ie^{-\frac{\pi i}{3}} & (1+2i)e^{-\frac{\pi i}{3}} \\ ie^{\frac{\pi i}{3}} & (1-2i)e^{\frac{\pi i}{3}} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \left(-\frac{\sqrt{3}}{4} - \frac{i}{4}\right)\left(\frac{1+\sqrt{5}}{2} + \frac{5\pi}{4\sqrt{3}}\right) + (1+2i)\left(\frac{1}{4} - i\frac{\sqrt{3}}{4}\right)\left(-1 - \frac{\pi}{12} + \frac{\pi}{2\sqrt{3}} - \frac{\sqrt{3}}{4}\right) \\ \left(-\frac{\sqrt{3}}{4} + \frac{i}{4}\right)\left(\frac{1+\sqrt{5}}{2} + \frac{5\pi}{4\sqrt{3}}\right) + (1-2i)\left(\frac{1}{4} + i\frac{\sqrt{3}}{4}\right)\left(-1 - \frac{\pi}{2} + \frac{\pi}{2\sqrt{3}} - \frac{\sqrt{3}}{4}\right) \end{pmatrix}$$

2,3) $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$

which you can think of as



$$\det\begin{pmatrix} 2-\lambda & * & * \\ 0 & -7-\lambda & * \\ 0 & 0 & -13-\lambda \end{pmatrix} = (2-\lambda)(-7-\lambda)(-13-\lambda)$$ because only doesn't include one of the lower triangular 0's.

$\lambda = 2 \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =: v_1$ is an eigenvector

$\lambda = -7 \Rightarrow \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} =: v_2$ is an eigenvector

$$\lambda = -13 \Rightarrow \begin{cases} 15 v_{31} - 3 v_{32} + 5 v_{33} = 0 \\ 6 v_{32} + N v_{33} = 0 \end{cases}$$

Take $v_{33} = -6$ (any nonzero choice will work). Then $v_{32} = 11$.

$15 v_{31} - 33 - 30i = 0 \Rightarrow v_{31} = \frac{11}{5} + 2i$ (I will multiply v_3 by 5 to clear denoms)

$$\Rightarrow \begin{pmatrix} 2 & -3 & 5i \\ & -7 & 11 \\ & & -13 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 11+10i \\ 0 & 3 & 55 \\ 0 & 0 & -30 \end{pmatrix} \begin{pmatrix} 2 & & \\ & -7 & \\ & & -13 \end{pmatrix} \begin{pmatrix} 1 & 1 & 11+10i \\ 0 & 3 & 55 \\ 0 & 0 & -30 \end{pmatrix}^{-1}$$

$$\Rightarrow e^{tA} = \begin{pmatrix} 1 & 1 & 11+10i \\ & 3 & 55 \\ & & -30 \end{pmatrix} \begin{pmatrix} e^{2t} & & \\ & e^{-7t} & \\ & & e^{-13t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 11+10i \\ & 3 & 55 \\ & & -30 \end{pmatrix}^{-1}$$

One way to compute $\begin{pmatrix} 1 & 1 & 11+10i \\ & 3 & 55 \\ & & -30 \end{pmatrix}^{-1}$ is row-reducing to I and keeping track of what would happen to I at the same time!

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 11+10i & 1 & & \\ & 3 & 55 & & 1 & \\ & & -30 & & & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & \frac{11+10i}{30} \\ & 3 & 0 & & 1 & \frac{55}{30} \\ & & -30 & & & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{3} & \frac{11+10i}{90} - \frac{55}{90} \\ & 3 & 0 & & 1 & \frac{55}{30} \\ & & -30 & & & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & & & 1 & -\frac{1}{3} & \frac{11+10i}{90} - \frac{55}{90} \\ & 1 & & & \frac{1}{3} & \frac{55}{90} \\ & & 1 & & & -\frac{1}{30} \end{array}\right]$$

This works because row reduction can be viewed as multiplying on the left by elementary matrices: $\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$ to switch rows,

$\begin{pmatrix} 1 & & & \\ & m & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ to scale, and $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & m & 1 & \\ & & & 1 \end{pmatrix}$ to add one row to another

If $E_n E_{n-1} \cdots E_1 A = I$, then $E_n E_{n-1} \cdots E_1 = A^{-1}$.

A particular solution is

$$\begin{pmatrix} 1 & 1 & 11+10i \\ & 3 & 55 \\ & & -30 \end{pmatrix} \begin{pmatrix} e^{2t} & & \\ & e^{-7t} & \\ & & e^{-13t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{-2u} & & \\ & e^{7u} & \\ & & e^{13u} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{3} & \frac{11+10i}{90} - \frac{55}{90} \\ & -\frac{1}{3} & \frac{25}{90} \\ & & -\frac{1}{30} \end{pmatrix} \begin{pmatrix} 17e^{2u} \\ -19e^{u} \\ 23e^{8u} \end{pmatrix} du$$

$$= \frac{1}{90} \begin{pmatrix} 1 & 1 & 11+10i \\ & 3 & 55 \\ & & -30 \end{pmatrix} \begin{pmatrix} e^{2t} & & \\ & e^{-7t} & \\ & & e^{-13t} \end{pmatrix} \int_0^t \begin{pmatrix} 90e^{-2u} & -30e^{-2u} & (-44+10i)e^{-2u} \\ & -30e^{7u} & 55e^{7u} \\ & & -3e^{13u} \end{pmatrix} \begin{pmatrix} 17e^{2u} \\ -19e^{u} \\ 23e^{8u} \end{pmatrix} du$$

$$= \frac{1}{90} \cdots \int_0^t \begin{pmatrix} 90 + 570e^{-u} + (-1012+230i)e^{6u} \\ 570e^{8u} + 1265e^{15u} \\ -69e^{21u} \end{pmatrix} du$$

$$= \frac{1}{90} \begin{pmatrix} 1 & 1 & 11+10i \\ & 3 & 55 \\ & & -30 \end{pmatrix} \begin{pmatrix} e^{2t} & & \\ & e^{-7t} & \\ & & e^{-13t} \end{pmatrix} \begin{pmatrix} 90t + 570 - 570e^{-t} + \frac{506-115i}{3}(1-e^{6t}) \\ \frac{235}{4}(e^{8t}-1) + \frac{253}{3}(e^{15t}-1) \\ \frac{23}{7}(1-e^{21t}) \end{pmatrix}$$

$$= \frac{1}{90} \begin{pmatrix} 1 & 1 & 11+10i \\ & 3 & 55 \\ & & -30 \end{pmatrix} \begin{pmatrix} 90te^{2t} - 570e^{t} - \frac{506-115i}{3}e^{8t} + \frac{2216-115i}{3}e^{2t} \\ \frac{235}{4}e^{t} + \frac{253}{3}e^{8t} - \left(\frac{235}{4} + \frac{253}{3}\right)e^{-7t} \\ -\frac{23}{7}e^{8t} + \frac{23}{7}e^{-13t} \end{pmatrix}$$

When $t=0$, $T=0$, because $\int_0^t = \int_0^0$

So the solution is $x(t) = e^{tA}\begin{pmatrix} 109 \\ 1009 \\ 10009 \end{pmatrix} +$

$$= \frac{1}{90} \begin{pmatrix} 1 & 1 & 11+10i \\ & 3 & 55 \\ & & -30 \end{pmatrix} \begin{pmatrix} e^{2t} & & \\ & e^{-7t} & \\ & & e^{-13t} \end{pmatrix} \begin{pmatrix} 90 & -30 & -44+10i \\ & -30 & 55 \\ & & -3 \end{pmatrix} \begin{pmatrix} 109 \\ 1009 \\ 10009 \end{pmatrix} +$$

4.1) $\mathcal{L}\{e^{777t}\} = \int_0^\infty e^{(777-s)t}\,dt = \dfrac{e^{(777-s)t}}{777-s}\Big|_0^\infty$

When $Re(s) > 777$, $= \dfrac{e^{-\infty}}{777-s} - \dfrac{e^0}{777-s} = \dfrac{1}{s-777}$

4.2) $\int_0^\infty te^{-(s+2)t}\,dt \overset{IBP}{=} \dfrac{te^{-(s+2)t}}{-(s+2)}\Big|_0^\infty - \int_0^\infty \dfrac{e^{-(s+2)t}}{-(s+2)}\,dt$

If $Re(s) > -2$

$= 0 - 0 + \dfrac{1}{s+2}\int_0^\infty e^{-(s+2)t}\,dt = \dfrac{1}{(s+2)^2}$

4.3) $\int_0^\infty \begin{cases} 1 & 0<a<t<b \\ 0 & o.w \end{cases} e^{-st}\,dt = \int_a^b e^{-st}\,dt = \dfrac{e^{-st}}{-s}\Big|_a^b$

$= \dfrac{e^{-as} - e^{-bs}}{s}$

4.4) $\int_0^\infty \begin{cases} 0 & t<1 \\ e^{3t} & t>1 \end{cases} e^{-st}\,dt = \int_1^\infty e^{-(s-3)t}\,dt = \dfrac{e^{-(s-3)}}{s-3}$ if $Re(s) > 3$

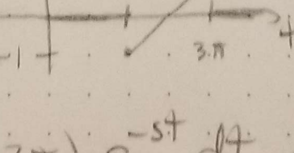
4.5) $\int_0^\infty t\sin \pi t\, e^{-st}\,dt = \int_0^\infty t\,\dfrac{e^{\pi i t} - e^{-\pi i t}}{2i}\, e^{-st}\,dt$

$= \dfrac{1}{2i}\int_0^\infty te^{-(s-\pi i)t} - te^{-(s+\pi i)t}\,dt$

IBP, $Re(s) > 0$

$= 0 - 0 - \dfrac{1}{2i}\int_0^\infty \dfrac{e^{-(s-\pi i)t}}{-(s-\pi i)} - \dfrac{e^{-(s+\pi i)t}}{-(s+\pi i)}\,dt$

$= \dfrac{1}{2i}\left(\dfrac{1}{(s-\pi i)^2} - \dfrac{1}{(s+\pi i)^2}\right)$

4.6) Define $f(t) :=$ 

$= \dfrac{1}{\pi}(t-2\pi)\,\mathbb{1}_{\{3\pi > t > \pi\}}$

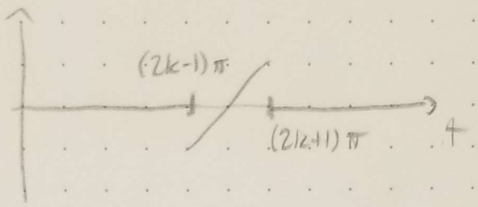
$\mathcal{L}\{f(t)\} = \int_\pi^{3\pi} \dfrac{1}{\pi}(t-2\pi)e^{-st}\,dt = -2\int_\pi^{3\pi} e^{-st}\,dt + \dfrac{1}{\pi}\int_\pi^{3\pi} te^{-st}\,dt$

$\overset{(4.3)}{\underset{(IBP)}{=}} -2\,\dfrac{e^{-\pi s} - e^{-3\pi s}}{s} + \dfrac{1}{\pi}\,\dfrac{te^{-st}}{-s}\Big|_\pi^{3\pi} + \dfrac{1}{\pi}\int_\pi^{3\pi} \dfrac{e^{-st}}{s}\,dt$

$$= -\frac{2}{s}\left(e^{-\pi s} - e^{-3\pi s}\right) + \frac{e^{-\pi s} - 3e^{-3\pi s}}{s} + \frac{1}{\pi}\,\frac{e^{-st}}{-s^2}\Big|_{\pi}^{3\pi}$$

$$= -\frac{e^{-\pi s} + e^{-3\pi s}}{s} + \frac{e^{-\pi s} - e^{-3\pi s}}{\pi s^2}$$

$$= e^{-\pi s}\left(\frac{1}{\pi s^2} - \frac{1}{s}\right) - e^{-3\pi s}\left(\frac{1}{\pi s^2} + \frac{1}{s}\right)$$

Now define $f_k(t) :=$ 

$= \frac{1}{\pi}(t - 2\pi k)$

We see that $f(t) = f_1(t)$, and $s(t) = \sum_{k=-\infty}^{\infty} f_k(t)$

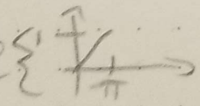
$\mathcal{L}\{s(t)\} = \mathcal{L}\{$ 

$\} + \sum_{k=1}^{\infty} \mathcal{L}\{f_k(t)\}$

$\mathcal{L}\{f_k(t)\} = e^{-2\pi(k-j)s}\,\mathcal{L}\{f_j(t)\}$, so ← Geometric series

$$\mathcal{L}\Big\{\sum_{k=1}^{\infty} f_k(t)\Big\} = \mathcal{L}\{f_1(t)\}\sum_{k=1}^{\infty} e^{-2\pi(k-1)s}$$

$$= \left[e^{-\pi s}\left(\frac{1}{\pi s^2} - \frac{1}{s}\right) - e^{-3\pi s}\left(\frac{1}{\pi s^2} + \frac{1}{s}\right)\right]\frac{1}{1 - e^{-2\pi s}}$$

$\mathcal{L}\{$ 

$\} = \int_0^{\pi} t e^{-st}\,dt = \frac{te^{-st}}{-s}\Big|_0^{\pi} + \int_0^{\pi} \frac{e^{-st}}{s}\,dt$

$$= -\frac{e^{-\pi s}}{s} + \frac{e^{-st}}{-s}\Big|_0^{\pi} = \frac{1}{s}\left(1 - 2e^{-\pi s}\right)$$

$$\Rightarrow \mathcal{L}\{s(t)\} = \frac{1}{s}\left(1 - 2e^{-\pi s}\right) + \frac{1}{1 - e^{-2\pi s}}\left(e^{-\pi s}\left(\frac{1}{\pi s^2} - \frac{1}{s}\right) - e^{-3\pi s}\left(\frac{1}{\pi s^2} + \frac{1}{s}\right)\right)$$