Math 218 — Assignment 8

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Due 2024/11/25

1. Consider the differential equation

$$t^5 y' + \frac{1}{y} = 0, \qquad -10 < t < 10, t \neq 0. \tag{1}$$

1.1) Give the general solution to (1). Be careful to ensure that your solution is indeed the most general possible.
1.2) Is there a solution to the initial value problem given by (1) with $y(2) = 3$ and $y(-4) = -\frac{1}{16}$? If so, find such a solution and check that it works. If not, explain why not.

2. Give the general solution $y(x)$ to

$$y' x^3 \cos y + 3x^2 \sin y - x = 0, \qquad 1 < x < 2.$$

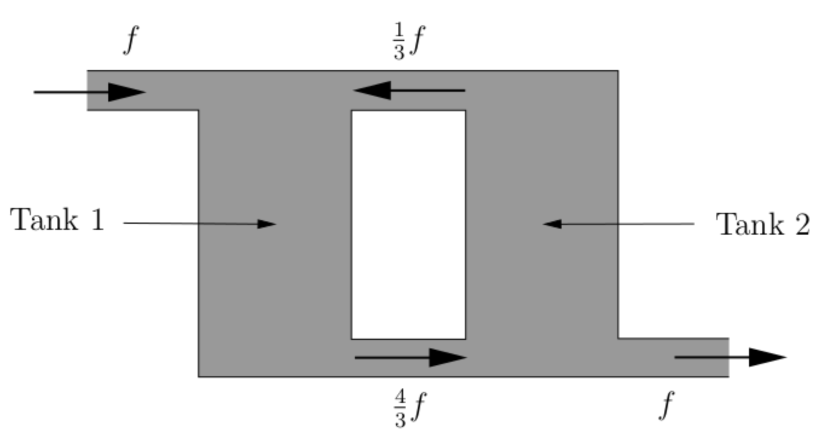
3. Give the general solution $y(x)$ to

$$\frac{xy'}{y} + \frac{1}{2}\log y = \sqrt{x}e^{-19x}, \qquad x > 0.$$

4. Consider the coupled constant volume mixing tank system as shown with inflow concentration $c_{in}(t)$, with state vector

$$\mathbf{x} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix},$$

where m_1 and m_2 denote the mass of chemical in tanks 1 and 2 respectively. Let V be the volume of each tank.



4.1) Show that the vector DE governing the state of the system is

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}, \qquad A = \begin{pmatrix} -4b & b \\ 4b & -4b \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 3bVc_{in} \\ 0 \end{pmatrix},$$

where $b = \frac{f}{3V}$ (this simplifies the algebra).

4.2) Find the solution to the homogeneous DE $\mathbf{x}' = A\mathbf{x}$.

4.3) Find the solution for the following initial conditions, assuming $c_{in}(t) = 0$:

i) $m_1(0) = M$, $m_2(0) = 0$ ii) $m_1(0) = \frac{1}{3}M$, $m_2(0) = \frac{2}{3}M$ iii) $m_1(0) = 0$, $m_2(0) = M$.

In each case give a qualitative sketch of the mass functions $m_1(t)$ and $m_2(t)$ on the same axes. Use the graphs to give a physical interpretation of the behaviour of the system, discussing whether the mass of chemical in each tank is increasing or decreasing and whether the masses are ever equal.

4.4) Referring to 4.3, in which case does the system flush most rapidly, i.e. in which case does the total mass in the system tend to zero most rapidly? First make an "educated guess", and then give a mathematical analysis.

4.5) Sketch typical orbits of the DE in $\mathbb{R}^2$, subject to the restriction $m_1 \geq 0$, $m_2 \geq 0$.

(i) Mark the orbits corresponding to the three solutions in part 4.3 on your sketch.

(ii) Consider an initial state with $m_2(0) < m_1(0)$. Use the sketch to describe the future evolution of the system.

(iii) Do the same for an initial state with $m_2(0) > 4m_1(0)$.

4.6) Find the solution of the non-homogeneous DE assuming $c_{in}(t) = c$, a constant, and an arbitrary initial state $\mathbf{x}(0) = \mathbf{a}$. What is the asymptotic behaviour as $t \to +\infty$?