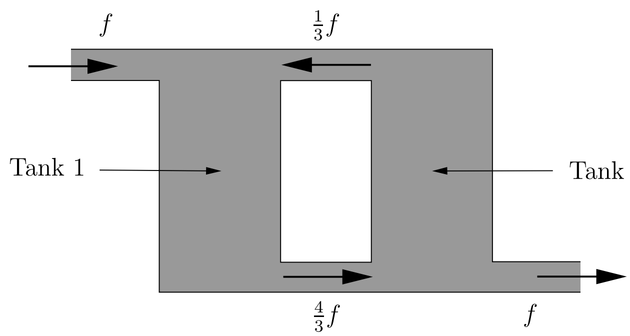
Consider the coupled constant volume mixing tank system as shown with inflow concentration $c_{in}(t)$, with state vector

$$\mathbf{x} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix},$$

where m_1 and m_2 denote the mass of chemical in tanks 1 and 2 respectively. Let V be the volume of each tank.



(a) Show that the vector DE governing the state of the system is

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}, \qquad A = \begin{pmatrix} -4b & b \\ 4b & -4b \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 3bV c_{in} \\ 0 \end{pmatrix},$$

where $b = \frac{f}{3V}$ (this simplifies the algebra).

(b) Find the solution to the homogeneous DE $\mathbf{x}' = A\mathbf{x}$.

(c) Find the solution for the following initial conditions, assuming $c_{in}(t) = 0$:

i) $m_1(0) = M$, $m_2(0) = 0$ ii) $m_1(0) = \frac{1}{3}M$, $m_2(0) = \frac{2}{3}M$ iii) $m_1(0) = 0$, $m_2(0) = M$.

In each case give a qualitative sketch of the mass functions $m_1(t)$ and $m_2(t)$ on the same axes. Use the graphs to give a physical interpretation of the behaviour of the system, discussing whether the mass of chemical in each tank is increasing or decreasing and whether the masses are ever equal.

(d) Referring to (c), in which case does the system flush most rapidly, i.e. in which case does the total mass in the system tend to zero most rapidly? First make an "educated guess", and then give a mathematical analysis.

(e) Sketch typical orbits of the DE in $\mathbb{R}^2$, subject to the restriction $m_1 \geq 0$, $m_2 \geq 0$.

(i) Mark the orbits corresponding to the three solutions in part (c) on your sketch.

(ii) Consider an initial state with $m_2(0) < m_1(0)$. Use the sketch to describe the future evolution of the system.

(iii) Do the same for an initial state with $m_2(0) > 4m_1(0)$.

(f) Find the solution of the non-homogeneous DE assuming $c_{in}(t) = c$, a constant, and an arbitrary initial state $\mathbf{x}(0) = \mathbf{a}$. What is the asymptotic behaviour as $t \to +\infty$?

Solution

(a) Using conservation of mass we have

$$\begin{aligned} \frac{dm_1}{dt} &= f c_{\text{in}} + \frac{1}{3} f\left(\frac{m_2}{V}\right) - \frac{4}{3} f\left(\frac{m_1}{V}\right) \\ \frac{dm_2}{dt} &= \frac{4}{3} f\left(\frac{m_1}{V}\right) - \frac{1}{3} f\left(\frac{m_2}{V}\right) - f\left(\frac{m_2}{V}\right) \end{aligned}$$

which we can rewrite as

$$\begin{aligned} m_1' &= \frac{f}{3V}(m_2 - 4m_1 + 3V c_{\text{in}}) \\ m_2' &= \frac{f}{3V}(4m_1 - 4m_2) \end{aligned}$$

which means that, in vector form,

$$\mathbf{x}' = b \begin{bmatrix} -4 & 1 \\ 4 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 3bV c_{\text{in}} \\ 0 \end{bmatrix}$$

(b) For $A = \begin{bmatrix} -4b & b \\ 4b & -4b \end{bmatrix}$ we have $A - \lambda I = \begin{bmatrix} -4b - \lambda & b \\ 4b & -4b - \lambda \end{bmatrix}$ so the characteristic equation is

$$\begin{aligned} (-\lambda - 4b)^2 - 4b^2 &= 0 \\ \lambda^2 + 8b\lambda + 12b^2 &= 0 \\ (\lambda + 2b)(\lambda + 6b) &= 0 \end{aligned}$$

Thus $\lambda = -2b$ and $\lambda = -6b$. Next we find the associated eigenvectors.

$\boxed{\lambda = -2b}$

Solving $(A + 2bI)\vec{v} = \vec{0}$ gives:

$$\begin{bmatrix} -2b & b \\ 4b & -2b \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad -2bv_1 + bv_2 = 0 \quad \Rightarrow \quad v_2 = 2v_1$$

Hence an eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\boxed{\lambda = -6b}$

Solving $(A + 6bI)\vec{v} = \vec{0}$ gives:

$$\begin{bmatrix} 2b & b \\ 4b & 2b \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad 2bv_1 + bv_2 = 0 \quad \Rightarrow \quad v_2 = -2v_1$$

Hence an eigenvector is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

From this our solution is

$$\vec{x} = c_1 e^{-2bt} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(c) (i) With $m_1(0) = M, m_2(0) = 0$ we get the system

$$\begin{bmatrix} M \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} M = c_1 + c_2 \\ 0 = 2c_1 - 2c_2 \end{matrix}$$

which has solutions $c_1 = c_2 = \frac{M}{2}$

The solutions are thus

$$\begin{aligned} m_1(t) &= \frac{M}{2}(e^{-2bt} + e^{-6bt}) \\ m_2(t) &= M(e^{-2bt} - e^{-6bt}) \end{aligned}$$

Note that:

$$m_1'(t) = \frac{M}{2}(-2be^{-2bt} - 6be^{-6bt}) = Mb(-e^{-2bt} - 3e^{-6bt})$$

which is always negative.

Whereas

$$m_2'(t) = Mb(-2e^{-2bt} + 6e^{-6bt}) = 0 \text{ if } 6e^{-6bt} = 2e^{-2bt}$$

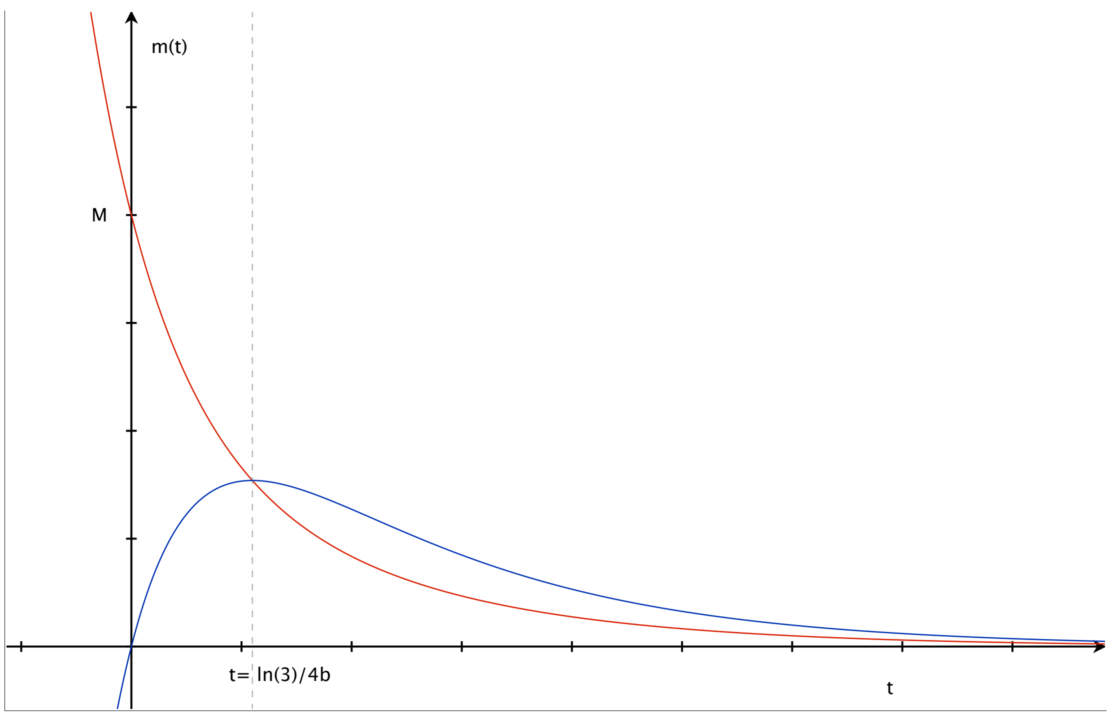
which happens when $t = \frac{\ln 3}{4b}$.

Thus, m_1 is always decreasing but m_2 increases (to approximately $0.4M$ and then decreases. To see if they intersect we solve $m_1 = m_2$ to get

$$\begin{aligned}\frac{M}{2}(e^{-2bt} + e^{-6bt}) &= M(e^{-2bt} - e^{-6bt}) \\ 3e^{-6bt} &= e^{-2bt} \\ t &= \frac{\ln 3}{4b}\end{aligned}$$

i.e. when m_2 is a maximum.

The plot is shown below (m_1 is red, m_2 is blue)



(ii) With $m_1(0) = \frac{1}{3}M, m_2(0) = \frac{2}{3}M$ we get the system

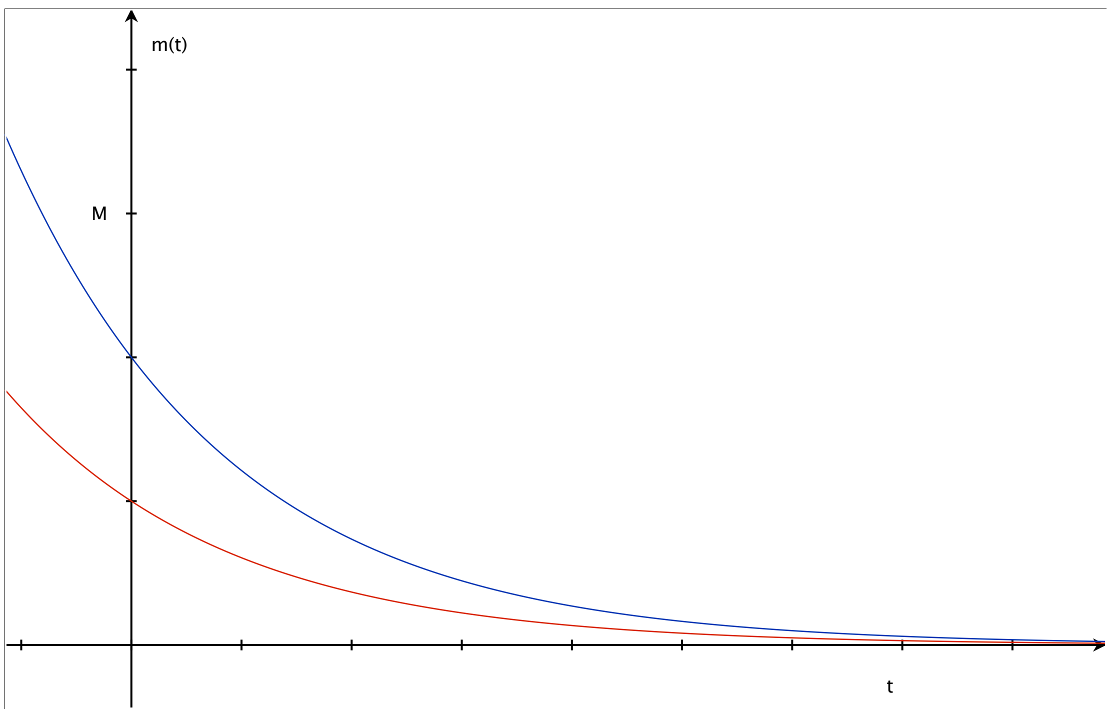
$$\begin{bmatrix} \frac{M}{3} \\ \frac{2M}{3} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \frac{M}{3} &= c_1 + c_2 \\ \frac{2M}{3} &= 2c_1 - 2c_2 \end{aligned}$$

which has solutions $c_1 = \frac{M}{3}$ and $c_2 = 0$.

The solutions are thus

$$m_1(t) = \frac{M}{3}e^{-2bt}$$
$$m_2(t) = \frac{2M}{3}e^{-2bt}$$

Both functions decrease and m_2 is always twice m_1 (so they don't intersect): The plot is shown below (m_1 is red, m_2 is blue)



(iii) With $m_1(0) = 0, m_2(0) = M$ we get the system

$$\begin{bmatrix} 0 \\ M \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} 0 &= c_1 + c_2 \\ M &= 2c_1 - 2c_2 \end{aligned}$$

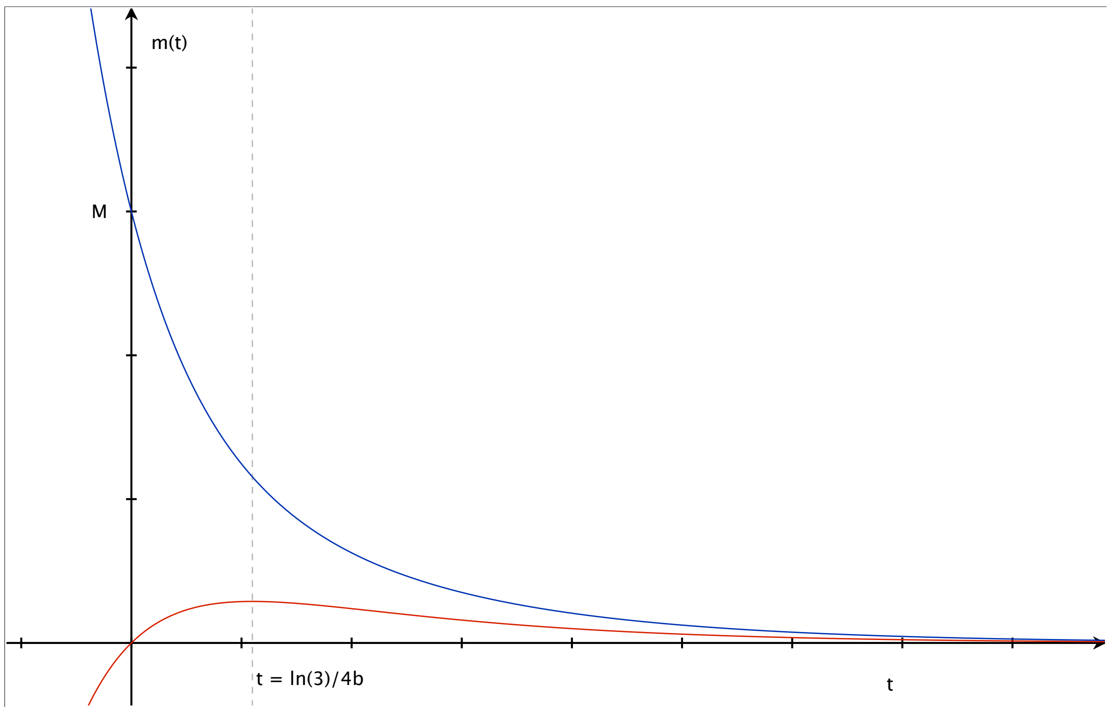
which has solutions $c_1 = \frac{M}{4}$ and $c_2 = -\frac{M}{4}$.
The solutions are thus

$$m_1(t) = \frac{M}{4}(e^{-2bt} - e^{-6bt})$$
$$m_2(t) = \frac{M}{2}(e^{-2bt} + e^{-6bt})$$

In this case $m_2'(t) < 0$ for all t and $m_1'(t) = 0$ when $-2be^{-2bt} + 6be^{-6bt} = 0$ which once again happens at $t = \frac{\ln(3)}{4b}$.

Also, in this setup it we get that m_2 is always greater than m_1 (compare with part (i) where the smaller coefficient, $\frac{M}{2}$ was with the solution that adds the exponentials whereas here the smaller coefficient, $\frac{M}{4}$ is attached to the solution with subtracting exponentials).

The plot is shown below (m_1 is red, m_2 is blue)



(d) The trick here is to realize that the "slower" exponential e^{-2bt} is the one that matters. The faster one e^{-6bt} will decay to zero very rapidly so we need to find out which case has the fastest decaying "slow" term e^{-2bt}.

In terms of leading order behaviour (i.e. e^{-2bt}) we have that:

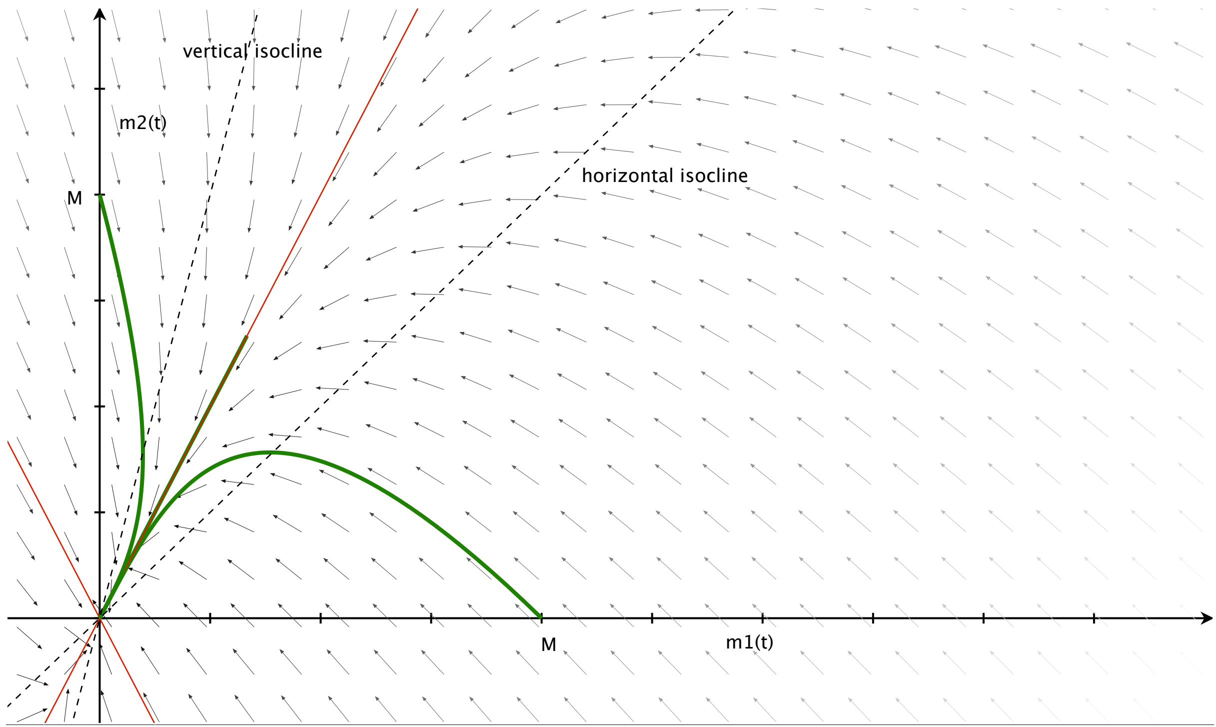
Case (i) gives $m_1 + m_2 = Me^{-2bt} + \frac{M}{2}e^{-2bt} = \frac{3M}{2}e^{-2bt}$

Case (ii) gives $m_1 + m_2 = \frac{M}{3}e^{-2bt} + \frac{2M}{3}e^{-2bt} = Me^{-2bt}$

Case (iii) gives $m_1 + m_2 = \frac{M}{4}e^{-2bt} + \frac{M}{2}e^{-2bt} = \frac{3M}{4}e^{-2bt}$

Thus the third case will decrease the fastest which probably makes sense when you think that if all of the mass is initially in tank 2 then a large chunk will flush out right away.

(e) The orbits are shown below



(i) The orbits corresponding to the scenarios above are marked in green.

(ii) The horizontal isoclines are given by $m_2'(t) = 0 \quad \Rightarrow \quad m_2 = m_1$, this is shown in the plot. When $m_2(0) < m_1(0)$ then m_1 decreases for all t whereas m_2 increases for a bit before decreasing. At that time m_2 will become greater than m_1.

(iii) The vertical isoclines are given by $m_1'(t) = 0 \quad \Rightarrow \quad m_2 = 4m_1$, this is shown in the plot. When $m_2(0) > 4m_1(0)$ then m_2 decreases for all t while m_1 hits a maximum before decreasing. m_1 becomes $> \frac{m_2}{4}$ at that time but the masses are never equal.

(f) Our system is now

$$\begin{bmatrix} m_1'(t) \\ m_2'(t) \end{bmatrix} = \begin{bmatrix} -4b & b \\ 4b & -4b \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \begin{bmatrix} 3bVc \\ 0 \end{bmatrix}$$

We already solved for the homogenous solution in part (b). We will now find the particular solution $\mathbf{x_p}$ by using undeterminded coefficients.

Let $\mathbf{x_p} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$. Upon substitution into the system we get

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4b & b \\ 4b & -4b \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + \begin{bmatrix} 3bVc \\ 0 \end{bmatrix}$$

$$\begin{aligned} 0 &= -4bk_1 + bk_2 + 3bVc \\ 0 &= 4bk_1 - 4bk_2 \end{aligned}$$

The second equation implies $k_1 = k_2$. Using this in the first we get $k_1 = k_2 = cV$. Thus

$$\mathbf{x_p} = \begin{bmatrix} cV \\ cV \end{bmatrix}$$

and the full solution is

$$\mathbf{x} = c_1 e^{-2bt} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} cV \\ cV \end{bmatrix}$$

For an arbitrary initial condition $\mathbf{x}(0) = \mathbf{a}$ we have

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ -2c_2 \end{bmatrix} + \begin{bmatrix} cV \\ cV \end{bmatrix}$$

$$\begin{aligned} a_1 &= c_1 + c_2 + cV \\ a_2 &= 2c_1 - 2c_2 + cV \end{aligned}$$

Solving for c_1 and c_2 gives

$$\begin{aligned} c_1 &= \frac{a_1}{2} + \frac{a_2}{4} - \frac{3cV}{4} \\ c_2 &= \frac{a_1}{2} - \frac{a_2}{4} - \frac{cV}{4} \end{aligned}$$

The full solution, for an arbitrary initial condition, is thus

$$\mathbf{x} = \left(\frac{a_1}{2} + \frac{a_2}{4} - \frac{3cV}{4} \right) e^{-2bt} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \left(\frac{a_1}{2} - \frac{a_2}{4} - \frac{cV}{4} \right) e^{-6t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} cV \\ cV \end{bmatrix}$$

As $t \to \infty$ we get the exponentials disappear and we are left with $\mathbf{x} = \begin{bmatrix} cV \\ cV \end{bmatrix}$ which makes sense considering c is mass/volume and V is volume, thus cV is a mass.