Math 218 — Assignment 9

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Due 2024/12/04

1. Consider the differential equation

$$(x-2)y'' + xy' - y = 0. \tag{1}$$

1.1) Suppose $y(x)$ has power series $\sum_{n=0}^{\infty} a_n x^n$ and solves (1). Express a_{n+2} in terms of a_{n+1} and a_n.

1.2) Suppose $y(x)$ is a solution to (1) and that $y(0) = 2, y'(0) = 3$. Give the power series expansion of y about $x = 0$ to order 7 (i.e. terms up to and including x^7).

1.3) Let y_7 denote the degree 7 approximation from problem 1.2. Plot (using a tool like Desmos), on the same set of axes and in different colours,

- y_7
- $-xy_7'$
- $-(x-2)y_7''$
- $-(x-2)y_7'' - xy_7' + y_7$.

2. Consider the differential equation

$$y' = \frac{1}{1-x} + \cos \omega y. \tag{2}$$

2.1) Let $y(x)$ be a solution to (2). If $y(0) = 1$, roughly how big do you expect $y(1-\delta)$ to be (where $0 < \delta \ll 1$)?

2.2) If $y(0) = 1$ and $0 < \delta \ll 1$, what step size Δx should you set when using Euler's method to approximate $y(1-\delta)$?

2.3) For $\omega = 10^2$ and $\delta = 10^{-3}$, carry out the first 7 steps of Euler's method by hand with this Δx (using a calculator).

3. Recall that we are defining the Fourier transform as

$$\hat{f}(k) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx.$$

With this convention, the inverse Fourier transform is

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{2\pi i k x} dk.$$

3.1) Compute the Fourier transform of e^{-ax^2} for $a > 0$. Hint (rot13): Pbzcyrgr gur fdhner.

3.2) Compute the inverse Fourier transform of e^{-bk^2} for $b > 0$.[1]

Recall the convolution theorem: $\widehat{f * g} = \hat{f}\hat{g}$. Here $*$ denotes the convolution operation: $f * g$ is the function mapping x to $\int_{-\infty}^{\infty} f(u)g(x-u)\,du$.

Let X and Y be random numbers with probability distributions f and g respectively (meaning roughly that the probability of observing the outcome $X = x$ is proportional to $f(x)$). It is a fact that the probability distribution of $X + Y$ (i.e. generate the random number X, then generate the random number Y, then add them up) is equal to $f * g$. (Think about rolling two six-sided dice: how many ways can their values sum to 7? 8? What is $\mathbb{1}_{\{|x|<1\}} * \mathbb{1}_{\{|x|<1\}}$?)

3.3) Suppose X and Y are normally distributed with mean 0 and standard deviations σ_1 and σ_2 respectively, meaning that f and g are the "bell curves"

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{x^2}{2\sigma_1^2}\right) \qquad \text{and} \qquad g(x) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{x^2}{2\sigma_2^2}\right).$$

What is the distribution of the random number $X + Y$?

Many, many things in the world are normally distributed, like test scores. This is because, when a number is a sum of many small random numbers, the *central limit theorem* states that that sum will be normally distributed.

3.4) Explain how, via the Fourier transform calculations you performed above, you have gotten some insight into why the central limit theorem is true.[2]

[1] A nice visualization of what you find was prepared by Kabir Dubey in their note *The Fourier uncertainty principles* §4.

[2] This is in fact essentially how the central limit theorem is proved.