Math 218 — Extra problems 1

Alex Cowan
alex.cowan@uwaterloo.ca

1 First order differential equations

1.1. Determine the general solution to the following differential equations using an integrating factor. (Show all of your work as usual.)
a) $y' + 3y = t + e^{-2t}$
b) $y' + y = te^{-t} + 1$
c) $ty' + 2y = \sin t$, for $t > 0$
d) $y' + 2ty = 2te^{-t^2}$

1.2. Determine the general solution to the following differential equations using the method of separation of variables.
a) $y' = \frac{x^2}{y}$
b) $y' + y^2 \sin x = 0$
c) $\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}$
d) $y' = 4\frac{y^3}{x} + xy^3$

1.3. Solve the following initial value problems.
a) $y' - y = 2te^{2t}$ and $y(0) = 1$
b) $y' = \frac{2\cos 2x}{3+2y}$ and $y(0) = -1$
c) $y' + \frac{2}{t}y = \frac{\cos t}{t}$ and $y(\pi) = 0$, for $t > 0$
d) $y' = 2(1 + x)(1 + y^2)$ and $y(1) = 0.3$

1.4. Draw direction fields for the following differential equations. Describe the behaviour of the solutions as a function of initial conditions.
a) $y' + 3y - 4 = 0$
b) $w' = w^2 - 1$
c) $\dot{y} = t - y - 2$
d) $2yy' + y + 2t = 0$
e) $\frac{dq}{dr} = \frac{3r^2}{3q^2 - 4}$

1.5. Give a pair of values $a, b \in \mathbb{R}$ such that the differential equation $y' = ay + b$ has the constant function $y(x) = 2$ as a solution, and such that all other solutions...
a) ...diverge from $y = 2$ as $x \to \infty$.
b) ...converge to $y = 2$ as $x \to \infty$.

1.6. Explain how the following questions can be answered without doing any calculations.
a) What are the "equilibrium" (i.e. constant) solutions to the differential equation $y' = \cos(y)$?
b) Non-constant solutions to $y' = \cos(y)$ converge to certain equilibrium solutions but not others. Which equilibrium solutions have other solutions that converge to them, and which do not?

1.7. Find a differential equation which has the constant functions $y = -1$, $y = 0$, and $y = 1$ as solutions, as well as the property that non-constant solutions...
a) ...converge to $y = -1$ or $y = 1$.

b) ...either converge to $y = 0$ or diverge to ∞ or $-\infty$.

1.8. Sketch direction fields for the differential equations you found in 1.7a) and 1.7b). Describe the behaviour of the solutions as a function of initial conditions.

1.9. Find a differential equation for which $x(t) \coloneqq t^4 + 1$ is a solution.

1.10. Express the general solution to the differential equation $y' + \frac{y}{\log x} = 0$ in terms of $\text{Li}(x) \coloneqq \int_2^x \frac{dt}{\log t}$.[1]

1.11. Let $W_0(x)$ denote the inverse function of $x \mapsto xe^x$ for $x > 0$ (i.e. $W_0(x)e^{W_0(x)} = x$ for all $x > 0$). The function W_0 is called the *Lambert W function*. Express the general solution to the differential equation $y' = W_0(x)y + x$ in terms of W_0 and antiderivatives of related functions.

1.12. Solve the differential equation $y' = \frac{y-4x}{x-y}$ by making the change of variables $u(x) = \frac{y}{x}$.

1.13. How many free parameters does the initial value problem $y' = 2(1+x)(1+y^2)$ and $y(1) = 0.3$ from 1.3d) (defined on the largest possible real domain) have?

2 Linear second order differential equations with constant coefficients

2.1. Determine the general solution to the following differential equations.
a) $y'' + 2y' - 3y = 0$
b) $6y'' - y' - y = 0$
c) $y'' + 5y' = 0$
d) $y'' + 3y' + 2y = 0$
e) $y'' + 2y' + 3y = 0$
f) $y'' + 3y' + 4y = 0$
g) $y'' + 2y' + 1y = 0$
h) $y'' + \omega_0^2 y = 0$ with $\omega_0 > 0$
i) $y'' = 0$

2.2. For 2.1 a) — i), qualitatively describe the behaviour of the solutions.

2.3. Consider the differential equation

$$y'' + 2\lambda y' + \omega_0^2 y = 0 \tag{1}$$

with $\lambda, \omega_0 > 0$.
a) What is the general solution to (1)?
b) For what values of λ and ω_0 is the system modelled by (1) *undamped*, *underdamped*, *overdamped*, and *critically damped*?
c) Qualitatively describe the behaviour of undamped, underdamped, overdamped, and critically damped systems.

2.4. Solve the following initial value problems.
a) $y'' + 4y' + 3y = 0$ with $y(0) = 2$ and $y'(0) = -1$
b) $y'' + y' + 1.25y = 0$ with $y(0) = 3$ and $y'(0) = 1$
c) $y'' + 2y' + 2y = 0$ with $y(\frac{\pi}{4}) = 2$ and $y'(\frac{\pi}{4}) = -2$
d) $m\ddot{y} + ky = 0$ with $\dot{y}(0) = 3$

2.5. Consider the differential equation

$$y'' + 4y' + 9y = e^{2\pi i t}. \tag{2}$$

[1] Here and elsewhere log denotes the natural logarithm, i.e. log base e.

a) How many free parameters does the general solution to (2) have?
b) What is the definition of the homogeneous solution to (2)?
c) Determine the homogeneous solution to (2).
d) Find any particular solution to (2).
e) Determine the general solution to (2).
f) Explain why your answer in e) follows logically from a) — d) above.

2.6. Derive answers to the following questions.
a) What is a particular solution to $y'' + y' + y = e^x$?
b) What is a particular solution to $y'' + y' + y = \sin x$?
c) What is a particular solution to $y'' + y' + y = e^x + \sin x$?
d) What is the general solution to $y'' + y' + y = e^x + \sin x$?
e) Let $s(x)$ denote the sawtooth wave pictured below.

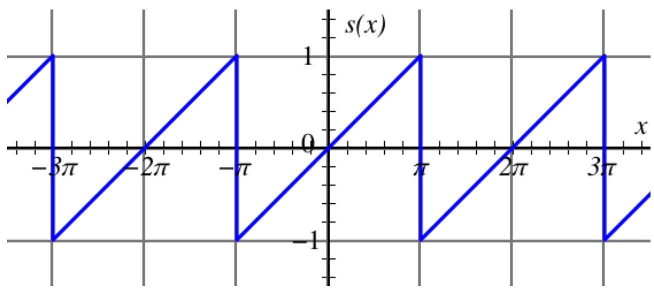

Figure 1: https://en.wikipedia.org/wiki/File:Sawtooth_pi.svg

I.e. $s(x) = \frac{x}{\pi}$ for $-\pi < x < \pi$ and then extended to other x values in a way that makes the function periodic. One can show, and you may assume, that

$$\begin{aligned} s(x) &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \\ &= \frac{2}{\pi}\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right). \end{aligned}$$

What is the general solution to $y'' + y' + y = s(x)$?

2.7.
a) What is the general solution to $y'' + 3y' + 2y = e^{-2t}$?
b) What is the general solution to $y'' + 2y' + y = e^{-2t}$?
c) What is the general solution to $y'' + 2y' + y = e^{-t}$?
d) What is the general solution to $y''' + 3y'' + 3y' + y = e^{-t}$?

2.8.
a) Write each of e^{it} and e^{-it} as linear combinations of cos and sin.
b) Write each of $\cos t$ and $\sin t$ as linear combinations of exponential functions.
c) Fix $a, b, c \in \mathbb{R}$ such that $b^2 - 4ac < 0$. For every $\gamma \in \mathbb{C}$ such that $a\gamma^2 + b\gamma + c \neq 0$, let y_γ denote a solution to the differential equation $ay'' + by' + cy = e^{\gamma t}$. For any $A, B, \alpha, \beta \in \mathbb{R}$, express the general solution to

$$ay'' + by' + cy = A\cos\alpha t + B\sin\beta t$$

in terms of particular solutions y_γ. (The notation means that if you write e.g. $y_{i\alpha}$, that denotes a particular solution to $ay'' + by' + cy = e^{i\alpha t}$.)

2.9. Give an example of a physical system that could reasonably be modelled by a linear second order differential equation with constant coefficients. Be specific in describing exactly what your dependent and independent variables are.

2.10. Consider a physical system modelled by the differential equation

$$y'' + 2\lambda y' + \omega_0^2 y = f_0 e^{i\omega t} \tag{3}$$

with $\lambda, \omega_0, \omega, f_0 > 0$.
a) What is the general solution to (3)?
b) Note that the homogeneous solution to (3) decays exponentially. Roughly how large must t be for particular solutions to (3) to be 1000 times as large as the homogeneous solution?
Define

$$A(\omega) := \frac{1}{\omega_0^2 - \omega^2 + i2\lambda\omega}. \tag{4}$$

c) Explain why (i.e. in what sense) maximizing $|A(\omega)|$ is the same as minimizing $|A(\omega)|^{-2}$.
d) Give an expression for $|A(\omega)|^{-2}$.
e) Show that, if $\omega_0 > \frac{1}{\sqrt{2}}\lambda$, then $\frac{d}{d\omega}|A(\omega)| = 0$ for $\omega = \omega_r := \sqrt{\omega_0^2 - 2\lambda^2}$. ($\omega_r$ is the *resonance frequency*).
f) What is the value of $|A(\omega_r)|$?
g) Show that, if $\omega_0 \leq \frac{1}{\sqrt{2}}\lambda$, then $\frac{d}{d\omega}|A(\omega)| = 0$ if and only if $\omega = 0$.

2.11. Say that a system of the form (3) *undergoes resonance* if there exists an $\omega > 0$ such that $|A(\omega)| > |A(0)|$, where A is as defined in (4).
True or false? Why?:
a) Undamped systems always undergo resonance.[2]
b) Underdamped systems always undergo resonance.
c) Overdamped systems never undergo resonance.
d) Some critically damped systems undergo resonance and some do not.

The *transient solution* to (3) is the part of the general solution found in 2.11a) which comes from the homogeneous solution. It's called this because, as quantified in 2.11b), it decays exponentially. The remaining part of the general solution to (3) is called the *steady-state solution*.

2.12. Let y_s denote the steady-state solution to (3). Show that $\Re(y_s)$ is a solution to $y'' + 2\lambda y' + \omega_0^2 y = f_0 \cos \omega t$, and that $\Im(y_s)$ is a solution to $y'' + 2\lambda y' + \omega_0^2 y = f_0 \sin \omega t$.[3]

2.13. For each system 2.1 a) — i), find the resonance frequency, or explain why there is no resonance frequency.

3 Numerical methods

3.1. Let $y(t)$ be a solution to the initial value problem $y' = F(y, t)$ with $y(t_0) = y_0$.
Write pseudocode which takes as input $y_0, t_0, t_{\max}, \Delta t \in \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}$ and outputs a reasonable numerical approximation to the values $y(t_0 + k\Delta t)$ for $0 < k < \frac{t_{\max} - t_0}{\Delta t}$. You can assume the inputs will satisfy $t_{\max} > t_0$ and $\Delta t > 0$, and F will be such that you won't run into division by 0 errors or similar.

3.2. Let $y(t)$ be a solution to the initial value problem $y'' = F(y, y', t)$ with $y(t_0) = y_0$ and $y'(t_0) = y_0'$ (in an abuse of notation; I don't mean $y_0' = \frac{d}{dt} y_0 = 0$).
Write pseudocode which takes as input $y_0, y_0', t_0, t_{\max}, \Delta t \in \mathbb{R}$ and $F : \mathbb{R}^3 \to \mathbb{R}$ and outputs a reasonable numerical approximation to the values $y(t_0 + k\Delta t)$ for $0 < k < \frac{t_{\max} - t_0}{\Delta t}$. You can assume the inputs will satisfy $t_{\max} > t_0$ and $\Delta t > 0$, and F will be such that you won't run into division by 0 errors or similar.

3.3. Let $x(t)$ and $y(t)$ be solutions to the initial value problems $\dot{x} = F(x, y, t)$ and $\dot{y} = G(x, y, t)$ with $x(t_0) = x_0$ and $y(t_0) = y_0$.

[2] Wainwright–West says "When $|A(\omega)|$ attains a maximum greater than $|A(0)|$, one says the the system undergoes resonance." What is the answer to a) with this definition?

[3] In general it is not true that $\Re(zw) = \Re(z) \cdot \Re(w)$, and indeed here $\Re(y_s) \neq \Re(A(\omega)) \cdot \Re(f_0 e^{i\omega t})$; beware!

Write pseudocode which takes as input $x_0, y_0, t_0, t_{\max}, \Delta t \in \mathbb{R}$ and $F, G : \mathbb{R}^3 \to \mathbb{R}$ and outputs reasonable numerical approximations to the values $x(t_0 + k\Delta t)$ and $y(t_0 + k\Delta t)$ for $0 < k < \frac{t_{\max}-t_0}{\Delta t}$. You can assume the inputs will satisfy $t_{\max} > t_0$ and $\Delta t > 0$, and F and G will be such that you won't run into division by 0 errors or similar.

3.4. I want to run your code for problem 3.1 with the following inputs:
a) $F(y,t) = \cos\cos\cos\cos\cos\cos\cos ty$, $y_0 = 0$, $t_0 = 0$, $t_{\max} = 1$, $\Delta t = 10^{-6}$
b) $F(y,t) = \frac{ty}{t^2+y^2}$, $t_0 = 1$, $t_{\max} = 10$, $\Delta t = 0.01$, for every $y_0 = -1, -0.99, -0.98, \ldots, 0.99, 1$.
c) $F_a(y,t) = \frac{1+ay+t^2}{1+t+y^2}$, $t_0 = 0$, $t_{\max} = 100$, $\Delta t = 0.01$, for every $y_0 = -5, -4.99, -4.98, \ldots 4.99, 5$ and every $a = 1, 1.01, 1.02, \ldots, 1.99, 2$
d) $F(y,t) = \log(1 + \frac{1}{2}\cos(1000(t^4 + y^4)))$, $y_0 = 1$, $t_0 = 1$, $t_{\max} = 100$, Δt as large as possible while still getting a reasonably accurate answer.
For each of these ambitions a) — d), do you think the code will 1. run basically instantaneously, 2. take a noticeable but short amount of time to run, 3. be realistic to run but require consideration to be practical, or 4. be unrealistic to run without access to major computing resources[4]? Why?

3.5. You wish to run the code that solves problem 3.1 for the following differential equations. In each case, identify what features of that specific differential equation might cause numerical simulation to be difficult (i.e. why it might be hard to get a reasonably accurate answer with code that'd be realistic to run on a modern laptop).
a) $y' = \log(1 + \frac{1}{2}\cos(1000(t^4 + y^4)))$
b) $y' = t^y$
c) $y' = -\frac{t^{-0.2024}}{\cos y}$
d) $y' = \frac{1}{\frac{1}{t^2+1} + e^{-y}}$

3.6.
a) Entirely by hand, use Euler's method[5] with $y_0 = 1$, $t_0 = 0$, and $\Delta t = 0.1$ to numerically approximate the solution to $y' = -ay$ for $t \leq 1$, for $a = 1$, 5, 15, and $a = 25$.
b) What is the critical value a_0 such that the numerical solution converges for $0 < a < a_0$ and diverges for $a > a_0$? What happens when $a = a_0$?
c) What happens when $a = 10$?
d) What is the exact solution to the initial value problem in a)?

3.7.
a) Let y be a solution to the initial value problem $y' = \sqrt{y+t}$ with $y_0 > 0$ and $t_{\max} > t_0 > 0$. Consider using Euler's method to determine $y(t_{\max})$. Will the output of the algorithm always overestimate the true value of $y(t_{\max})$, or always underestimate it, or are both outcomes possible depending on $y_0, t_0, t_{\max}$? Why?
b) For the initial value problem 3.6a) on the interval $0 \leq t < \infty$, for what values of $a \in \mathbb{R}$ is the numerical approximation to y from Euler's method always less than the exact solution?

3.8.
a) Give peudocode for generating direction fields.
b) Write code that generates the direction fields from problem 1.4.

3.9. Design and implement a version of Euler's method that uses second order Taylor series expansions instead of only first order expansions.

[4]Note that this does not imply that it would be realistic to run with access to these resources.
[5]Probably what you used for problem 3.1