Math 218 — Extra problems 3

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1. In this question we are working over the complex numbers. Let A be an arbitrary square matrix of any size. Let $v_1, \ldots, v_k$ be a complete list of eigenvectors of A, and $\lambda_1, \ldots, \lambda_k$ their associated eigenvalues.

   True or false? If true, demonstrate why. If false, give a counterexample and explain why it's a counterexample.

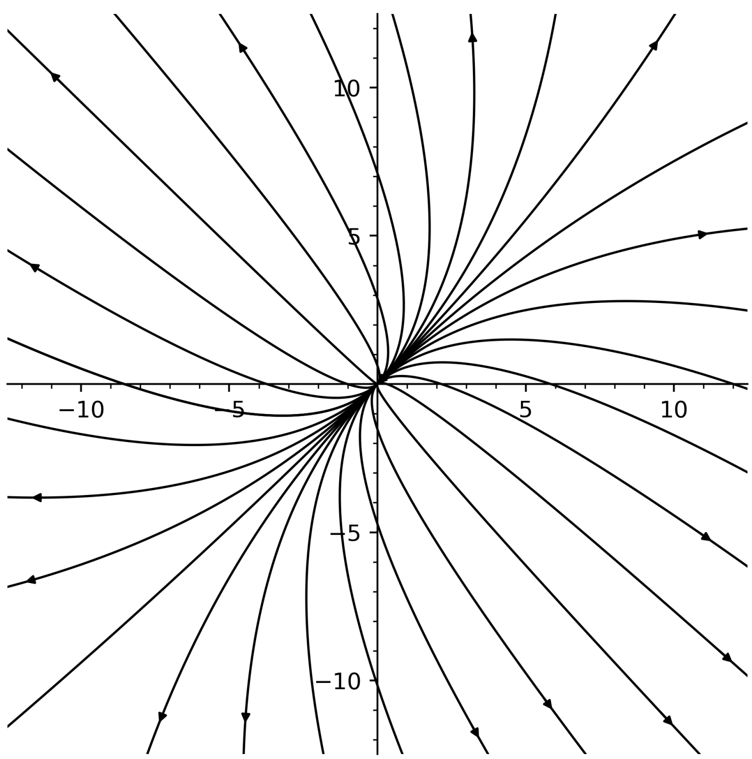
   (a) For every possible choice of complex numbers $c_1, \ldots, c_k$, the function

   $$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_k e^{\lambda_k t} v_k \tag{1}$$

   is a solution to the differential equation $x' = Ax$.

   (b) Every solution to the differential equation $x' = Ax$ is of the form (1).

2. The 2×2 matrix A has integer entries all less than or equal to 5 in absolute value. Below is a plot of phase space for the differential equation $x' = Ax$. Determine A. (Part marks: list properties A must posses.)



3. Consider the differential equation

$$y'' + 8y' + 16y = 0. \tag{2}$$

(a) By defining $x_1 := y$ and $x_2 := y'$, write (2) in the form $x' = Ax$.

(b) Find the general solution to the differential equation $x' = Ax$ from part (a).

(c) Using your answer from part (b), give the general solution to (2).

4. Consider the differential equation

$$y'' + 8y' + 16y = 0. \tag{3}$$

Find the general solution to (3) by taking a Laplace transform.

5. Consider the differential equation

$$y'' + 8y' + 16y = 0. \tag{4}$$

(a) Write $y(t) =: \sum_{n=0}^{\infty} a_n t^n$. Relate each a_n to $a_0, a_1, \ldots, a_{n-1}$ so that y is a general solution to (4).

(b) Suppose $y(0) = -2$ and $y'(0) = -3$. Give the power series of $y(t)$ around $t = 0$ to degree up to and including 4.

6. Consider the differential equation

$$y'' + 6y' + 34y = 4e^{-3t}\cos 5t. \tag{5}$$

(a) By defining $x_1 := y$ and $x_2 := y'$, write (5) in the form $x' = Ax + f(t)$.

(b) Find the general solution to the differential equation $x' = Ax + f(t)$ from part (a).

(c) Using your answer from part (b), give the general solution to (5).

(d) Find the general solution to (5) using a different method.

7. Consider the differential equation

$$x' = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} x + f(t). \tag{6}$$

(a) Find a solution to (6) when $f(t) = \vec{0}$ and $x(0) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

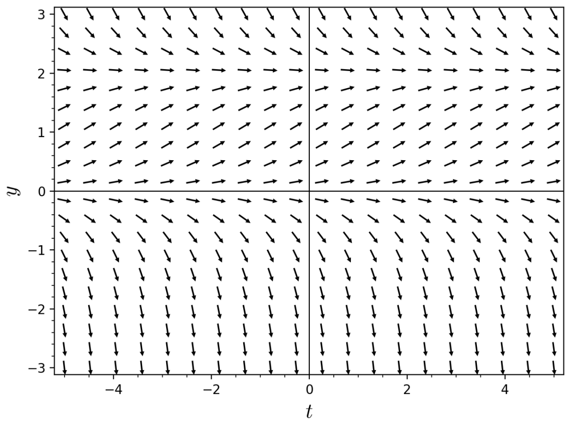
(b) Find a solution to (6) when $f(t) = \begin{pmatrix} e^{2\pi it} \\ e^{2\pi it} \end{pmatrix}$ and $x(0) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

(c) Find a solution to (6) when $f(t) = \begin{pmatrix} e^{2\pi it} \\ -e^{2\pi it} \end{pmatrix}$ and $x(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

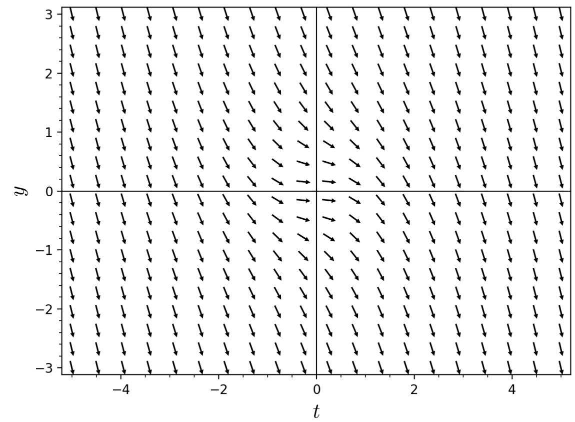
(d) Find a solution to (6) when $f(t) = \begin{pmatrix} 3\cos 2\pi t \\ 3\sin 2\pi t \end{pmatrix}$ and $x(0) = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$.

8. Match the following differential equations $y' = F(t, y)$ for each of the following F to their direction fields. One of the direction field plots is blank; sketch what it should be. Explain your reasoning throughout.

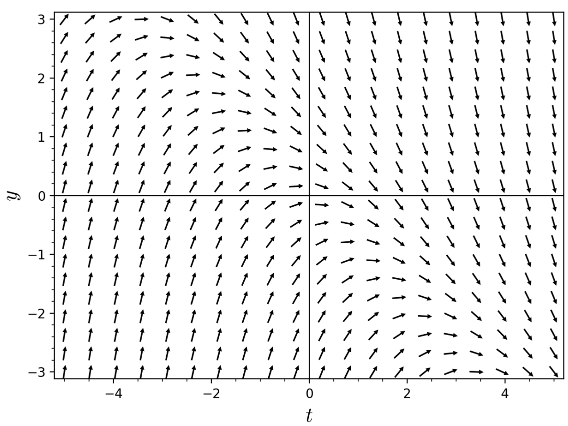
(i) $y - \dfrac{y^2}{2}$ (ii) $\log \dfrac{1}{1 + t^2 + y^2}$ (iii) $-\dfrac{2t}{3} - \dfrac{3y}{4}$ (iv) $\dfrac{y}{t^2 - 9}$



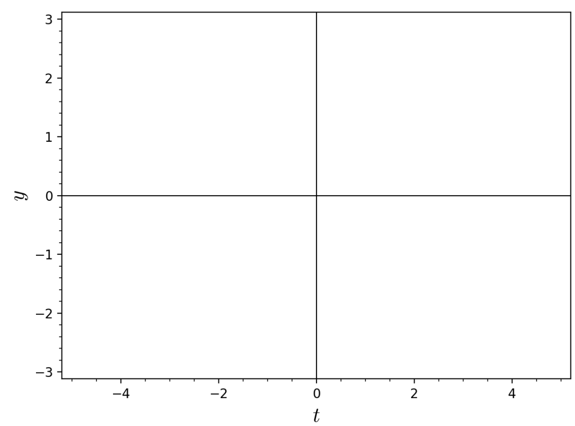
(a)



(b)



(c)



(d)

9. Suppose A is a matrix such that

$$e^{tA} = \begin{pmatrix} e^{2t} \cos 3t & -e^{2t} \sin 3t \\ e^{2t} \sin 3t & e^{2t} \cos 3t \end{pmatrix}.$$

(a) What are the eigenvalues of A?

(b) What is A?

10. Calculate

$$\exp\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

for $a \neq 0$ using the series definition of exp.

11. (a) Express $\mathscr{L}\{tf(t)\}$ in terms of $\mathscr{L}\{f(t)\}$ (with proof).
    (b) Express $\mathscr{L}\{g(t+k)\}$ in terms of $\mathscr{L}\{g(t)\}$ (with proof).
    (c) Solve the initial value problem

$$y'' + y' + 4y = (t+1)e^{2t+2}, \qquad y(0) = 2,\ y'(0) = 0.$$

    on the interval $t > -\frac{1}{2}$ by taking a Laplace transform.

12. For arbitrary complex numbers α and β, compute $e^{\alpha t} * e^{\beta t}$...
    (a) ...using the definition of $*$.
    (b) ...by taking a Laplace transform.

13. Find a function $f(t)$ satisfying

$$f(t) + \int_0^t (t-u)f(u)\,du = 1.$$

14. If $\mathscr{L}\{y\}(s) = e^{-\pi s} + \frac{\pi}{\pi+s}$, find $\int_0^\infty y(t)\,dt$.

15. Define $F(s) := \mathscr{L}\{t^2 \sin t\}$.
    (a) What is the domain of F?
    (b) Determine what $F(0)$ "should be"[1] using properties of the Laplace transform.

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[1]There exists a complex-differentiable, a.k.a. holomorphic function $\tilde{F}$ which agrees with F on all of the latter's domain, while being defined on some strictly larger open connected domain Ω. It turns out, by the black magic of complex analysis, that $\tilde{F}$ is unique among functions defined on Ω. This principle, called *analytic continuation*, is not even close to true when working only over $\mathbb{R}$, and is really spectacular. For example, the Riemann zeta function $\zeta(s) := \sum_{n=1}^\infty n^{-s}$ is defined by its series only for $\mathrm{Re}(s) > 1$, but can be analytically continued to all of $\mathbb{C}$ except for the point $s = 1$, and there is no ambiguity about which analytic continuation is meant; there is only one!

However, when this happens, you can't just plug in values of s into F to find the value of $\tilde{F}$. For example, while it's true that $\zeta(0) = -\frac{1}{2}$ (where now ζ denotes the aforementioned analytic continuation), it's nonsense to say $\sum_{n=1}^\infty n^{-0} = 1+1+1+1+\cdots = -\frac{1}{2}$. Similarly,

$$\zeta(-1) = -\tfrac{1}{12} \neq \sum_{n=1}^\infty n = 1+2+3+4+\ldots$$

Instead, to evaluate the analytically continued function $\tilde{F}$, one typically uses functional equations which relate the value of F to the value of it or other functions at other points where they can be evaluated. For example, $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$.

Another example is that of the gamma function $\Gamma(s) := \int_0^\infty e^{-x}x^s \frac{dx}{x}$, which satisfies the relation $s\Gamma(s) = \Gamma(s+1)$, easily verified using integration by parts. (This relationship means that $\Gamma(n) = (n-1)!$, so Γ basically lets you take factorials of complex numbers.) The integral defining Γ only converges for $\mathrm{Re}(s) > 0$, but it can be analytically continued to all of $\mathbb{C}$ save for $s = 0, -1, -2, \ldots$ by using the functional equation $\Gamma(s) = \frac{1}{s}\Gamma(s+1)$.

16. Let $s(x)$ be the sawtooth wave from extra problems 1 problem 2.6e). Compute the Fourier transform of $s(x)$ (heuristically). Compare with assignment 6 problem 4.6. (One way to do this is to recall how the Laplace transform interacts with derivatives; the Fourier transform is basically a Laplace transform, and you can show it interacts with derivatives in basically the same way.)

17. (a) State the definition of the Fourier transform $\hat{f}$.
    (b) State the Fourier inversion theorem. (I.e. given $\hat{f}$, what formula recovers f?) Note the symmetry between a function and its Fourier transform.
    (c) State the convolution theorem for Fourier transforms.

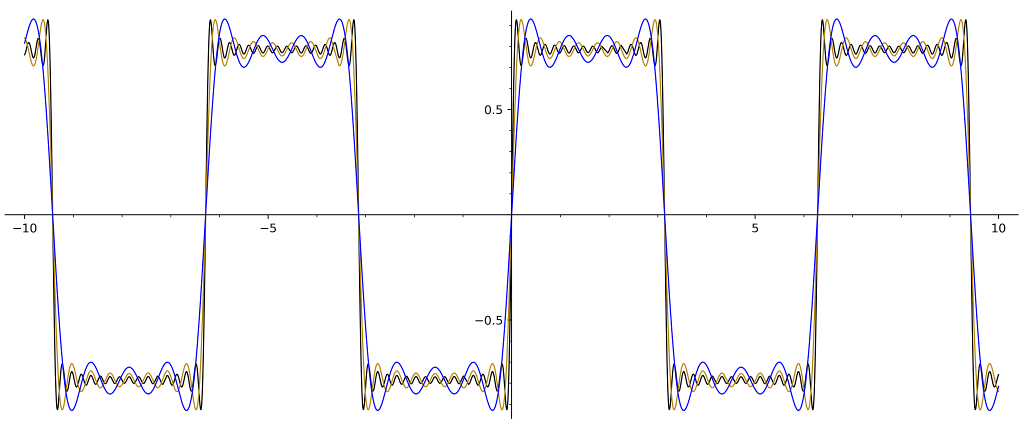
    Define

$$r(x) := \sum_{j=0}^{\infty} \frac{\sin((2j+1)x)}{2j+1}. \tag{7}$$

The function $r(x)$ is a *square wave*. One can show by manipulating some geometric series that r is periodic with period 2π and that

$$r(x) = \begin{cases} 1 & \text{if } x \in (0, \pi) \\ -1 & \text{if } x \in (-\pi, 0). \end{cases}$$

Using the Fourier series (7), I plot $r(x)$ below, truncating the sum, i.e. I sum only $0 < j < n$ for some finite n. Here is a plot of (7) truncated at $j < n = 4$ (blue), 8 (gold), and 16 (black).



Examining the figure above, one sees that, away from the jumps at the multiples of π, taking more terms in (7) causes $r(x)$ to better approximate its limit of the constant function 1 or -1 on those intervals. In contrast however, at the discontinuities of $r(x)$, the three approximations (7) truncated at $n = 4$, 8, and 16 do not appear to be converging to the limiting function of alternating flat 1's and -1's. This is known as *Gibbs' phenomenon.*

    (d) By viewing truncation of the Fourier series (7) as multiplication by the indicator function $\mathbb{1}_{\{|2k+1|<j\}}$, use the convolution theorem to explain why Gibbs' phenomenon occurs.

18. Solve the the initial value problem $\frac{y'}{r(x)} = 1 + y$ and $y(0.1) = 0.2$ (defined on the largest possible real domain; what is this domain?). How many free parameters does the solution have?