Differential Equations for Engineers

Course Notes for Math 218

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Contents

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Preface

Goals of the course

This course is intended to serve two purposes:

i) To prepare you for future courses.

ii) To provide an introduction to the discipline of Applied Mathematics, namely, the formulation and analysis of mathematical models of real-world phenomena. Sincemany models are based on differential equations, an introductory course in DEs provides a natural vehicle for this purpose.

What you need to know

Success in the course depends on having a good knowledge of *single variable calculus* — derivatives, antiderivatives, qualitative curve-sketching and improper integrals, in particular. In the final chapter, some knowledge of *linear algebra* is also required – matrices, eigenvalues and eigenvectors – but only for the two-dimensional case. You'll find that the exponential function plays a major role in the subject of differential equations, and so it is important that you have a good grasp of *exponentials* and *logarithms*. Some whatsurprisingly, *complex numbers* are used in the course, even though all the unknown functions are real-valued. The reason for the appearance of complex numbers is that the roots of real polynomials are in general complex. So in the course you'll find yourself using the famous *Euler formula*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Some concepts from physics arise in the applications, the most important being *Newton's Second Law of Motion*. However, in order to keep the course accessible, the background needed for the applications, most of which arise in everyday life, will be given in the course.

Learning the course material

The lecture notes give a discussion of the theoretical matters, and contain a selection of worked examples(which differ from those that will be given in the lectures). Some of the solutions to the examples are given in full detail, but when reading many parts of the notes you will need a pencil and paper in order to fill in some of the missing steps. Each section contains exercises with answers to help you get started in learning the course. *We recommend that you do all of these exercises.*

The lectures may be less theoretical than the notes and will contain fully worked examples.

The course notes also contain a Review Problem Set, in case your knowledge of Calculus needs refreshing. There are further problem sets, one for each chapter in the notes, from which assigned questions will be selected.

Chapter 1

First Order Differential Equations

An equation involving an unknown function and some of its derivatives is called a differential equation (DE). The order of the equation is the order of the highest derivative appearing in it. For example, an equation involving an unknown function and its first derivative is called a first order DE. We motivate the need to study such equations by first showing how they arise in various applications.

1.1 DEs and Mechanics

1.1.1 Newton's Second Law of Motion

Newton's Second Law of Motion leads to differential equations when applied to various problems in mechanics. We consider the simplest case of motion of a particle in a straight line. The relevant physical quantities are

m, the mass of the particle (which may vary with time),
$v(t)$, the velocity of the particle at time t,
F, the total force acting (time dependent, in general).

Newton's Second Law states that the rate of change of momentum $mv(t)$ equals the total force, i.e.

$$\frac{d}{dt}[mv(t)] = F. \tag{1.1}$$

If the mass is constant this equation can be written

$$m\frac{dv}{dt} = F, \tag{1.2}$$

i.e. *mass times acceleration equals force acting.* Before (1.1) or (1.2) can be used to describe the motion of a particle, the force F has to be specified. In the example to follow, F depends on t through the velocity v, i.e. $F = F(v)$, in which case (1.2) assumes the form

$$m\frac{dv}{dt} = F(v),$$

which is a first order differential equation (DE) for the unknown function $v(t)$.

Comment: Newton's second law has been tested in countless experiments, and accurately predicts the motion of particles, subject to one limitation, namely that the velocity of the particle is small compared to the velocity of light i.e.

$$\frac{v}{c} \ll 1.$$

The velocity of light is $c \approx 3 \times 10^8 m/s \approx 10^9$ km/hr. For sufficiently high velocities, as occur for example in high energy particle accelerators, Newton's second law has to be replaced by its relativistic counterpart, which is part of Einstein's theory of relativity.

Example: (Terminal velocity of a sky-diver)

Gravity exerts a downward force on the sky-diver, and air-resistance exerts an upward force, which increases as the velocity increases. Eventually we expect that air-resistance will balance the force due to gravity, so that the total force acting is zero. Then by (1.2) the sky-diver will fall with constant velocity, called the *terminal velocity.* In order to describe the motion in detail we need to specify the force due to air-resistance. This is a complicated matter; we make the simple assumption, supported by experiment for subsonic velocities, that this force is proportional to the velocity and acts so as to oppose the motion. Taking the downward direction as the positive direction, the total force acting on the sky-diver will be

$$F = mg - \alpha v, \tag{1.3}$$

where g is the constant acceleration due to gravity (at the earth's surface). The constant α, which depends on the physical characteristics of the sky-diver or falling object, is called the *drag coefficient.* □

Comment: Strictly speaking, the acceleration due to gravity is not constant, and depends on the distance from the centre of the earth. Since the height h above the earth's surface is small compared to the earth's radius R, i.e.

$$\frac{h}{R} \ll 1,$$

it is a reasonable approximation to treat g as a constant. □

With (1.3), Newton's law (1.2) assumes the form

$$m\frac{dv}{dt} = mg - \alpha v, \tag{1.4}$$

a first order DE for $v(t)$. This DE can be solved by separation of variables (see Section 1.2.2), but for now we draw a conclusion directly from the DE. As stated earlier we expect the sky-diver to eventually reach a constant terminal velocity. Since a constant velocity gives $\frac{dv}{dt} = 0$, equation (1.4) implies that

$$v_{\text{terminal}} = \frac{mg}{\alpha}. \tag{1.5}$$

Note that the terminal velocity depends on the three physical parameters m, g and α, being inversely proportional to the drag coefficient α (opening a parachute will *increase* α, thereby *reducing* v_{terminal}). There is one other parameter associated with the physical system, namely the sky-diver's initial velocity $v(0)$, i.e. the vertical velocity when leaving the plane at time $t = 0$. It is of interest that v_{terminal} does not depend on $v(0)$, although we expect that $v(t)$, the velocity at time t, will depend on $v(0)$. The relation between $v(t)$ and $v(0)$ will become clear when you solve the DE (1.4) [see Problem Set 1].

1.1.2 Mixing Problems

Various problems in biology and engineering can be put in the following framework. Consider a tank containing a chemical solution. The contents are kept well-mixed, so that the concentration is uniform. There is an *inflow* of the chemical solution of specified concentration, and and *outflow* of chemical solution, whose concentration at time t equals the concentration of solution in the tank at time t.

The goal is to predict the amount of chemical in the tank at time t, or perhaps to adjust the inflow and outflow as as to achieve a desired concentration in the tank.

Let $m(t)$ denote the amount (mass) of chemical in the tank at time t. The rate of change of $m(t)$ equals the difference between the rate of inflow and rate of outflow:

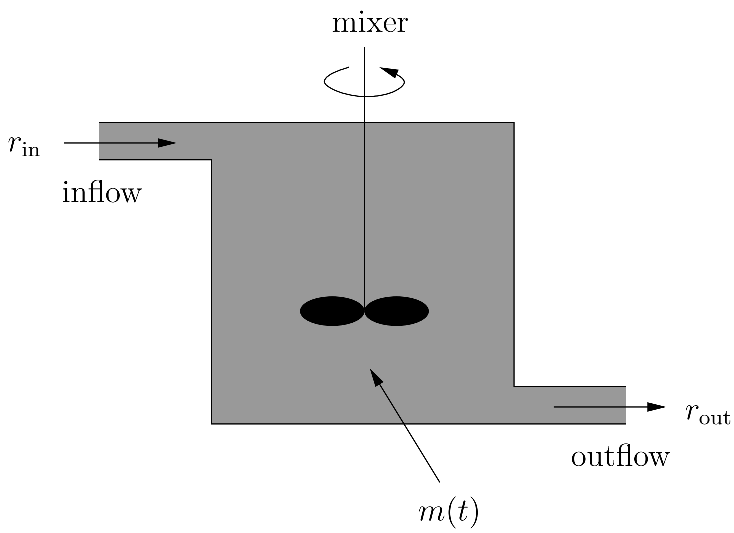


Figure 1.1: A mixing tank.

$$\frac{dm}{dt} = r_{\text{in}} - r_{\text{out}}, \tag{1.6}$$

where r_{in} is the rate at which chemical is added by the inflow and r_{out} is the rate at which chemical is removed by the outflow. Each term in equation (1.6) will have the same units, for example kg/min (i.e. mass of chemical per unit time). We shall refer to equation (1.6) as the *mass balance equation* for a mixing tank.

Example: A tank contains m_0 kg of salt dissolved in 100 litres of water. A salt solution containing $\frac{1}{4}$ kg per litre is added at 3 litre/min., and the well-stirred mixture leaves the tank at the same rate. Find the amount of salt in the tank at time t.

Solution: From the given data, the rate at which salt is added to the tank is

$$r_{\text{in}} = (3)\left(\tfrac{1}{4}\right) \text{kg/min..}$$

Let $m(t)$ denote the amount of salt in the tank at time t. Then the concentration at time t is

$$\frac{m(t)}{100} \quad \text{kg/litre}$$

Thus, the rate at which salt leaves the tank is

$$r_{\text{out}} = (3)\left(\frac{m(t)}{100}\right) \text{kg/min.}$$

The mass balance equation (1.6) gives

$$\frac{dm}{dt} = \tfrac{3}{4} - \tfrac{3}{100}m = -\tfrac{3}{100}(m - 25). \tag{1.7}$$

Also, since we are given that there are m_0 kg of salt in the tank at $t = 0$, we have

$$m(0) = m_0. \tag{1.8}$$

We will solve (1.7) together with condition (1.8) in section 1.3.1.

1.1.3 Newton's Law of Gravitation

Will a projectile fired vertically upwards on the earth's surface (or on the surface of the moon) eventually fall back to earth, or will it continue travelling away indefinitely? The answer is that it depends on the velocity with which the projectile is fired. If this initial velocity exceeds a certain threshold called the *escape velocity*, then the projectile will travel away indefinitely and "escape" from the earth's gravitational field. As with the sky-diver, this problem is governed by Newton's Second Law, and also involves gravity. The key difference is the distance scale. In the present case the distance from the earth's surface will not be small compared to the earth's radius, and so it is unreasonable to treat the acceleration due to gravity as a constant. We thus need to use *Newton's Law of Gravitation*. On the other hand, in giving a simple analysis, it is reasonable to neglect air-resistance, since the thickness of the earth's atmosphere is small compared to the earth's radius (of course air resistance is totally absent on the moon).

Newton's Law of Gravity states that the force of attraction between two point particles of mass m_1 and m_2 is proportional to the masses and inversely proportional to the square of the distance r between them. For motion in one dimension the force is given by

$$F = \frac{Gm_1m_2}{r^2}, \tag{1.9}$$

where G is a constant of proportionality called the *gravitational constant*.

Comment: We can idealize the projectile as a point particle, but not the earth. However, it can be shown that the gravitational force exerted by a finite homogeneous sphere on a particle is the same as if all the mass of the sphere was concentrated at the centre of the sphere. Thus, as regards Newton's Law of gravitation, the earth can be idealized as a point particle. □

For a vertically moving projectile the total force acting is

$$F = -\frac{Gm\mathcal{M}}{r^2}, \tag{1.10}$$

where m is the mass of the projectile and $\mathcal{M}$ is the earth's mass, and r is the distance of the projectile from the centre of the earth. We have chosen the positive direction to be vertically up.

Comment: The acceleration g due to gravity near the earth's surface can be related to the gravitational constant G. Set $r = R$, the earth's radius, in (1.10) to obtain

$$F = -\frac{Gm\mathcal{M}}{R^2} = -mg,$$

giving

$$g = \frac{G\mathcal{M}}{R^2}. \qquad \Box \tag{1.11}$$

The equation of motion for the projectile is now obtained by substituting (1.10) in Newton's Second Law (1.2), giving

$$m\frac{dv}{dt} = -\frac{Gm\mathcal{M}}{r^2},$$

where v is the velocity of the projectile. We simplify this equation by cancelling m, and using (1.11) to express G in terms of g, giving

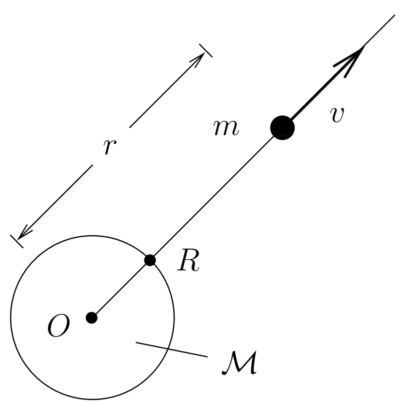


Figure 1.2:

$$\frac{dv}{dt} = -\frac{gR^2}{r^2}. \tag{1.12}$$

Comment: In cancelling m, we are tacitly using a fundamental experimental fact about gravity. The mass m that appears in Newton's Second Law (1.2) is the *inertial mass* m_I of the particle, while the mass m that appears in Newton's Law of Gravity is the *gravitational mass* m_G. It has been determined experimentally to a high degree of accuracy that

$$m_I = m_G,$$

although no explanation is known. $\square$

Equation (1.12) contains two unknown functions $v(t)$ and $r(t)$. We can write it as a DE in one unknown in two ways.

First, since $v = \frac{dr}{dt}$, we can write

$$\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2},$$

which is a *second order* DE for $r(t)$. The initial conditions at the launch at time $t = 0$ are

$$r(0) = R, \qquad \frac{dr}{dt}(0) = v_{\text{init}},$$

where v_{init} is the initial velocity.

Second, if we consider velocity v as a function of distance r, we can write (1.12) as a first order DE, which is more convenient for determining the escape velocity. By the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \frac{dv}{dr}v,$$

so that (1.12) becomes

$$v\frac{dv}{dr} = -\frac{gR^2}{r^2}, \tag{1.13}$$

a first order DE for $v = v(r)$, with initial condition

$$v(R) = v_{\text{init}}. \tag{1.14}$$

The problem now is: for which values of v_{init} will the velocity $v(r)$ satisfy $v(r) > 0$ for all $r \geq R$. The escape velocity is the smallest value of v_{init} with this property. In order to solve this problem we need to solve the DE (1.13) [see Problem Set 1].

1.2 Mathematical aspects of first order DEs

1.2.1 Solution of a first order DE

The general form of a first order DE is

$$\frac{dy}{dx} = f(x, y), \tag{1.15}$$

where f is a function of two variables.[1] The variable y represents the *unknown function*, $y = y(x)$, and x is the *independent variable*. A *solution of the DE* (1.15) is a differentiable function ϕ such that $y = \phi(x)$ satisfies (1.15) for all x in some interval.

[1] It is usually assumed that f has continuous partial derivatives (i.e. is of class C^1); technical details such as these are not important in Math 218.

Exercise: Verify that

$$y = Ce^{-x^2} + x^2 - 1, \quad x \in \mathbb{R},$$

where C is a constant, is a solution of the DE

$$\frac{dy}{dx} = -2xy + 2x^3. \qquad \square$$

1.2.2 Finding solutions

A common error

Consider a first order DE

$$\frac{dy}{dx} = f(x, y). \tag{1.16}$$

Knowing that

$$\int \frac{dy}{dx} dx = y + C,$$

a naive person might be tempted to try to solve (1.16) by taking the antiderivative of both sides with respect to x, obtaining

$$y + C = \int f(x, y) dx.$$

This attempt to solve (1.16) fails in general because the antiderivative on the right hand side contains y, which depends on x and hence *cannot be treated as a constant* (y is of course the unknown function which we are trying to find).

We note that this simple-minded approach only works in the very special case where the DE (1.16) has the form

$$\frac{dy}{dx} = f(x), \tag{1.17}$$

i.e. *the right hand side is independent of* y. We shall say that the DE (1.17) is *directly solvable*, because it can be solved simply by taking the antiderivative.

In general, it is not possible to actually find solutions of the DE (1.15), even though we know they exist. Fortunately, the first order DEs that arise in many applications are of two special types that can be solved, namely *separable and linear.*

Separable first order DEs

The general form of a first order *separable DE* is

$$\frac{dy}{dx} = A(x)B(y), \tag{1.18}$$

where $A(x)$ and $B(y)$ are arbitrary functions.

Example 1: The sky-diver DE in Section 1.1.1,

$$m\frac{dv}{dt} = mg - \alpha v, \tag{1.19}$$

is of the form

$$\frac{dv}{dt} = A(t)B(v),$$

with

$$A(t) = 1, \quad B(v) = g - \frac{\alpha}{m}v,$$

and hence is *separable*. □

Example 2: The escape velocity DE in Section 1.1.3,

$$v\frac{dv}{dr} = -\frac{gR^2}{r^2}, \tag{1.20}$$

is of the form

$$\frac{dv}{dr} = A(r)B(v),$$

with

$$A(r) = \frac{R^2}{r^2}, \quad B(v) = -\frac{g}{v},$$

and hence is *separable*. □

Linear first order DEs

The general form of a first order *linear* DE is

$$\frac{dy}{dx} + p(x)y = f(x), \tag{1.21}$$

where $p(x)$ and $f(x)$ are arbitrary functions.

Example: The DE

$$\frac{dy}{dx} + xy = e^{-x}$$

is *linear*, but the DE

$$\frac{dy}{dx} + xy^2 = e^{-x}$$

is *non-linear*. Note that the sky-diver DE (1.19) and the mixing tank DE (1.7) are linear, but the escape-velocity DE (1.20) is non-linear. □

There are two special types of linear DEs:

- *homogeneous*,

$$\frac{dy}{dx} + p(x)y = 0, \text{ i.e. } \frac{dy}{dx} = -p(x)y \tag{1.22}$$

  This DE is also separable.

- *constant coefficient*,

$$\frac{dy}{dx} + ky = f(x), \qquad k = \text{constant}. \tag{1.23}$$

A linear DE that is homogeneous AND has a constant coefficient has the form

$$\frac{dy}{dx} + ky = 0, \text{ i.e. } \frac{dy}{dx} = -ky, \qquad k = \text{constant}. \tag{1.24}$$

This is *the world's simplest and most important first order DE*. It is so simple that it can be *solved by inspection*: the derivative of the unknown function y is $-k$ times y, and the only functions with this property are

$$y = Ce^{-kx}, \qquad C = \text{constant}. \tag{1.25}$$

This is important enough to write out formally.

Proposition: Any solution of the DE

$$\frac{dy}{dx} = -ky, \qquad k = \text{constant}$$

is given by

$$y = Ce^{-kx},$$

where C is a constant.

Aside: If $k > 0$, this DE describes *exponential decay* and if $k < 0$, it describes *exponential growth*.

Proof: Multiply (1.24) by e^{kx} and rewrite as

$$\frac{d}{dx}\left(e^{kx}y\right) = 0.$$

Hence,

$$e^{kx}y = C,$$

giving (1.25). $\square$

Exercise: Write the general form of a linear DE that is also separable.

Answer: $\dfrac{dy}{dx} = p(x)(ay + b)$, where a and b are constants.

1.2.3 Solving separable DEs

A separable DE

$$\frac{dy}{dx} = A(x)B(y) \tag{1.26}$$

can be solved by *separation of variables*, as follows. Divide (1.26) by $B(y)$ and take the antiderivative with respect to x:

$$\int \frac{1}{B(y)}\frac{dy}{dx}dx = \int A(x)dx.$$

By the change of variable algorithm (substitution method), this can be written

$$\int \frac{1}{B(y)}dy = \int A(x)dx.$$

Provided that both antiderivatives can be evaluated in terms of elementary functions, one obtains a *one-parameter family of solutions* i.e. depending on one constant of integration. □

Example: Solve

$$\frac{dy}{dx} = -2xy.$$

First, you may notice that $y = 0$ is a solution. To find the other solutions, we assume that $y \neq 0$ and use separation of variables by rewriting the DE as

$$\frac{1}{y}\frac{dy}{dx} = -2x$$

and integrating both sides:

$$\int \frac{1}{y}\frac{dy}{dx}dx = \int -2xdx,$$

or

$$\int \frac{1}{y}dy = \int -2xdx,$$

giving

$$\ln|y| + C_1 = -x^2 + C_2$$

where C_1 and C_2 are constants. Next we 'absorb' these constants together (this will be done frequently in this course) by letting $C = C_2 - C_1$, giving

$$\ln|y| = -x^2 + C$$

Exponentiating both sides gives

$$|y| = e^{-x^2+C} = e^C e^{-x^2} \tag{1.27}$$

Noting that e^C is just a constant, we write the solution as

$$y(x) = ke^{-x^2},$$

where k is an arbitrary constant.[2]

Equilibrium solutions:

A separable DE may have certain exceptional solutions that can be found by inspection (in the previous example, $y = 0$ is one such solution). If the function $B(y)$ in (1.26) is zero at $y = b$, i.e.

$$B(b) = 0,$$

[2]A note for the detail-oriented student: equation (1.27) leads to $y = \pm e^C e^{-x^2}$ (the sign depending on whether y is positive or negative). Setting $k = \pm e^C$ means k could be any *nonzero* constant; however, if $k = 0$, then this just gives $y = 0$, which we know to be a solution. So k can be *any* real number.

then *the constant function* $y = b$ *is a solution of the DE* (1.26). Because the unknown function is a constant function, this solution is called an *equilibrium solution* – one thinks of the physical system as being in a state of equilibrium. For example the separable DE

$$\frac{dy}{dx} = y(1-y)$$

has equilibrium solutions $y = 0$ and $y = 1$. When solving a separable DE, *always begin by finding the equilibrium solutions* (if any), because they are excluded by the general procedure since one divides by $B(y)$.

Example: The separable DE

$$\frac{dy}{dx} = \frac{xy}{1+y^2}$$

has an equilibrium solution $y = 0$. To find the general solution, rewrite the DE as

$$\frac{1+y^2}{y}\frac{dy}{dx} = x.$$

This leads to

$$\int \left(\frac{1}{y} + y\right) dy = \int x dx,$$

giving

$$\ln \mid y \mid + \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C. \qquad \square$$

Comment: When solving a separable DE it may not be possible to isolate y in the general solution, as happens in the above example.

Exercise: Solve the DE

$$\frac{dy}{dx} = 2xe^{-y}.$$

Answer: $y = \ln(x^2 + C)$, where C is a constant.

1.2.4 Solving linear DEs

A linear DE

$$\frac{dy}{dx} + p(x)y = f(x) \tag{1.28}$$

can be solved by finding an auxiliary function called an *integrating factor*. The method is illustrated in the following example.

Example: Solve the DE

$$\frac{dy}{dx} = y - e^{-x}. \tag{1.29}$$

Solution: Transfer the y-term to the left side and multiply throughout by a function $I(x)$,

$$I\frac{dy}{dx} - Iy = -Ie^{-x}. \tag{1.30}$$

Choose I to satisfy

$$\frac{dI}{dx} = -I. \tag{1.31}$$

The reason for doing this is that (1.30) becomes

$$I\frac{dy}{dx} + \frac{dI}{dx}y = -Ie^{-x},$$

which, using the Product Rule for derivatives, can be written

$$\frac{d}{dx}(Iy) = -Ie^{-x}. \tag{1.32}$$

The DE (1.31) for I (the "world's simplest") can be solved by inspection:

$$I = Ce^{-x}.$$

Since we only want a particular solution, we choose $I = e^{-x}$ (i.e. $C = 1$) for simplicity. The DE (1.32) assumes the form

$$\frac{d}{dx}\left(e^{-x}y\right) = -e^{-2x}. \tag{1.33}$$

Take the antiderivative of both sides with respect to x:

$$\int \frac{d}{dx}\left(e^{-x}y\right) dx = \int -e^{-2x} dx,$$

giving

$$e^{-x}y = \tfrac{1}{2}e^{-2x} + C.$$

Solving for y:

$$y = \tfrac{1}{2}e^{-x} + Ce^{x}, \tag{1.34}$$

where C is an arbitrary constant. *Equation* (1.34) *gives the family of all solutions of the DE* (1.29). □

Comment: The function $I(x)$ in the previous solution is called an *integrating factor* for the DE. It is always determined by solving a separable DE, which in the above example could be solved by inspection (see equation (1.31)). In the general case, the integrating factor for the DE (1.28) is given by $I(x) = e^{\int p(x)dx}$. The purpose of finding the integrating factor is to write the given DE in the form (1.32), since in this form it can be solved directly by taking the antiderivative of both sides. Note that any DE of the form

$$\frac{d(I(x)y)}{dx} = g(x)$$

can be solved for y directly, by taking the antiderivative of both sides with respect to x. □

Exercise: Solve the linear DE

$$x\frac{dy}{dx} - y = x^3, \quad x > 0.$$

Comment: It is essential to divide through by x so as to isolate $\frac{dy}{dx}$, before attempting to find the integrating factor.

Answer: The integrating factor is $I(x) = \frac{1}{x}$, and the general solution is $y = \frac{1}{2}x^3 + Cx$. □

1.2.5 Qualitative sketches of families of solutions

Two basic problems in the theory of DEs are

A: Solve the DE, i.e. find all solutions of the DE (only possible for certain classes of DE).

B: Describe the behaviour of typical solutions of a DE, e.g. how does the solution $y(x)$ behave as $x \to +\infty$?

In 1.2.2 and 1.2.3 we discussed the two most important special classes of first order DEs that can be solved. In order to understand the behaviour of solutions, however, it is necessary to give *a qualitative sketch of the family of solutions*, which will depend on one parameter (the constant of integration). In this section, we discuss how to draw such a sketch.

When sketching the solutions one can obtain useful information directly from the DE, without solving it, as follows. We think of the DE

$$\frac{dy}{dx} = f(x, y)$$

as specifying a *slope* at each point of the xy-plane, namely the slope of the tangent line to the solution $y = y(x)$ through that point. In other words, if $y = y(x)$ is the solution curve through the point (x_0, y_0), then the slope of the curve at (x_0, y_0) is

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f(x_0, y_0).$$

One can then draw the **direction field** of the DE — that is, a field of slopes in the xy-plane — and then draw curves which 'follow' the direction field.

Example 1: Consider DE

$$\frac{dy}{dx} = -2y. \tag{1.35}$$

The slope at (x_0, y_0) is $-2y_0$, which is positive in the lower half of the plane ($y_0 < 0$), negative in the upper half of the plane ($y_0 > 0$), and zero on the x-axis ($y_0 = 0$). Also, the farther from the x-axis, the steeper the slope. The direction field is sketched in figure 1.3. The special property of this DE is that the slope at (x, y) depends on y but not on x.

On the other hand, one can see by inspection of the DE that the family of solutions is

$$y = Ce^{-2x}.$$

Note that $y = 0$ is an equilibrium solution, with slope 0 at each point. The solution curves are shown in Figure 1.4. □

Here is an example where the pattern of the solution curves is more complicated.

Example 2: Give a qualitative sketch of the solution curves of the DE

$$\frac{dy}{dx} = y - e^{-x}.$$

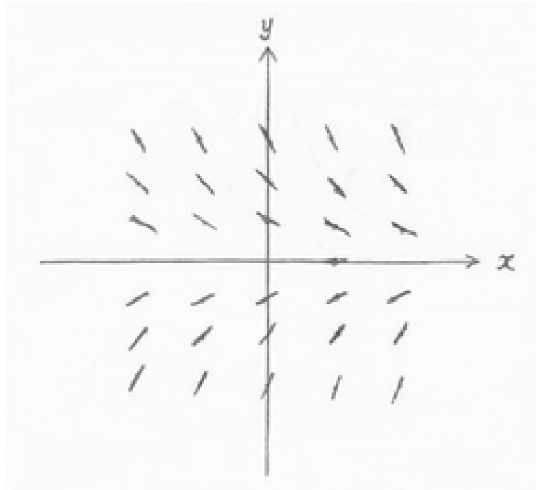


Figure 1.3: Direction field from example 1.

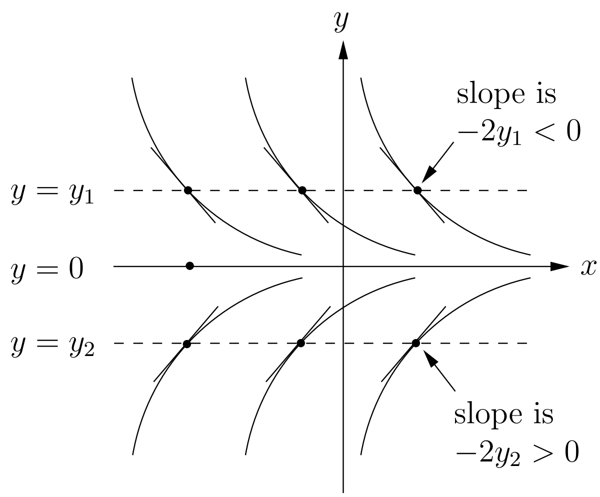


Figure 1.4: Family of solutions from example 1.

From Section 1.2.3, equation (1.34), the solutions are

$$y = \tfrac{1}{2}e^{-x} + Ce^{x}.$$

Solution: We first make use of the DE itself. Sketching a direction field is tedious in this case, so we use a more clever approach. The DE implies that the *slope is zero* at all points of the curve

$$y = e^{-x}$$

(drawn as the dashed curve; see Figure 1.5). The solution with $C = 0$ plays a special role (draw it in colour). As $x \to -\infty$, $y \approx \frac{1}{2}e^{-x}$, i.e. all solutions approach the $C = 0$ solution. As $x \to +\infty$, $y \approx Ce^{x}$, and so the shape of the solution curve depends on whether $C > 0$ or $C < 0$. □

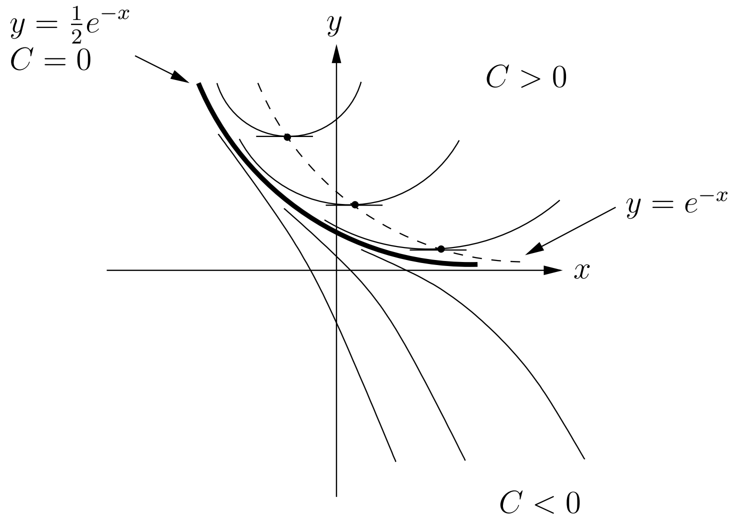


Figure 1.5: Family of solutions from example 2.

Exceptional solutions:

The key to sketching the family of solution curves is to identify any *exceptional solutions*. In example 1, $y = 0$ is an exceptional solution, and in example 2, $y = \frac{1}{2}e^{-x}$ is exceptional. In each case the exceptional solution divides the whole family of solutions into two subclasses such that the members of a subclass have the same qualitative properties (i.e. the same overall shape). Another feature of exceptional solutions is that they often "attract" other solution curves, either as $x \to +\infty$ or as $x \to -\infty$.

Note that for a separable DE $\frac{dy}{dx} = A(x)B(y)$, any *equilibrium solution*, i.e. $y(x) = b$, with $B(b) = 0$, is an exceptional solution (as in example 1). □

A fundamental property of families of solutions:

Consider the DE

$$\frac{dy}{dx} = f(x, y).$$

A fundamental theorem in the theory of DEs (the Existence-Uniqueness theorem, discussed AMath 351 and proved in Birkhoff and Rota) states that if the function $f(x, y)$ has continuous partial derivatives (i.e. f is of class C^1, in brief), then through a given point (x_0, y_0) there passes one and only one solution curve of the DE. This means that

if f is of class C^1, solution curves of the DE $\frac{dy}{dx} = f(x, y)$ cannot intersect.

This fact is very helpful when sketching families of solution curves.

Here are two examples where f is not of class C^1 and intersections do occur.

Example 3: Consider the DE

$$\frac{dy}{dx} = \frac{y}{x}$$

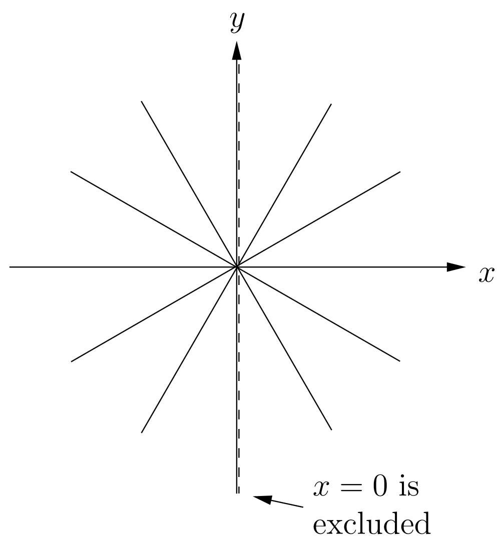


Figure 1.6: Solution curves from example 3.

The solution curves are $y = Cx$ (see Figure 1.5). Here $f(x, y) = \frac{y}{x}$, which is not C^1 when $x = 0$ ($f(0, y)$ is not even defined). $\square$

Example 4: Consider the DE

$$\frac{dy}{dx} = 3y^{2/3}$$

The solution curves are

$$y = (x + C)^3,$$

together with the equilibrium solution

$$y = 0.$$

Here $f(x, y) = 3y^{2/3}$, which is not C^1 when $y = 0$ since

$$\frac{\partial f}{\partial y} = \frac{2}{y^{1/3}}.$$

Intersections occur on the x-axis. See Figure 1.7. $\square$

1.2.6 First order linear DEs with constant coefficient

The general form of a linear DE with a constant coefficient is

$$\frac{dy}{dx} + ky = f(x), \tag{1.36}$$

where *k is a constant.* The general solution of such a DE can be found by obtaining an integrating factor as in Section 1.2.3. For certain functions $f(x)$ that arise commonly in applications, however, there is a quicker method for finding a particular solution (i.e. a

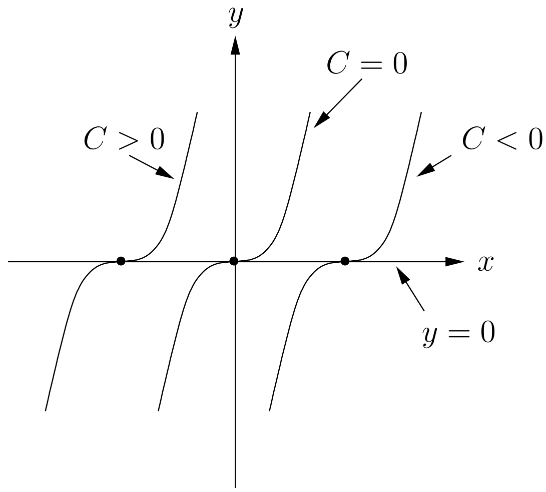


Figure 1.7: Solution curves from example 4.

single solution) of the DE (1.36), called the *method of undetermined coefficients*.[3] We first illustrate this method, and then show how to use it to find the *general* solution of (1.36) efficiently.

The method of undetermined coefficients:

Example 1: Find a particular solution of

$$\frac{dy}{dx} + 2y = x. \tag{1.37}$$

Solution: Consider a trial function of the form

$$y = Ax + B, \tag{1.38}$$

where A and B are constants (the "undetermined coefficients"). Substitute (1.38) in (1.37):

$$A + 2(Ax + B) = x.$$

Equate coefficients of x^1 and $x^0 = 1$, giving

$$2A = 1$$
$$A + 2B = 0.$$

Solve for A and B: $A = \frac{1}{2}$, $B = -\frac{1}{4}$.

By (1.38), a particular solution of the DE (1.37) is

$$y_p = \tfrac{1}{2}x - \tfrac{1}{4}. \qquad \square$$

Comment:

[3]This method can also be applied to second order linear DEs with constant coefficients. See Section 2.2.4.

1) The trial function contains a number of constants, and the DE leads to a system of *linear algebraic* equations to be solved for these constants.

2) Choosing a trial function requires some experience. If one doesn't include enough terms and constants, the system of linear equations will be *incompatible*, and one has to try again. The method can only be applied if $f(x)$ is composed of powers of x, $\sin \omega x$, $\cos \omega x$ and e^{rx}. The table below shows some simple cases for the DE (1.36).

$f(x)$	trial function y
x	$Ax + B$
x^2	$Ax^2 + Bx + C$
$\sin \omega x$ or $\cos \omega x$	$A \sin \omega x + B \cos \omega x$
e^{rx}	Ae^{rx} or* Axe^{rx}

*In the special case $r = -k$ (the same k in the DE (1.36)), one needs to 'multiply by x'—the reason will be explained in section 2.2.4.

Exercise: Find a particular solution of

$$\frac{dy}{dx} + 2y = \cos x.$$

Answer: $y_p = \frac{1}{5}(2 \cos x + \sin x)$. □

Suppose we have found a particular solution y_p of the DE (1.34). We can then find the general solution of this DE *without any extra work*. The following Proposition shows how. The proposition is more general than we need, since it does not depend on the coefficient k in the DE being constant.

Proposition: If y_p is a particular solution of the DE

$$\frac{dy}{dx} + k(x)y = f(x),$$

and y is *any* solution of this DE, then the difference

$$y_h = y - y_p \tag{1.39}$$

is a solution of the *homogeneous* DE

$$\frac{dy}{dx} + k(x)y = 0.$$

Proof: We are given that

$$\frac{dy}{dx} + k(x)y = f(x),$$

and

$$\frac{dy_p}{dx} + k(x)y_p = f(x).$$

Subtract these two equations to get

$$\frac{d}{dx}(y - y_p) + k(x)(y - y_p) = 0,$$

which is the desired result. □

It follows from (1.39) that the general solution y is the sum of two terms, $y = y_p + y_h$. This is the key result, which we now write out in full.

General solution of a linear DE:

The general solution of the linear DE

$$\frac{dy}{dx} + k(x)y = f(x) \tag{1.40}$$

has the form

$$y(x) = y_p(x) + y_h(x), \tag{1.41}$$

where $y_p(x)$ is *a particular* solution of the *inhomogeneous* DE (1.40), and $y_h(x)$ is the *general* solution of the *homogeneous* DE

$$\frac{dy}{dx} + k(x)y = 0. \qquad \square \tag{1.42}$$

Comment: The proposition depends in an essential way on the fact that the DE is *linear*.

In the *constant coefficient case*, the homogeneous DE (1.42) is

$$\frac{dy}{dx} + ky = 0,$$

(the "world's simplest") whose general solution we know to be

$$y = Ce^{-kx}.$$

Thus, once we have obtained a particular solution $y_p(x)$ of (1.36) using the method of undetermined coefficients, we can immediately write down the general solution using (1.41). □

Return to example 1:

Find the general solution of

$$\frac{dy}{dx} + 2y = x. \tag{1.43}$$

Solution: We have found a particular solution

$$y_p(x) = \tfrac{1}{2}x - \tfrac{1}{4}.$$

The general solution of the homogeneous DE (the "world's simplest")

$$\frac{dy}{dx} + 2y = 0$$

is

$$y_h(x) = Ce^{-2x}.$$

Thus by (1.41), the general solution of the DE (1.43) is

$$y(x) = \left(\tfrac{1}{2}x - \tfrac{1}{4}\right) + Ce^{-2x}. \qquad \square$$

Exercise: Find the general solution of

$$\frac{dy}{dx} + 2y = \cos x.$$

Answer: $y = \frac{1}{5}(2\cos x + \sin x) + Ce^{-2x}.$ $\square$

Exercise: Find the general solution of

$$\frac{dy}{dx} - 3y = e^{2x}.$$

Answer: $y = -e^{2x} + Ce^{3x}.$ $\square$

1.2.7 An important special case

Consider the DE

$$\frac{dy}{dx} = ky + b, \tag{1.44}$$

where k and b are constants, with $k \neq 0$. This DE is *linear* (since the only y-dependence on the right side is y itself), has a *constant coefficient* (since k, the coefficient of y, is a constant), and is also *separable* (since b is a constant).

This DE can thus be solved using each of the techniques that we have introduced so far, and we suggest that you do this as an exercise:

1) Solve (1.44) as a linear DE by finding an integrating factor.

2) Solve as a separable DE.

3) Find a particular solution using the method of undertermined coefficients (use $y = A$, a constant as the trial function), and then write down the general solution.

The recommended method for solving (1.44), and by far the quickest method, is to convert the DE into the "world's simplest" form and solve by inspection as follows. Since b and k are constants and $k \neq 0$ we can rewrite (1.44) in the form

$$\frac{d}{dx}\left(y + \frac{b}{k}\right) = k\left(y + \frac{b}{k}\right). \tag{1.45}$$

If we define

$$u = y + \frac{b}{k} \tag{1.46}$$

this DE becomes

$$\frac{du}{dx} = ku,$$

whose solution is

$$u = Ce^{kx}, \tag{1.47}$$

where C is a constant. Using (1.46), we get

$$y + \frac{b}{k} = Ce^{kx}, \tag{1.48}$$

Thus

$$y = -\frac{b}{k} + Ce^{kx}$$

is the general solution of the DE (1.44).

In practice, there is no need to formally introduce u. Having rewritten the DE in the form (1.45) one can immediately write down the solution (1.48).

1.2.8 Initial value problems

The sky-diver DE,

$$m\frac{dv}{dt} = mg - \alpha v,$$

determines the velocity of a sky-diver as a function of time t. We have seen that a first order DE has a one-parameter family of solutions (the constant of integration is the parameter). When applying the sky-diver DE, the physical process being described will begin at the time when the sky-diver jumps from the plane. We label this time as $t = 0$, and so we have the *initial condition*

$$v(0) = 0.$$

This initial condition will determine the constant of integration, leading to a unique solution that gives the velocity of the sky-diver at time t.

In general, for a DE

$$\frac{dy}{dx} = f(x, y), \tag{1.49}$$

the initial condition will be of the form

$$y(x_0) = y_0. \tag{1.50}$$

where x_0 and y_0 are given constants. The equations (1.49) and (1.50) are said to define an *initial value problem.*

Example: Find the unique solution of the initial value problem

$$\frac{dy}{dx} = -2y, \quad y(0) = 3.$$

Solution: By inspection the general solution of the DE is

$$y = Ce^{-2x}$$

Substituting $x = 0$ and $y = 3$ gives

$$3 = Ce^0,$$

i.e. $C = 3$. Thus the unique solution is

$$y = 3e^{-2x}. \qquad \square$$

Comment: The general solution of a first order DE (1.49) corresponds to a one-parameter family of solution curves. The initial condition (1.50) picks out a unique curve, namely the curve that passes through the point (x_0, y_0).

Exercise: Find the unique solution of the initial value problem

$$\frac{dy}{dx} = -2xy^2, \quad y(1) = \tfrac{1}{3}.$$

Answer: $y = \dfrac{1}{x^2 + 2}$. $\square$

Exercise: Find the unique solution of the initial value problem

$$x^2 \frac{dy}{dx} = -2xy + 1, \quad y(1) = 0.$$

Answer: $y = \frac{1}{x} - \frac{1}{x^2}$. $\square$

1.3 Other applications of first order DEs

1.3.1 Mixing problems (continued)

Here we continue the example from section 1.1.2. Recall that the initial condition is

$$m(0) = m_0.$$

The DE (1.7), which was

$$\frac{dm}{dt} = -\tfrac{3}{100}(m - 25)$$

can be written

$$\frac{d}{dt}(m - 25) = -\tfrac{3}{100}(m - 25),$$

and hence can be solved by inspection:

$$m - 25 = Ce^{-\frac{3}{100}t}.$$

On setting $t = 0$, the initial condition above gives $C = m_0 - 25$, leading to the solution

$$m(t) = 25 + (m_0 - 25)e^{-\frac{3}{100}t}. \tag{1.51}$$

Interpretation: Based on the physical set-up, one expects that as time passes the concentration of the solution in the tank will approach the concentration of the inflow i.e. $\frac{1}{4}$ kg/litre. Thus the amount of salt in the tank will approach $\left(\frac{1}{4}\right)(100) = 25$ kg as $t \to \infty$, in agreement with equation (1.51). $\square$

Comment: One can imagine problems such as the above arising in different contexts, e.g.

(1) nutrients flowing into and out of a cell (which plays the role of the tank),

(2) carbon-monoxide seeping into a room and then being dispersed.

Overview:

In a mixing tank problem, the unknown function is the mass of chemical in the tank at time t, denoted by $m(t)$. There are two flow rates, the inflow rate f_{in} and the outflow rate f_{out}. The flow rates f_{in} and f_{out} are given, and in general could be functions of time t, but in simple problems they will be constants, and may even be equal. If they are equal, then the volume V of solution in the tank will be constant in time.

There are two concentrations, the concentration of the inflow c_{in} and the concentration of the outflow c_{out}. The inflow concentration is given, and will be constant in the simplest situation. The outflow concentration is the concentration of the solution in the tank at time t, and is hence given by the key relation

$$c_{\text{out}} = \frac{m(t)}{V(t)},$$

where $V(t)$ is the volume at time t. Finally, the rates of mass inflow r_{in} and mass outflow r_{out} that appear in the mass balance equation (1.6) are given by

$$r_{\text{in}} = c_{\text{in}} f_{\text{in}}, \quad r_{\text{out}} = c_{\text{out}} f_{\text{out}}.$$

References: Boyce & Diprima, pg. 51, #18-22.
Braun, pg. 56, #6-11.

1.3.2 Population growth

Let $x(t)$ be the population of some species (e.g. fish, bacteria, etc.) at time t. The simplest hypothesis is the Malthusian Law i.e. that the rate of change of x at time t is proportional to the population at time t:

$$\frac{dx}{dt} = rx, \tag{1.52}$$

where r is a constant called the *rate of growth* (*rate of decline*, if $r < 0$). By inspection the general solution of the DE (1.52) is

$$x(t) = Ce^{rt},$$

and the initial condition $x(0) = x_0$ leads to

$$x(t) = x_0 e^{rt},$$

describing *exponential growth* ($r > 0$) or *decay* ($r < 0$).

It is clear that exponential growth can only continue for a restricted time due to resource limitations. A more realistic model takes into account that a given environment can support at most a finite number of a particular species, denoted by K, and called the *carrying capacity*. A realistic model thus requires that the rate of change $\frac{dx}{dt}$ approach zero as x gets close to K. A simple way to accomplish this is to assume that the rate of growth is not simply a constant r as in (1.52), but depends on x according to

$$r\left(1 - \frac{x}{K}\right),$$

where K is the constant carrying capacity. The DE governing the population $x(t)$ then assumes the form

$$\frac{dx}{dt} = r\left(1 - \frac{x}{K}\right)x, \tag{1.53}$$

called the *logistic equation*, a separable, non-linear DE.

Exercise: Solve the DE (1.53) using two different methods:

(i) by separation of variables (you will find that letting $u = \frac{x}{K}$, a simple rescaling of x, simplifies the algebra);

(ii) by making the change of variable $y = \frac{K}{x}$. This has the effect of changing (1.53) into the simple form $\frac{d}{dt}(y-1) = -r(y-1)$, which can be solved by inspection.

Answer: The solution can be written in the form

$$x(t) = \frac{x_0 e^{rt}}{1 - \frac{x_0}{K} + \frac{x_0}{K}e^{rt}},$$

after imposing the initial condition $x(0) = x_0$.

1.3.3 Efflux

Consider the cylindrical tank with cross sectional area A_c as shown in figure 1.8. There is a hole at the bottom with area A_h letting liquid out. According to Torricelli's law, under ideal conditions, the velocity v of the liquid on the way out of the hole is $v = \sqrt{2gh}$ where g is the acceleration due to gravity and h is the current height of the liquid in the tank.

The volume of liquid in the tank is given by $V = A_c h$, where h is a function of time. The rate of change of this volume, $\dfrac{dV}{dt} = A_c\dfrac{dh}{dt}$ is equal (in size) to the amount leaving through the hole, that is $A_h v = A_h\sqrt{2gh}$.

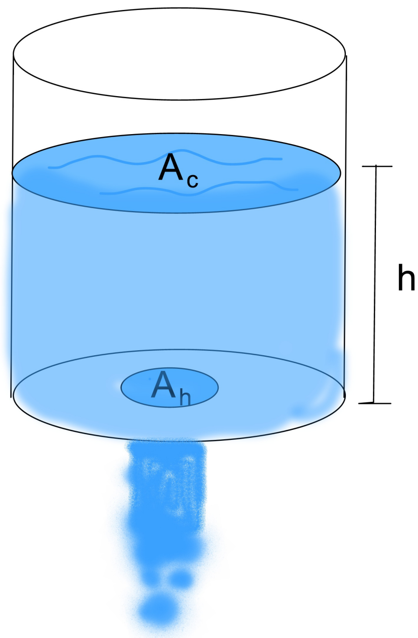


Figure 1.8: A tank with liquid draining out of the bottom

Since the volume in the tank is decreasing we have [4]

$$\begin{aligned}\frac{dV}{dt} &= -A_h v \\ A_c\frac{dh}{dt} &= -A_h\sqrt{2gh} \\ \frac{dh}{dt} &= -\frac{A_h}{A_c}\sqrt{2gh}\end{aligned}$$

which is a separable DE.

Generally the area of the hole doesn't change but the cross sectional area could change if the tank is not uniform. That is we could have A_c being a function of height. This would turn the DE into

$$\frac{dh}{dt} = -\frac{A_h}{A_c(h)}\sqrt{2gh}$$

which is still separable.

Finally, Torricelli's law gives an ideal speed of efflux. If we include viscosity of the liquid (e.g. syrup would flow out much slower than water), friction, contraction and other such losses then the amount leaving the tank is modified by including a discharge coefficient $C \in [0, 1]$ such that the outflow volume is given by $CA_h\sqrt{2gh}$ and the DE, in the most general case, becomes

$$\frac{dh}{dt} = -C\frac{A_h}{A_c(h)}\sqrt{2gh}$$

Reference: Zill, page 23 & 99.

[4](Note: A_h has units of length^2 and v has units $\frac{\text{length}}{\text{time}}$ thus $A_h v$ has units $\frac{\text{length}^3}{\text{time}}$, the same as $\frac{dV}{dt}$).

1.3.4 Chemical Reactions

Using the law of mass action we can create a differential equation that governs the creation of a new substance from the reaction of various initial substances. For example, suppose two chemicals X & Y react to form a new chemical Z. Further suppose that each gram of Z is made up of r parts of X and q parts of Y. We can write this relationship as

$$\frac{dZ}{dt} = k\left(X_0 - \frac{rZ}{r+q}\right)\left(Y_0 - \frac{qZ}{r+q}\right)$$

where, X_0, Y_0 and Z are measured in grams and X_0 and Y_0 are the initial masses of each of chemical X and Y respectively. The first bracket represents how many grams of X remain after Z grams are created whereas the second bracket represents how many grams of Y remain after Z grams are created. The parameter k is determined by experiment/observation.

This DE is separable and can be solved using partial fractions once the initial amounts X_0 and Y_0 are known and a measurement is taken on the amount of Z at some future point in time (to determine k).

It should be noted that not all reactions satisfy this equation. It is simply a model that can be used when certain assumptions are satisfied (see Deuflhard section 1.3).

Reference: Zill, page 24.

1.3.5 Epidemics

Consider a population of N individuals containing a number of individuals having an infectious disease. The goal is to determine how rapidly the disease will spread if no control measures are taken.

Let $x(t)$ denote the number of infectious individuals at time t, and let $N - x(t)$ denote the number of susceptible individuals. Assume that the rate of spread of the disease, $\frac{dx}{dt}$, is proportional to the number of contacts between infectious and susceptible individuals. Assume that both groups mingle freely so that the number of contacts is proportional to $x(N - x)$. Then $x(t)$ satisfies

$$\frac{dx}{dt} = \alpha x(N - x), \tag{1.54}$$

where α is a constant. Equation (1.54) is a first order DE for $x(t)$, with initial condition $x(0) = x_0$, which can be solved by separation of variables.

Reference: Boyce & DiPrima, page 65.

1.3.6 Cooling problems

Consider an object whose temperature at time t is $T(t)$, in surroundings whose temperature is T_A (called the *ambient temperature*). If $T(t)$ is larger than T_A the object will lose heat to its surroundings and its temperature will decrease. Newton's Law of Cooling states that the rate of decrease is proportional to the difference between $T(t)$ and the ambient temperature T_A:

$$\frac{dT}{dt} = -k(T - T_A), \tag{1.55}$$

where k is a constant. (Newton's Law is an approximation which is reasonable provided the temperature differences are not too great.)

If the *ambient temperature is constant* (not necessary in general), the DE (1.55) can be written

$$\frac{d}{dt}(T - T_A) = -k(T - T_A),$$

which can be solved by inspection:

$$T(t) - T_A = Ce^{-kt},$$

giving an exponential rate of cooling.

Reference: Boyce & DiPrima, page 48.

1.3.7 Pursuit problems

This class of problems refers to the situation in which one participant tries to catch a second moving participant, e.g. a bull charging a running person, a hawk chasing a flying pigeon, a gunboat chasing a smuggler. These problems are difficult. Here is an example.

Example: A dog swims across a river towards his master on the far bank, but is carried downstream by the current. The dog always paddles towards his master. What path will the dog follow? Assume that the speed of the river is w and of the dog in still water is u, with $w < u$ (w and u are constants).

Solution: Refer to figure 1.9 Since the dog always swims towards its master, its velocity is

$$\mathbf{v}_{\text{dog}} = (-u\cos\theta, -u\sin\theta).$$

The velocity of the river is

$$\mathbf{v}_{\text{river}} = (0, w).$$

The rate of change of the dog's position $\mathbf{x}(t) = (x(t), y(t))$ is the sum of the two velocities:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}_{\text{dog}} + \mathbf{v}_{\text{river}}.$$

In components,

$$\frac{dx}{dt} = -u\cos\theta, \quad \frac{dy}{dt} = -u\sin\theta + w. \tag{1.56}$$

But

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}. \tag{1.57}$$

We can write the dog's path in the form $y = y(x)$. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

By (1.56) and (1.57),

$$\frac{dy}{dx} = \frac{uy - w\sqrt{x^2+y^2}}{ux}.$$

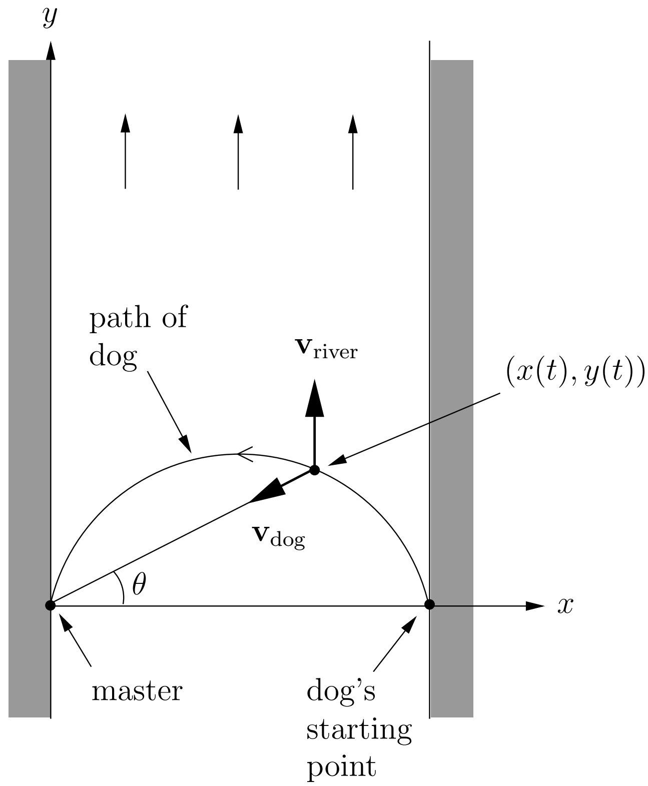


Figure 1.9:

Let $k = \frac{w}{u}$. Since $x > 0$ we can rearrange the DE to get

$$\frac{dy}{dx} = \frac{y}{x} - k\sqrt{1 + \frac{y^2}{x^2}}, \tag{1.58}$$

a *first order DE for the path* $y = y(x)$ *of the dog.* The initial condition is $y(b) = 0$, where b is the width of the river.

Exercise: The form of the DE (1.58) suggests that we introduce a new dependent variable z defined by

$$z = \frac{y}{x}.$$

Then $y = xz$, so that $\frac{dy}{dx} = z + x\frac{dz}{dx}$ by the product rule. Substituting this into (1.58), subtracting z from both sides and dividing by x gives gives

$$\frac{dz}{dx} = -k\frac{\sqrt{1+z^2}}{x},$$

which is separable. The final solution for $y = y(x)$ is

$$y = \tfrac{1}{2}b\left[\left(\frac{x}{b}\right)^{1-k} - \left(\frac{x}{b}\right)^{1+k}\right].$$

The algebra involved in deriving this is somewhat tricky. An intermediate step is

$$z + \sqrt{1+z^2} = \left(\frac{x}{b}\right)^{-k},$$

which can be solved for z giving

$$z = \tfrac{1}{2}\left[\left(\frac{x}{b}\right)^{-k} - \left(\frac{x}{b}\right)^{k}\right]. \qquad \square$$

Reference: Borelli & Coleman, pages 112-5.

1.3.8 Electrical circuits

We consider simple electrical circuits which are constructed from a voltage source and three basic elements, *resistors* (of resistance R), *inductors* (of inductance L) and *capacitors* (of capacitance C), indicated symbolically in Figure 1.10.

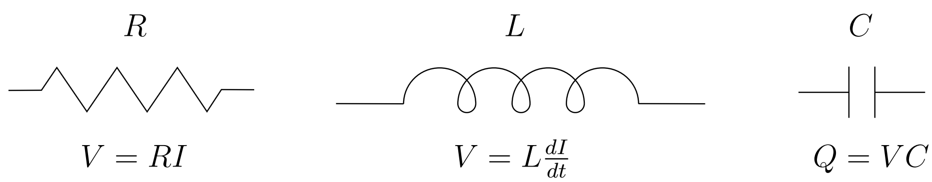


Figure 1.10: Resistor, inductor & capacitor

In the figure, I denotes the electric current through an element, V is the electrical potential difference across the element, and Q is the electric charge on the plate of the conductor onto which the current I is assumed to flow; I is therefore related to Q by

$$I = \frac{dQ}{dt}. \tag{1.59}$$

The three formulas in the figure may be regarded as defining R, L and C.

The current I in the electrical circuit varies with time t, and in simple cases is governed by a DE. The form of the DE is determined by one of *Kirchhoff's Laws*, which states that the potential difference across the terminals (i.e. the voltage source) must equal the sum of the potential differences across the various elements:

$$V_{\text{terminals}} = \sum V_{\text{elements}}. \tag{1.60}$$

Example: The RC circuit.

By (1.60) and the equations in Figure 1.10,

$$V(t) = V_R + V_C = RI + \frac{Q}{C}. \tag{1.61}$$

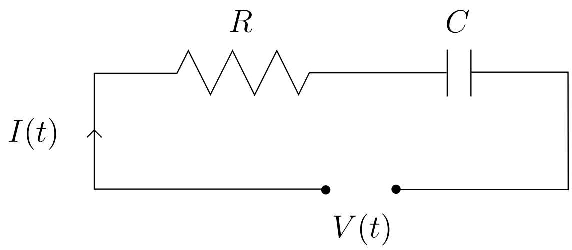


Figure 1.11:

Differentiate (1.61) with respect to t and use (1.59) to eliminate $\frac{dQ}{dt}$, giving

$$\frac{dI}{dt} + \frac{1}{RC}I = \frac{1}{R}\frac{dV}{dt}. \tag{1.62}$$

Equation (1.62) is a first order linear DE for $I(t)$, the source voltage $V(t)$ being regarded as given. □

Example: The RL circuit.

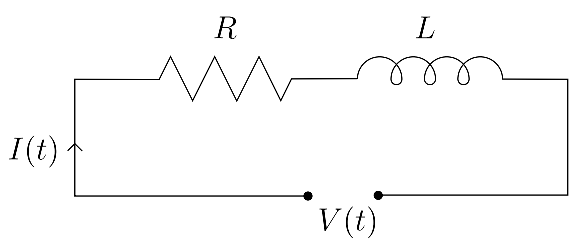


Figure 1.12:

By (1.60) and the equations in Figure 1.10,

$$V(t) = V_R + V_L = RI + L\frac{dI}{dt},$$

that is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}V(t). \tag{1.63}$$

Equation (1.54) is a first order linear DE for $I(t)$. □

Example: The RLC circuit.

By (1.60) and the equations in Figure 1.10,

$$V(t) = V_R + V_L + V_C = RI + L\frac{dI}{dt} + \frac{Q}{C}.$$

Differentiate with respect to t and use (1.59), as in passing from (1.61) to (1.62). We obtain

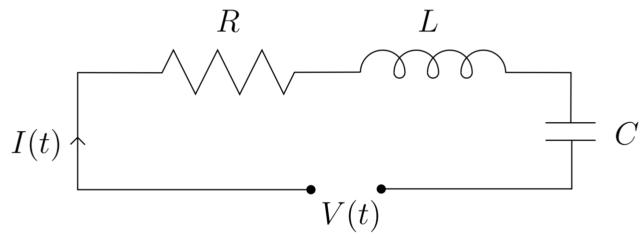


Figure 1.13:

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}\frac{dV}{dt}, \tag{1.64}$$

a *second order* linear DE (to be studied in the next chapter).

References: Boyce & DiPrima, pages 180-1.
Borelli & Coleman, pages 176-86.

Chapter 2

Second Order Linear DEs

2.1 Introduction

2.1.1 Oscillations and Second Order DEs

Oscillations are the single most important physical phenomenon described by second order DEs. As motivation, we consider a simple mechanical system consisting of a spring to which is attached a trolley that can run smoothly on a straight track.

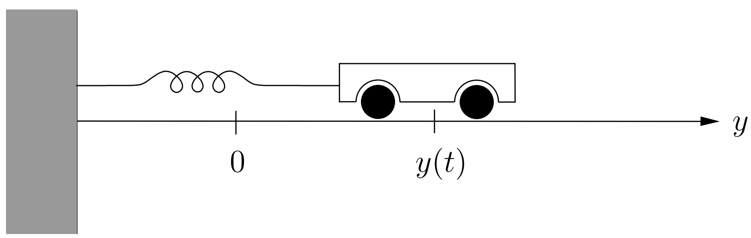


Figure 2.1: A mechanical oscillator

If the trolley is displaced from it equilibrium position $y = 0$ and released, the spring will cause it to run to and fro on the track. It is this motion that we wish to describe.

Let $y(t)$ be the displacement of the trolley from its equilibrium position at time t. When a spring is stretched or compressed it exerts a restoring force which, for small displacements can be assumed to be proportional to the displacement (Hooke's Law). The constant of proportionality is called the *stiffness constant*, and is denoted by k. The force exerted by the spring on the trolley is thus

$$F_{\text{spring}} = -ky. \tag{2.1}$$

We also assume a *damping force* (due to frictional effects, air resistance) that is proportional to the velocity, and acts so as to slow the motion:

$$F_{\text{damp}} = -c\frac{dy}{dt}, \tag{2.2}$$

where $c > 0$ is the *damping constant*.

Comment: If $\frac{dy}{dt} > 0$, the trolley is moving to the right and the force acts to the left, while if $\frac{dy}{dt} < 0$ the trolley is moving to the left and the force acts to the right, in both cases opposing the motion, as required.

The motion of the trolley is governed by Newton's Second Law. Since its mass m is constant, we have

$$m\frac{d^2y}{dt^2} = F_{\text{spring}} + F_{\text{damp}}.$$

On account of (2.1) and (2.2) this equation gives

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0. \tag{2.3}$$

We shall show how to solve this DE (see Section 2.2.3) and shall find that if the damping constant c is non-zero, the solutions show two types of behaviour as $t \to \infty$:

i) exponential monotone decay to zero, or

ii) exponential oscillatory decay to zero.

We shall also consider the situation in which an external force $F(t)$ acts on the trolley, in which case (2.3) is replaced by

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t). \tag{2.4}$$

Of particular interest is the case in which $F(t)$ is *periodic*, which can lead to a variety of interesting behaviour (see Section 2.3).

Comment: The DEs (2.3) and (2.4) describes many different physical systems e.g. a pendulum with small amplitude, a buoy bobbing up and down in the ocean, electrical circuits, two connected mixing tanks, etc.

2.1.2 The world's simplest second order DE

On setting the damping constant c to zero in the DE (2.3) in Section 2.1.1, we obtain

$$m\frac{d^2y}{dt^2} + ky = 0.$$

On defining $\omega^2 = k/m$, this becomes

$$\frac{d^2y}{dt^2} + \omega^2 y = 0. \tag{2.5}$$

The solutions of this DE can be found by inspection, by recalling the differentiation property of sin and cos (and using the chain rule):

$$\frac{d^2}{dt^2}(\sin\omega t) = -\omega^2\sin\omega t, \quad \frac{d^2}{dt^2}(\cos\omega t) = -\omega^2\cos\omega t.$$

Thus $y_1 = \cos\omega t$ and $y_2 = \sin\omega t$ are solutions of (2.5). It is easy to verify that

$$y = c_1 \cos\omega t + c_2 \sin\omega t, \tag{2.6}$$

for any constants c_1 and c_2, is also a solution of (2.5). We shall see later (Section 2.2.1) that (2.6) is in fact the general solution of the DE (2.5). We summarize this most important result:

the general solution of the DE

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

is

$$y = c_1 \cos(\omega t) + c_2 \sin(\omega t),$$

where c_1 and c_2 are arbitrary constants. □

Comment: Since this solution is periodic of period $\frac{2\pi}{\omega}$, the solution describes the trolley moving to and fro with constant period. This type of motion is referred to as *simple harmonic motion*.

2.1.3 Types of Second Order DEs

The *general form* of a second order DE is

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right). \tag{2.7}$$

The simplest class are the *linear DEs*, which are characterized by $f\left(x, y, \frac{dy}{dx}\right)$ being linear in y and $\frac{dy}{dx}$, i.e.

$$f\left(x, y, \frac{dy}{dx}\right) = F(x) + A(x)y + B(x)\frac{dy}{dx}.$$

A linear DE is usually written with the y and $\frac{dy}{dx}$ terms on the left side. Replacing $B(x)$ by $-P(x)$ and $A(x)$ by $-Q(x)$, the DE (2.7) becomes

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x),$$

or more concisely,

$$y'' + P(x)y' + Q(x)y = F(x). \tag{2.8}$$

This is the general form of a *non-homogeneous linear second order DE.*

If $F(x) = 0$, the DE is said to be homogeneous:

$$y'' + P(x)y' + Q(x)y = 0. \tag{2.9}$$

This is the general form of a *homogeneous linear second order DE.*

Finally, if $P(x) = p$ and $Q(x) = q$, where p and q are constants, (2.8) and (2.9) become

$$y'' + py' + qy = F(x), \tag{2.10}$$

and

$$y'' + py' + qy = 0. \tag{2.11}$$

This is the general form of a second order linear DE with *constant coefficients*, (2.10) being non-homogeneous and (2.11) being homogeneous. In this course *we shall be concerned almost exclusively with this type of second order DE.*

Note that the DE (2.4), namely

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t),$$

is of the form (2.10) – just divide by m to put this DE in standard form. Finally the world's simplest second order DE, namely

$$\frac{d^2y}{dt^2} + \omega^2 y = 0,$$

is of the form (2.11) with $p = 0$ and $q = \omega^2$.

2.1.4 The Initial Value Problem for Second Order DEs

Recall the spring and trolley problem from Section 2.1.1, described by the DE

$$my'' + cy' + ky = 0, \tag{2.12}$$

where $y(t)$ is the displacement of the trolley at time t, from the equilibrium position and $'$ denotes $\frac{d}{dt}$.

What are the different ways of setting the trolley in motion? The simplest way is to pull the trolley from its equilibrium position, hold it at rest, and then release it at time $t = 0$. This procedure corresponds to the initial conditions

$$y(0) = y_0, \quad y'(0) = 0, \tag{2.13}$$

i.e. the initial velocity is zero. Another possibility is to give the trolley a tap with a hammer while it is at rest in its equilibrium position. The impact will set the trolley in motion, and the initial condition will be

$$y(0) = 0, \quad y'(0) = v_0, \tag{2.14}$$

i.e. the initial velocity is non-zero. The most general possibility is to pull the trolley from equilibrium and give it a little jerk as you release it, thereby imparting an initial displacement and initial velocity to it. The initial conditions will be

$$y(0) = y_0, \quad y'(0) = v_0. \tag{2.15}$$

With initial conditions such as (2.13)-(2.15) we certainly expect the physical system to move in a uniquely determined way, and hence we expect the DE (2.12) to have a unique solution.

This simple example illustrates what holds in general for a linear second order DE:

$$y'' + P(x)y' + Q(x)y = F(x), \tag{2.16}$$

namely, that the appropriate initial conditions are

$$y(x_0) = y_0, \quad y'(x_0) = v_0, \tag{2.17}$$

and that these initial conditions determine a unique solution of the DE (2.16) (subject to appropriate restrictions on P, Q and F). This existence-uniqueness result is discussed in AMath 351. □

2.2 Solving Second Order Linear DEs with constant coefficients

2.2.1 A fundamental property of homogeneous second order linear DEs

Consider the homogeneous DE

$$y'' + P(x)y' + Q(x)y = 0. \tag{2.18}$$

The fact that this DE is linear AND homogeneous has an immediate and important consequence.

Proposition: If $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous linear DE (2.18), then

$$c_1y_1(x) + c_2y_2(x)$$

is also a solution, for any constants c_1 and c_2.

Proof: Since y_1 and y_2 are solutions of (2.18),

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \tag{2.19}$$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0. \tag{2.20}$$

Multiply (2.19) by c_1 and (2.20) by c_2 and add, to get

$$(c_1y_1 + c_2y_2)'' + P(x)(c_1y_1 + c_2y_2)' + Q(x)(c_1y_1 + c_2y_2) = 0,$$

as required. □

The *fundamental property* referred to in the title is this:

> Any solution of the homogeneous linear DE (2.18) has the form
>
> $$y = c_1y_1(x) + c_2y_2(x), \tag{2.21}$$
>
> where $y_1(x)$ and $y_2(x)$ are *linearly independent*[1] solutions of (2.18), and c_1, c_2 are constants.

[1] $y_1(x)$ and $y_2(x)$ are linearly independent means that no linear combination of $y_1(x)$ and $y_2(x)$ equals zero, for all x; equivalently, it means that $y_1(x)$ is not a constant multiple of $y_2(x)$.

Example: Consider the world's simplest second order DE

$$y'' + \omega^2 y = 0. \tag{2.22}$$

Two solutions are

$$y_1 = \sin \omega t, \qquad y_2 = \cos \omega t,$$

and these are linearly independent, since cos( ) is not a multiple of sin( ). The general solution of (2.22) is thus

$$y = c_1 \sin \omega t + c_2 \cos \omega t, \tag{2.23}$$

where c_1 and c_2 are constants. □

The fundamental property is a consequence of the existence and uniqueness theorem for second order linear DEs. The general solution (2.21) contains TWO arbitrary constants because the initial conditions are TWO in number:

$$y(x_0) = y_0, \qquad y'(x_0) = v_0.$$

When one solves an initial value problem, c_1 and c_2 are determined in terms of y_0 and v_0.

Exercise: Show that the unique solution of the initial value problem

$$y'' + \omega^2 y = 0; \quad y(0) = y_0, \quad y'(0) = v_0$$

is

$$y = \frac{v_0}{\omega} \sin \omega t + y_0 \cos \omega t. \qquad \square$$

2.2.2 General form of the solution

Consider a non-homogeneous second order linear DE

$$y'' + P(x)y' + Q(x)y = F(x). \tag{2.24}$$

The linearity of this DE has an important consequence.

Proposition: If y_1 and y_2 are solutions of the non-homogeneous DE (2.24), then the difference

$$y_1(x) - y_2(x)$$

is a solution of the associated homogeneous DE

$$y'' + P(x)y' + Q(x)y = 0. \tag{2.25}$$

Proof: Same as the proposition on page 16 for first order DEs. □

Suppose $y_p(x)$ is a particular solution of (2.24) and $y(x)$ is *any* solution of (2.24), i.e. it represents the general solution. Then by the Proposition,

$$y_h(x) = y(x) - y_p(x)$$

is a solution of the homogeneous DE (2.25). We can rewrite this equation as

$$y(x) = y_h(x) + y_p(x),$$

giving the general solution. □

General solution of a second order linear DE:

The general solution of the DE (2.24) is of the form

$$y(x) = y_h(x) + y_p(x),$$

where $y_h(x)$ is the general solution of the homogeneous DE (2.25), and $y_p(x)$ is a particular solution of the non-homogeneous DE (2.24). (compare with page 18).

It follows from Section 2.2.1 that

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

where y_1 and y_2 are two linearly independent solutions of the homogeneous DE (2.25).

We thus need to develop two algorithms:

(1) An algorithm to give the *general solution* of a *homogeneous* linear DE. This is possible only for the case of *constant coefficients*

$$y'' + py' + qy = 0,$$

where p and q are constants. We develop this algorithm in Section 2.2.3.

(2) An algorithm to find a *particular solution* of a *non-homogeneous* linear DE

$$y'' + py' + qy = f(x).$$

Here we simply extend the method of undetermined coefficients that we used in the first order case (see Section 1.2.5). This extension is discussed in Section 2.2.4. □

2.2.3 General Solution of the Homogeneous DE

A homogeneous second order linear DE *with constant coefficients* is of the form

$$y'' + py' + qy = 0, \tag{2.26}$$

where p and q are constants. Our goal is to find the *general solution* of any such DE.

We begin by considering a trial function of the form

$$y = e^{mx}, \tag{2.27}$$

where m is a constant. Since $y' = me^{mx}$ and $y'' = m^2 e^{mx}$, equation (2.26) yields

$$(m^2 + pm + q)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, (2.27) is a solution of (2.26) if and only if m is a solution of

$$m^2 + pm + q = 0. \tag{2.28}$$

This quadratic equation is called *the characteristic equation of the DE* (2.26). Its roots are

$$m_{1,2} = \tfrac{1}{2}\left[-p \pm \sqrt{p^2 - 4q}\right]. \tag{2.29}$$

There are three distinct cases, each of which has to be treated separately:

A) Distinct real roots ($p^2 > 4q$)

B) Distinct complex roots ($p^2 < 4q$)

C) Equal real roots ($p^2 = 4q$).

Case A: Distinct real roots

In this case, substituting the two roots (2.29) into (2.27), we get two solutions

$$e^{m_1 x} \quad \text{and} \quad e^{m_2 x}.$$

Since the ratio $\dfrac{e^{m_1 x}}{e^{m_2 x}} = e^{(m_1 - m_2)x}$ is not constant, these solutions are linearly independent. By Section 2.2.1, the *general solution* of the DE (2.26) is thus

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}, \tag{2.30}$$

where c_1 and c_2 are arbitrary constants and m_1, m_2 are the real roots of the characteristic equation (2.28). □

Example 1: Find the general solution of the DE

$$y'' + 5y' + 6y = 0. \tag{2.31}$$

Solution: Substituting the trial function $y = e^{mx}$ in (2.31) gives the characteristic equation

$$m^2 + 5m + 6 = 0,$$

which factors as

$$(m+2)(m+3) = 0.$$

There two distinct real roots $m = -2$ and $m = -3$, give two independent solutions

$$e^{-2x} \quad \text{and} \quad e^{-3x}.$$

The general solution is thus

$$y = c_1 e^{-2x} + c_2 e^{-3x}. \qquad \square$$

Case B: Distinct complex roots

The roots m_1 and m_2, as given by (2.29), are distinct and complex if and only if

$$p^2 - 4q < 0.$$

In this case it is convenient to write m_1 and m_2 in the form

$$m_1 = a + ib, \quad m_2 = a - ib,$$

where a and b are real. We substitute m_1 and m_2 into (2.27) to obtain two complex solutions which we can decompose into real and imaginary parts using Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The two solutions of the DE (2.26) are

$$e^{m_1 x} = e^{(a+ib)x} = e^{ax}e^{ibx} = e^{ax}(\cos bx + i\sin bx) \tag{2.32}$$

and

$$e^{m_2 x} = e^{(a-ib)x} = e^{ax}e^{-ibx} = e^{ax}(\cos bx - i\sin bx). \tag{2.33}$$

Since we want real solutions we consider the linear combinations

$$\begin{aligned} \tfrac{1}{2}(e^{m_1 x} + e^{m_2 x}) &= e^{ax}\cos bx \\ \tfrac{1}{2i}(e^{m_1 x} - e^{m_2 x}) &= e^{ax}\sin bx, \end{aligned}$$

as follows from (2.32) and (2.33). These solutions are linearly independent. Thus, *the general solution of the DE* (2.26) is

$$y = (c_1\cos bx + c_2\sin bx)e^{ax}, \tag{2.34}$$

where c_1 and c_2 are arbitrary constants, and

$$a \pm ib$$

are the roots of the characteristic equation. $\square$

Example 2: Find the general solution of the DE

$$y'' + 2y' + 5y = 0. \tag{2.35}$$

Solution: Substituting the trial function $y = e^{mx}$ into (2.35) gives

$$m^2 + 2m + 5 = 0.$$

Completing the square leads to

$$(m+1)^2 + 2^2 = 0,$$

giving the complex roots

$$m = -1 \pm 2i.$$

A complex solution is

$$e^{(-1+2i)x} = e^{-x}(\cos 2x + i\sin 2x),$$

(by Euler's formula), from which one can read off the two independent real solutions

$$e^{-x}\cos 2x \quad \text{and} \quad e^{-x}\sin 2x.$$

Thus, the general solution of the DE (2.35) is

$$y = (c_1 \cos 2x + c_2 \sin 2x)e^{-x}. \qquad \square$$

Case C: Equal real roots

The roots m_1 and m_2,as given by (2.29), are real and equal if and only if

$$p^2 - 4q = 0.$$

In this case, we obtain only one solution from (2.27) and (2.29), namely

$$y_1 = e^{mx},$$

where $m = -\frac{1}{2}p$.

To find a second linearly independent solution we consider a trial function of the form

$$y = v(x)e^{mx}. \tag{2.36}$$

We substitute (2.36) into the DE (2.26) to obtain

$$v'' + (2m + p)v' + (m^2 + pm + q)v = 0, \tag{2.37}$$

after collecting terms and cancelling a factor of e^{mx} (fill in the details). Now a miracle happens ... since $m = -\frac{1}{2}p$ and m is a solution of the characteristic equation $m^2+pm+q = 0$, equation (2.37) reduces to

$$v'' = 0.$$

Choose $v(x) = x$ as a particular solution, and then (2.36) gives

$$y_2 = xe^{mx} \tag{2.38}$$

as a second linearly independent solution. Thus, *the general solution of the DE* (2.26) is

$$y = (c_1 + c_2 x)e^{mx},$$

where c_1 and c_2 are arbitrary constants, and m is the single solution of the characteristic equation. $\square$

Example 3: Find the general solution of the DE

$$y'' + 6y' + 9y = 0. \tag{2.39}$$

Solution: Substituting the trial function $y = e^{mx}$ in (2.39) gives

$$m^2 + 6m + 9 = 0,$$

which is a perfect square,

$$(m + 3)^2 = 0$$

giving a single root $m = -3$, and one solution $y_1 = e^{-3x}$. A second solution is obtained from (2.38), which gives

$$y_2 = xe^{-3x}.$$

(There is no need to repeat the whole derivation – the second solution is simply x times the first one.) Thus, the general solution of (2.39) is

$$y = (c_1 + c_2 x)e^{-3x}. \qquad \square$$

2.2.4 The method of undetermined coefficients

We now show how to find a *particular solution* of the *non-homogeneous* linear DE with constant coefficients, whose general form is

$$y'' + py' + qy = f(x),$$

where p and q are constants, and f is a given function.

We use the method of undetermined coefficients, as introduced in Section 1.2.5, for first order DEs. This means that we restrict our considerations to the case where f is

- an exponential e^{bx}
- a sine or cosine
- a polynomial

or

- sums of such functions.

Example 1: Find the general solution of the DE

$$y'' + 5y' + 6y = e^{2x}. \tag{2.40}$$

Solution: We consider a trial function

$$y = Ae^{2x}, \tag{2.41}$$

where A is a constant. Since $y' = 2Ae^{2x}$ and $y'' = 4Ae^{2x}$, substituting (2.41) in (2.40) yields

$$[4A + 5(2A) + 6(A)]e^{2x} = e^{2x}.$$

The e^{2x} cancels, and solving for A gives $A = \frac{1}{20}$. Thus (2.41) gives the particular solution

$$y_p = \tfrac{1}{20}e^{2x}. \tag{2.42}$$

The general solution of (2.40) has the form

$$y = y_h(x) + y_p(x), \tag{2.43}$$

where y_h is the general solution of the homogeneous DE $y'' + 5y' + 6y = 0$. We solved this problem in Example 1 of Section 2.2.3:

$$y_h = c_1e^{-2x} + c_2e^{-3x}. \tag{2.44}$$

By (2.42), (2.43) and (2.44), the general solution of the DE (2.40) is

$$y = c_1e^{-2x} + c_2e^{-3x} + \tfrac{1}{20}e^{2x}. \qquad \square$$

It is of interest to consider the previous example with an exponential driving term, in greater generality. Consider

$$y'' + py' + qy = e^{ax} \tag{2.45}$$

where p, q and a are constants. As in the previous example, we consider a trial function

$$y = Ae^{ax}. \tag{2.46}$$

Substituting (2.46) in (2.45) yields

$$A(a^2 + pa + q)e^{ax} = e^{ax}$$

(fill in the details). Thus

$$A = \frac{1}{a^2 + pa + q},$$

giving a particular solution (2.46). However, it is clear that A is undefined for certain values of the constant a, namely, those values which satisfy

$$a^2 + pa + q = 0. \tag{2.47}$$

This is precisely the characteristic equation of the homogeneous DE associated with (2.45). Thus, *the trial function* (2.46) *does not give a solution if a is a root of the characteristic equation.*

How do we find a particular solution in this special case? Based on our experience in case C in the previous section when we had to multiply by x to find a second solution, we consider the trial function

$$y = Axe^{ax}. \tag{2.48}$$

On substituting (2.48) in (2.45) and simplifying, we get

$$A(a^2 + pa + q)xe^{ax} + A(2a + p)e^{ax} = e^{ax}.$$

The first term in brackets is zero, because we are assuming that a satisfies (2.47). Thus

$$A = \frac{1}{2a + p}.$$

Using (2.48), this value of A gives a particular solution unless

$$a = -\tfrac{1}{2}p. \tag{2.49}$$

What is the significance of this condition? Well, if the characteristic equation (2.47) has a double real root, i.e. if $p^2 = 4q$, then (2.47) becomes

$$\left(a + \tfrac{1}{2}p\right)^2 = 0.$$

Thus, (2.49) states that *a is a double root of the characteristic equation.* In this case we generalize (2.48) and use

$$y = Ax^2e^{ax} \tag{2.50}$$

as a trial function. On substituting (2.50) in (2.45) we get

$$A(a^2 + pa + q)x^2e^{ax} + 2A(2a + p)xe^{ax} + 2Ae^{ax} = e^{ax}.$$

Since we are assuming a is a double root of the characteristic equation, equations (2.47) and (2.49) hold, and hence the terms in brackets are zero, leaving

$$A = \tfrac{1}{2}.$$

Thus, in this case equation (2.50) gives a particular solution of the DE (2.45).

Summary:

Consider the DE

$$y'' + py' + qy = e^{ax}, \tag{2.51}$$

with characteristic equation

$$a^2 + pa + q = 0. \tag{2.52}$$

- If a is not a root of (2.52), (2.51) has a particular solution of the form $y = Ae^{ax}$.
- If a is a single root of (2.52), (2.51) has a particular solution of the form $y = Axe^{ax}$.
- If a is a double root of (2.52), (2.51) has a particular solution of the form $y = Ax^2e^{ax}$. □

We next consider the case of a *polynomial driving term*:

$$y'' + py' + qy = a_0 + a_1x + \cdots + a_nx^n. \tag{2.53}$$

Since the derivative of a polynomial is a polynomial, we consider a trial function of the form

$$y_p = A_0 + A_1x + \cdots + A_nx^n, \tag{2.54}$$

where $A_0, A_1, \ldots, A_n$ are the undetermined coefficients. This choice works unless $q = 0$ in (2.53), in which case one multiplies (2.54) by x. □

Exercise 1: Find a particular solution of

$$y'' + 5y' + 6y = 6x.$$

Answer: $y_p = -\frac{5}{6} + x$

Exercise 2: Find a particular solution of

$$y'' + y' = 2x.$$

Answer: $y_p = -2x + x^2$. □

We finally consider a *sine or cosine driving term*:

$$y'' + py' + qy = \sin\omega x. \tag{2.55}$$

Since the derivatives of $\sin\omega x$ are multiples of $\sin\omega x$ and $\cos\omega x$, we take a trial function of the form

$$y_p = A\sin\omega x + B\cos\omega x. \tag{2.56}$$

This choice works unless $p = 0$ and $\omega^2 = q$. This special case is

$$y'' + \omega^2 y = \sin\omega x. \tag{2.57}$$

To get a suitable trial function we multiply (2.56) by x:

$$y_p = x(A \sin \omega x + B \cos \omega x).$$

Remark: As a general rule, if the driving term (right hand side of the DE) and a term in the homogeneous solution $y_h(x)$ are linearly dependent, then *one should multiply the trial function by x.*

Exercise 3: Find a particular solution of

$$y'' - 3y' - 4y = 2 \sin x.$$

Answer: $y_p = -\frac{5}{17} \sin x + \frac{3}{17} \cos x.$

Exercise 4: Find a particular solution of

$$y'' + 4y = \sin 2x.$$

Answer: $y_p = -\frac{1}{4} x \cos 2x.$ □

If the *driving term is a sum* one can use the following proposition to find a particular solution.

Proposition:

If y_1 is a solution of $y'' + py' + qy = f_1(x)$,
and y_2 is a solution of $y'' + py' + qy = f_2(x)$,
then $y = y_1 + y_2$ is a solution of

$$y'' + py' + qy = f_1(x) + f_2(x).$$

Proof: Exercise (a consequence of linearity...). □

2.3 Analysis of the oscillator DE

2.3.1 Introductory remarks

In this section we describe the properties and physical interpretation of the solutions of the DE that describes the mass-spring system with damping and periodic driving force:

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{2.58}$$

We divide by m and write the DE in the form

$$y'' + 2\lambda y' + \omega_0^2 y = f_0 \cos \omega t, \tag{2.59}$$

where

$$\lambda = \frac{c}{2m}, \quad \omega_0^2 = \frac{k}{m}, \quad f_0 = \frac{F_0}{m}. \tag{2.60}$$

Since the DE (2.59) describes a variety of oscillating systems, both mechanical and electrical, we shall refer to it as *the oscillator DE*.

The parameter λ and ω_0 characterize the system itself (while f_0 and ω characterize the driving force), and are thus called *the system parameters*. λ is called the *damping parameter* and ω_0 is called the *natural frequency* (the latter name will be justified in Section 2.3.2). We shall see that the ratio $\frac{\lambda}{\omega_0}$ of these two parameters plays an important role in determining the behaviour of the solutions of the DE.

The DE (2.59) also describes the current in an RLC electrical circuit. In Section 1.3.6 we saw that the current in such a circuit satisfies the DE

$$LI'' + RI' + \frac{1}{C}I = V'(t),$$

where $V(t)$ is the applied voltage. With

$$V(t) = V_0 \sin(\omega t)$$

this DE becomes

$$LI'' + RI' + \frac{1}{C}I = V_0\omega \cos \omega t. \tag{2.61}$$

Dividing by L gives

$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{V_0\omega}{L} \cos \omega t,$$

which is of the form (2.59) with

$$\lambda = \frac{R}{2L}, \quad \omega_0^2 = \frac{1}{LC}, \quad f_0 = \frac{V_0\omega}{L}. \tag{2.62}$$

2.3.2 Zero driving force

In this section we discuss the oscillator DE (2.59) with zero force, i.e.

$$y'' + 2\lambda y' + \omega_0^2 y = 0. \tag{2.63}$$

There are four qualitatively different cases, depending on the ratio

$$\zeta = \frac{\lambda}{\omega_0}.$$

A) The undamped case: $\zeta = 0$

B) The underdamped case: $0 < \zeta < 1$

C) The critically damped case: $\zeta = 1$

D) The overdamped case: $\zeta > 1$

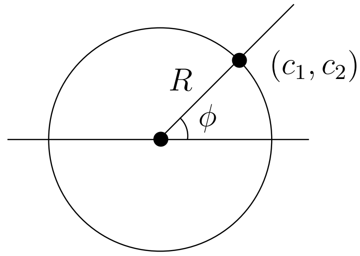


Figure 2.2: Definition of the phase angle ϕ.

These cases correspond to the different possibilities for the roots of the characteristic equation of the DE (2.63).

Case A: The undamped case ($\zeta = 0$)

The DE (2.63) specializes to

$$y'' + \omega_0^2 y = 0,$$

the "world's simplest". By inspection the general solution is (see Section 2.1.2)

$$y = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t). \tag{2.64}$$

In order to interpret the solution it is necessary to use the trig identity

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \tag{2.65}$$

to rewrite (2.64) as single cosine.

Think of the constants c_1, c_2 in (2.64) as defining a point (c_1, c_2) on a circle of radius $R = \sqrt{c_1^2 + c_2^2}$, and let ϕ be the radian measure of the angle defined by (c_1, c_2) (see Figure 3.2).

It follows that

$$c_1 = R\cos\phi, \qquad c_2 = R\sin\phi. \tag{2.66}$$

The solution (2.64) becomes

$$y = R[\cos(\omega_0 t)\cos\phi + \sin(\omega_0 t)\sin\phi],$$

which, using (2.65), can be written as

$$y = R\cos(\omega_0 t - \phi). \tag{2.67}$$

Thus, the system (e.g. the trolley attached to the spring) oscillates with *period*

$$P = \frac{2\pi}{\omega_0}$$

(recall that $\cos t$ is periodic of period 2π and hence $\cos(\omega_0 t - \phi)$ is periodic of period $\frac{2\pi}{\omega_0}$). We say that the system undergoes *simple harmonic motion* (SHM). The constant R in the solution (2.67), which represents the maximum displacement of the system, is called the *amplitude* of the SHM. The *frequency* of the SHM is

$$\nu = \frac{1}{P} = \frac{\omega_0}{2\pi},$$

and the constant ω_0 is called the *circular frequency.* The constant ϕ is called the *phase.*

It should be kept in mind that the frequency ω_0, which is one of the system parameters, is an intrinsic property of the system. Since the initial conditions

$$y(0) = y_0, \qquad y'(0) = v_0 \tag{2.68}$$

enter into the solution (2.67) through the constants c_1 and c_2, *the frequency ω_0 is independent of the initial conditions.*

On the other hand, the amplitude R and phase ϕ do depend on the initial conditions. It follows from equations (2.67) and (2.68) that

$$R = \sqrt{y_0^2 + \frac{v_0^2}{\omega_0^2}}, \tag{2.69}$$

and that ϕ is determined by

$$\cos\phi = \frac{y_0}{R}, \qquad \sin\phi = \frac{v_0}{\omega_0 R}, \tag{2.70}$$

(exercise on Problem Set 2).

The special case of "release from rest", i.e. $y_0 \neq 0$, $v_0 = 0$, gives

$$R = \mid y_0 \mid \quad \text{and} \quad \phi = \begin{cases} 0, & \text{if} \quad y_0 > 0 \\ \pi, & \text{if} \quad y_0 < 0 \end{cases}.$$

The other special case of "a kick while in equilibrium", i.e. $y_0 = 0$, $v_0 \neq 0$, gives

$$R = \frac{\mid v_0 \mid}{\omega_0} \quad \text{and} \quad \phi = \begin{cases} \frac{\pi}{2}, & \text{if} \quad v_0 > 0 \\ \frac{3\pi}{2}, & \text{if} \quad v_0 < 0 \end{cases}.$$

Case B: The underdamped case ($0 < \zeta < 1$)

The characteristic equation for the DE (2.63) is

$$m^2 + 2\lambda m + \omega_0^2 = 0.$$

The roots are

$$m = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2},$$

which we rewrite as

$$m = \omega_0 \left[-\zeta \pm i\sqrt{1-\zeta^2} \right]$$

using $\zeta = \frac{\lambda}{\omega_0}$. The general solution of the DE is therefore

$$y = e^{-\zeta\omega_0 t}\left[c_1 \cos\left(\sqrt{1-\zeta^2}\omega_0 t\right) + c_2 \sin\left(\sqrt{1-\zeta^2}\omega_0 t\right)\right],$$

(as in Section 2.2.3). As in Case A, we can write the expression in square brackets as a single cosine, giving

$$y = Re^{-\zeta\omega_0 t}\cos\left[\sqrt{1-\zeta^2}\omega_0 t - \phi\right]. \tag{2.71}$$

The constants R and ϕ are related to c_1 and c_2 by equation (2.66) and are thus determined by the initial conditions via equations analogous to, but more complicated than, equations (2.69) and (2.70). We do not need the specific form of these equations.

Equation (2.71) gives the general solution of the oscillator DE (2.63) in the underdamped case. The function $y(t)$ satisfies

$$-Re^{-\zeta\omega_0 t} \leq y \leq Re^{-\zeta\omega_0 t}.$$

The graph of $y(t)$ thus oscillates between these two exponential curves. The zeros are equally spaced, with the spacing determined by the period of the cosine function i.e. the spacing is

$$\frac{2\pi}{\omega_0\sqrt{1-\zeta^2}}.$$

Of course $y(t)$ itself is *not* periodic. We say that $y(t)$ describes a system that is performing *damped oscillations*. The graph of $y(t)$ is shown below.

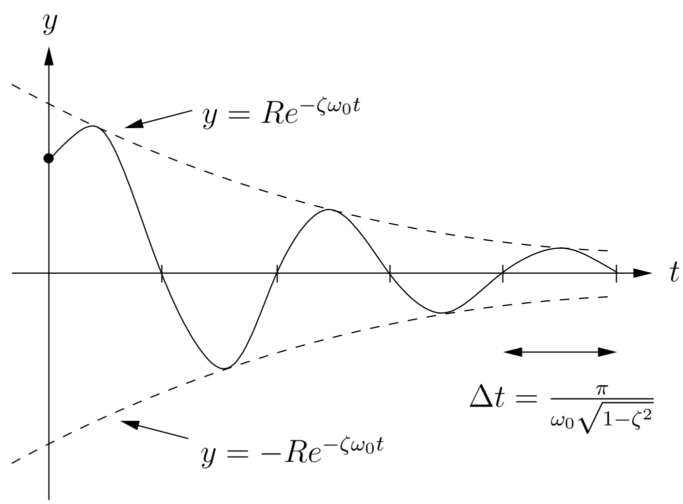


Figure 2.3: Graph of $y = Re^{-\zeta\omega_0 t}\cos\left[\sqrt{1-\zeta^2}\omega_0 t - \phi\right]$.

Comment: Damped oscillations can approximate SHM under the following conditions. If the constant

$$\zeta = \frac{\lambda}{\omega_0}$$

is very small, the system will perform a significant number of oscillations before the damping has had sufficient time to decrease the amplitude of the oscillations appreciably. For example, let $\Delta t_{1\%}$ be the time for the amplitude factor $e^{-\lambda t}$ to decay by 1%. Show that if $\zeta = 10^{-5}$, there will be approximately 160 oscillations during this time interval.

Case C: The critically damped case ($\zeta = 1$)

The characteristic equation for the DE (2.63) is

$$m^2 + 2\omega_0 m + \omega_0^2 = 0,$$

with a single repeated root $m = -\omega_0$. As in Section 2.2.3, the general solution is

$$y = (c_1 + c_2 t)e^{-\omega_0 t}.$$

It follows (Exercise) that the unique solution which satisfies the initial conditions $y(0) = y_0$ and $y'(0) = v_0$, is

$$y = [y_0 + (v_0 + \omega_0 y_0)t]\, e^{-\omega_0 t}. \tag{2.72}$$

For all initial conditions, $\lim_{t\to\infty} y = 0$, i.e. the system returns to a state of equilibrium. The shape of the graph of $y(t)$ (i.e. the qualitative behaviour) depends on y_0 and v_0. Figure 3.4 shows some different possibilities for $y_0 > 0$.

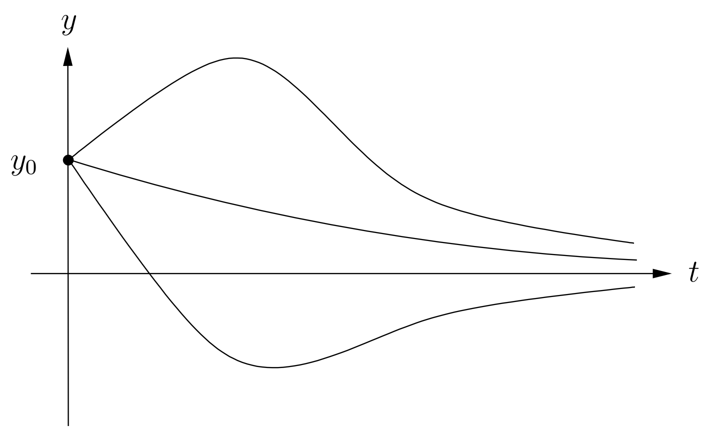


Figure 2.4: Displacement of a critically damped oscillator.

Case D: The overdamped case ($\zeta > 1$)

The characteristic equation for the DE (2.63) is

$$m^2 + 2\lambda m + \omega_0^2 = 0,$$

with distinct real roots

$$m_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}.$$

In terms of $\zeta = \lambda/\omega_0$,

$$m_{1,2} = \omega_0 \left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]. \tag{2.73}$$

The general solution of the DE (2.63) is

$$y = e^{-\zeta\omega_0 t}\left[c_1 e^{\sqrt{\zeta^2-1}\,\omega_0 t} + c_2 e^{-\sqrt{\zeta^2-1}\,\omega_0 t}\right]. \tag{2.74}$$

The constants c_1 and c_2 are determined by the initial conditions. The expressions are a bit complicated, and are not important. The essential point is that for all initial conditions, i.e. for all c_1 and c_2,

$$\lim_{t\to+\infty} y = 0,$$

i.e. the system eventually returns to a state of equilibrium.

Summary:

We have analyzed the solutions of the DE

$$y'' + 2\lambda y' + \omega_0^2 y = 0, \tag{2.75}$$

which governs the displacement of a mechanical oscillator and also the current in an electrical circuit. The first essential result is a consequence of the form of the general solutions (2.71), (2.72) and (2.74).

Proposition: If $\lambda > 0$ and $\omega_0 > 0$ then all solutions of the DE

$$y'' + 2\lambda y' + \omega_0^2 y = 0$$

satisfy

$$\lim_{t\to+\infty} y(t) = 0. \qquad \Box$$

The interpretation is as follows. The DE (2.75) admits the *equilibrium solution* $y(t) = 0$ for all t, corresponding to the physical system being in a state of equilibrium. The proposition implies that if the system starts at time $t = 0$ with the initial conditions $y(0) = y_0$ and $y'(0) = v_0$, then *it will eventually return arbitrarily closely to the equilibrium state* $y = 0$*, no matter what the initial conditions are*. The physical cause of this behaviour is the *damping*, either the mechanical damping or the electrical damping (the resistor R in the electrical circuit).

The second essential result is this: *the system undergoes (damped) oscillations while returning to equilibrium if an only if the system parameter* $\zeta = \frac{\lambda}{\omega_0}$ *satisfies* $0 < \zeta < 1$.

A final question of practical concern arises from the Proposition, namely, *how rapidly does the system approach equilibrium*, i.e. how rapidly does y tend to zero? The decay rate of y is governed by the exponential term in equations (2.71), (2.72) and (2.74):

$$y(t) \sim \begin{cases} e^{-\zeta\omega_0 t}, & \text{if} \quad 0 < \zeta < 1 \\ e^{-\omega_0 t}, & \text{if} \quad \zeta = 1 \\ e^{-(\zeta-\sqrt{\zeta^2-1})\omega_0 t}, & \text{if} \quad \zeta > 1 \end{cases} \tag{2.76}$$

Since $\zeta > 1$ implies $\zeta - \sqrt{\zeta^2 - 1} < 1$, it follows that *the displacement* y *decays most rapidly for* $\zeta = 1$*, i.e. in the critically damped case*. Thus, if one wishes to design a mechanical or electrical system that will return rapidly to a state of equilibrium after being disturbed, he/she should arrange that the damping is close to critical i.e. $\zeta = \frac{\lambda}{\omega_0} \approx 1$.

2.3.3 Non-zero driving force

The oscillator DE with non-zero driving force is

$$y'' + 2\lambda y' + \omega_0^2 y = f_0 \cos(\omega t), \tag{2.77}$$

[see equation (2.59) in Section 2.3.1]. We know from Section 2.2.2 that the general solution of this DE is of the form

$$y(t) = y_h(t; c_1, c_2) + y_p(t), \tag{2.78}$$

where y_h is the general solution of the homogeneous DE

$$y'' + 2\lambda y + \omega_0^2 y = 0,$$

(and hence depends on two arbitrary constants c_1 and c_2), and y_p is a particular solution of (2.77). We also know from Section 2.2.4 that $y_p(t)$ is of the form

$$y_p(t) = a_1 \cos \omega t + a_2 \sin \omega t,$$

(use the method of undetermined coefficients). This solution can also be written in the form

$$y_p(t) = A \cos(\omega t - \delta), \tag{2.79}$$

where the constants a_1 and a_2 have been replaced by the amplitude A and phase δ (see equations (2.64) and (2.67)).

One of the main results from Section 2.3.2 is that

$$\lim_{t \to \infty} y_h(t; c_1, c_2) = 0.$$

Thus, by (2.78) and (2.79), for sufficiently large t

$$y(t) \approx A \cos(\omega t - \delta),$$

i.e. *for sufficiently large t the response $y(t)$ of the system is periodic, having the same period as the driving force.* Heuristically, one can say that the driving force "overcomes" the effects of the system parameters, namely the natural frequency ω_0 and the damping constant λ, and compels the system to oscillate at the driving force's frequency, namely ω.

Referring to equation (2.78), the term y_h, which dies away, is called the *transient part* of the solution, while the term y_p, which is periodic and hence persists indefinitely, is called the *steady state* part of the solution. In many applications, the transient part will die out rapidly (depending on $\zeta = \lambda/\omega_0$ and ω_0; see equation (2.76)), in which case the behaviour of the physical system is essentially described by the steady state part, which we refer to as *the response of the system to the driving (i.e. external) force.*

The essential question then is: what is the amplitude A of the steady state term (2.79)? Or more precisely, *how does the amplitude A depend on the system parameters λ and ω_0, and on the driving force parameters, namely the frequency ω and the amplitude f_0?*

We shall show, by deriving the particular solution y_p, that A *depends in a critical way on the ratio $\frac{\omega}{\omega_0}$ and on the damping parameter $\zeta = \frac{\lambda}{\omega_0}$.* This dependence has important implications for the design of many physical systems.

1. Derivation of the steady state solution

A particular solution of the DE (2.77) can be found in the usual way, using "undetermined coefficients". Here we take the opportunity to show that the solution can be found more quickly by using *complex numbers.*

Consider the complex DE

$$z'' + 2\lambda z' + {\omega_0}^2 z = f_0 e^{i\omega t}. \tag{2.80}$$

Since $e^{i\omega t} = \cos\omega t + i\sin\omega t$, the DE (2.77) is the real part of (2.80), with $y = Re(z)$. Assume a trial function of the form

$$z(t) = Ae^{i(\omega t - \delta)} \tag{2.81}$$

where A and δ are the undetermined coefficients (they are assumed to be real). When (2.81) is substituted in (2.80) one finds after a non-trivial calculation that

$$\begin{gathered} A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\lambda^2\omega^2}}, \\ \cos\delta = \frac{1}{f_0}A(\omega_0^2 - \omega^2), \quad \sin\delta = \frac{2}{f_0}A\lambda\omega. \end{gathered} \tag{2.82}$$

[The details are an exercise on Problem Set 2.]

Taking the real part of (2.81) gives the particular solution of (2.77)

$$y(t) = A\cos(\omega t - \delta), \tag{2.83}$$

where A and δ are given by equations (2.82).

In summary, *the steady state solution of the oscillator DE*

$$y'' + 2\lambda y' + \omega_0^2 y = f_0\cos(\omega t),$$

is given by

$$y(t) = A\cos(\omega t - \delta), \tag{2.84}$$

with

$$\begin{gathered} A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\lambda^2\omega^2}}, \\ \cos\delta = \frac{1}{f_0}A(\omega_0^2 - \omega^2), \quad \sin\delta = \frac{2}{f_0}A\lambda\omega. \end{gathered} \tag{2.85}$$

Thus the steady state response is *simple harmonic motion* with *amplitude* A, *angular frequency* ω (the same as the driving force), and *phase* δ.

2. Analysis of the amplitude of the steady state solution

Consider a fixed system, with system parameters λ and ω_0, governed by the oscillator DE (2.77). We are interested in the response of the system to a driving force of frequency ω and amplitude f_0. The amplitude A of the response is given in (2.82):

$$A(\omega, \lambda) = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\lambda^2\omega^2}}. \tag{2.86}$$

For a given system, ω_0 and λ are fixed. The dependence of A on the amplitude f_0 of the driving force is simple: A is proportional to f_0, by (2.86). The dependence of A on the frequency ω of the driving force is complicated. One can imagine changing ω. How does A change? We will be able to answer this question at a glance if we sketch the graph of $A(\omega, \lambda)$ versus ω, for fixed λ. This graph is called the *frequency response curve of the system.*

To simplify the algebra, we first rewrite (2.86) as

$$\begin{aligned} A(\omega,\lambda) &= \frac{f_0}{\omega_0^2}\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+4\frac{\lambda^2}{\omega_0^2}\frac{\omega^2}{\omega_0^2}}} \\ &= \frac{f_0}{\omega_0^2}\frac{1}{\sqrt{(1-\Omega^2)^2+4\zeta^2\Omega^2}} \\ &= \frac{f_0}{\omega_0^2}\mathcal{A}(\Omega,\zeta) \end{aligned} \tag{2.87}$$

where we have defined the ratios

$$\Omega = \frac{\omega}{\omega_0} \tag{2.88}$$

and, as before,

$$\zeta = \frac{\lambda}{\omega_0}. \tag{2.89}$$

We shall analyze the frequency response curve of the function

$$\mathcal{A}(\Omega,\zeta) = \frac{1}{\sqrt{(1-\Omega^2)^2+4\zeta^2\Omega^2}} \tag{2.90}$$

which has the same qualitative properties of the amplitude function $A(\omega,\lambda)$ ($\mathcal{A}(\Omega,\zeta)$ is just $A(\omega,\lambda)$ re-scaled by a factor of $\frac{\omega_0^2}{f_0}$).

The graph of $\mathcal{A}(\Omega,\zeta)$, for various fixed values of ζ, can be drawn using the following information (see figures 2.5 and 2.6):

(i) $\mathcal{A}(0,\zeta) = 1$, for all $\zeta \geq 0$

(ii) $\lim\limits_{\Omega\to\infty} \mathcal{A}(\Omega,\zeta) = 0$, for all $\zeta \geq 0$.

These results follow immediately from (2.90).

(iii) If $2\zeta^2 \geq 1$, then $\mathcal{A} < 1$ for all $\Omega > 0$. This follows by writing (2.90) in the form

$$\mathcal{A} = \left[1+\Omega^4+2(2\zeta^2-1)\Omega^2\right]^{-\frac{1}{2}}. \tag{2.91}$$

Thus, if $2\zeta^2 \geq 1$, the maximum of $\mathcal{A}$ on the interval $0 \leq \Omega < \infty$ occurs at the endpoint $\Omega = 0$.

(iv) If $2\zeta^2 < 1$, then $\mathcal{A}$ equals 1 for $\Omega = 0$ and $\Omega = \sqrt{2(1-2\zeta^2)}$, and is greater than 1 between these values. This result follows by writing (2.91) in the form

$$\mathcal{A} = \left[1+\Omega^2\{\Omega^2-2(1-2\zeta^2)\}\right]^{-\frac{1}{2}}. \tag{2.92}$$

Thus, if $0 < 2\zeta^2 < 1$, $\mathcal{A}$ must have a maximum value greater than 1.

(v) If $0 < 2\zeta^2 < 1$ the maximum value of $\mathcal{A}$ is

$$\mathcal{A} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, \tag{2.93}$$

and occurs for

$$\Omega = \sqrt{1-2\zeta^2}. \tag{2.94}$$

This result is obtained by showing that the derivative of $\mathcal{A}$ with respect to Ω has the form

$$\frac{\partial \mathcal{A}}{\partial \Omega} = -2\Omega[\Omega^2 - (1-2\zeta^2)]\mathcal{A}^3,$$

so that

$$\frac{\partial \mathcal{A}}{\partial \Omega} = 0 \quad \text{implies} \quad \Omega = 0 \quad \text{or} \quad \Omega = \sqrt{1-2\zeta^2}.$$

Then substitute $\Omega = \sqrt{1-2\zeta^2}$ into (2.92) to get (2.93).

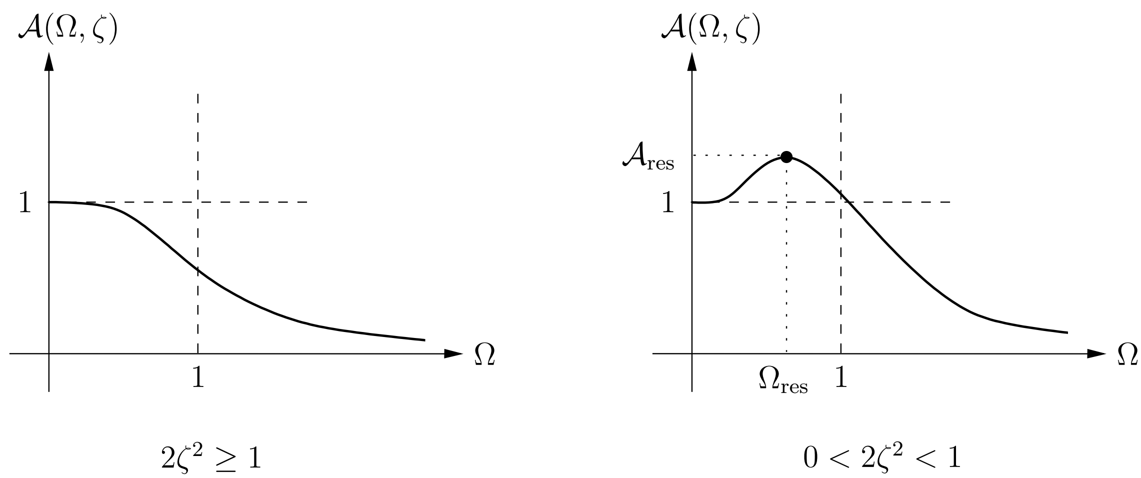


Figure 2.5: Typical frequency response curves in the cases $2\zeta^2 \geq 1$ and $0 < 2\zeta^2 < 1$.

When $\mathcal{A}$ attains a maximum greater than 1, one says the *the system undergoes resonance.* We thus label the values of $\mathcal{A}$ and Ω with a subscript "res", i.e. we write (2.93) and (2.94) as

$$\mathcal{A}_{\text{res}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, \quad \Omega_{\text{res}} = \sqrt{1-2\zeta^2}. \tag{2.95}$$

By (2.87) and (2.88), the physical amplitude and frequency at resonance can be written

$$A_{\text{res}} = \frac{f_0}{2\omega_0^2\zeta\sqrt{1-\zeta^2}}, \quad \omega_{\text{res}} = \omega_0\sqrt{1-2\zeta^2}, \tag{2.96}$$

with $\zeta = \lambda/\omega_0$. Observe that

$$\omega_{\text{res}} < \omega_0 \quad \text{for all} \quad \zeta \quad \text{with} \quad 2\zeta^2 < 1,$$

i.e. *the resonant frequency is less than the natural frequency.* The second result is that A_{res} *can be arbitrarily large if the damping parameter* ζ *is sufficiently close to* 0. Indeed, if $0 < \zeta \ll 1$, equation (2.96) gives the approximations

$$A_{\text{res}} \approx \frac{f_0}{2\omega_0^2 \zeta}, \qquad \omega_{\text{res}} \approx \omega_0. \tag{2.97}$$

It is instructive to draw the family of frequency response curves for different values of ζ, in the $\Omega\mathcal{A}$-plane. For this purpose it is useful to note that

$$\mathcal{A}_{\text{res}} = \frac{1}{\sqrt{1 - \Omega_{\text{res}}^4}}, \tag{2.98}$$

as follows from (2.95).

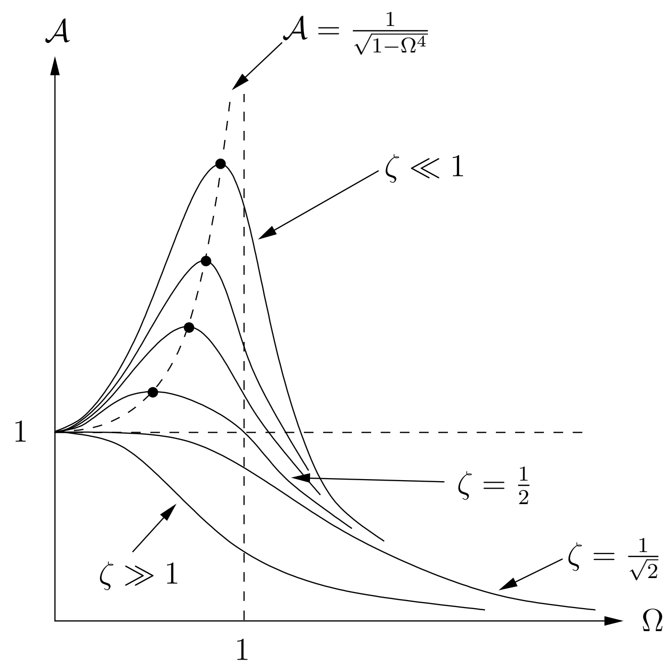


Figure 2.6: The frequency response diagram. Note that the maxima lie on the curve (2.98).

Comment: If $0 < \zeta \ll 1$, the frequency response curve has a steep and narrow peak, with

$$\mathcal{A}_{\text{res}} \approx \frac{1}{2\zeta}, \quad \omega_{\text{res}} \approx \omega_0$$

showing how *resonance provides amplification.* Engineers refer to the quantity

$$Q = \frac{1}{2\zeta}$$

as the *Q-factor of the system* (Q for Quality). In some situations, a high Q-factor is needed, as in a radio tuning circuit (an RLC circuit, see Section 2.3.1). In this situation, the driving frequency ω would be the frequency of the station you wish to receive, and you would "tune in" the station by varying $\omega_0 = \frac{1}{\sqrt{LC}}$ to obtain resonance i.e. the circuit selectively amplifies the signal of the station. In other situations, it is desirable to have a *flat* frequency response curve. The value $\zeta = \frac{1}{\sqrt{2}}$, i.e. $A = \frac{1}{\sqrt{2}}$, which is the value of ζ which just avoids resonance, gives the flattest response curve, i.e. very flat for $0 < \Omega < 1$ $(0 < \omega < \omega_0)$. Such a system would act as a *filter*, excluding frequencies with $\omega \gg \omega_0$.

3. Analysis of the phase of the steady state solution

Recall from equation (2.85) that the phase δ of the steady state solution is given by

$$\cos\delta = \frac{1}{f_0}A(\omega_0^2 - \omega^2), \quad \sin\delta = \frac{2}{f_0}A\lambda\omega \tag{2.99}$$

where

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\lambda^2\omega^2}}.$$

In terms of $\mathcal{A}(\Omega,\zeta)$, $\Omega = \frac{\omega}{\omega_0}$, and $\zeta = \frac{\lambda}{\omega}$, this becomes

$$\cos\delta = \mathcal{A}(1 - \Omega^2), \quad \sin\delta = 2\mathcal{A}\zeta\Omega, \tag{2.100}$$

where

$$\mathcal{A} = \left[(1-\Omega^2)^2 + 4\zeta^2\Omega^2\right]^{-\frac{1}{2}}.$$

We are going to sketch the graph of δ as a function of Ω, for various values of ζ (see figure 2.7).

We need the following properties of the function $\delta = \delta(\Omega,\zeta)$, which are a consequence of equation (2.100):

(i) $\delta(0,\zeta) = 0$ and $\delta(1,\zeta) = \frac{\pi}{2}$ for all $\zeta > 0$,

(ii) $\lim\limits_{\Omega\to\infty} \delta(\Omega,\zeta) = \pi$,

(iii) $\dfrac{\partial\delta}{\partial\Omega}(0,\zeta) = 2\zeta$ and $\dfrac{\partial\delta}{\partial\Omega}(1,\zeta) = \frac{1}{\zeta}$,

obtained by differentiating $\tan\delta = \dfrac{2\zeta\Omega}{1-\Omega^2}$ with respect to Ω.

The main results are:

(1) the response of the system always lags the driving force, since $\delta > 0$,

(2) if $\zeta \ll 1$, resonance occurs for $\Omega \approx 1$ and then the phase is close to $\frac{\pi}{2}$.

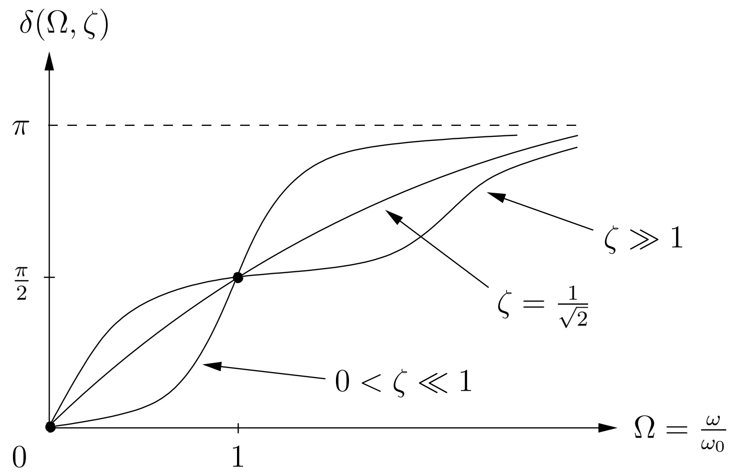


Figure 2.7: The phase response diagram.

4. The case of zero damping

We finally consider the idealized case of zero damping ($\lambda = 0$), so that the oscillator DE (2.77) reduces to

$$y'' + \omega_0^2 y = f_0 \cos(\omega t). \tag{2.101}$$

If $\omega \neq \omega_0$, the general solution is

$$y = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{f_0}{\omega_0^2 - \omega^2} \cos(\omega t), \tag{2.102}$$

and if $\omega = \omega_0$, the general solution is

$$y = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{f_0}{2\omega_0} t \sin(\omega_0 t). \tag{2.103}$$

(exercise using undetermined coefficients, preferably in complex form for efficiency).

The first significant difference from the damped case is that when $\omega = \omega_0$, i.e. the driving frequency equals the natural frequency, the response y grows without bound as $t \to +\infty$, due to the $t \sin(\omega_0 t)$ term in (2.103). This is an extreme but idealized form of resonance.

The second significant difference is that the general solution does not separate into a transient term and a steady state term, since there are no (decaying) exponential terms in the general solution. This implies that *the long term behaviour depends on the initial conditions.*

We consider in detail the case where the system starts in equilibrium, i.e. the initial conditions are

$$y(0) = 0, \qquad y'(0) = 0.$$

It follows from equations (2.102) and (2.103) that the unique solutions are

$$y = \frac{f_0}{\omega_0^2 - \omega^2}(\cos \omega t - \cos \omega_0 t), \quad \text{if } \omega \neq \omega_0, \tag{2.104}$$

and

$$y = \frac{f_0}{2\omega_0} t \sin(\omega_0 t), \quad \text{if } \omega = \omega_0. \tag{2.105}$$

In the second case, the response is a linearly growing oscillation, as shown in Figure 2.8.

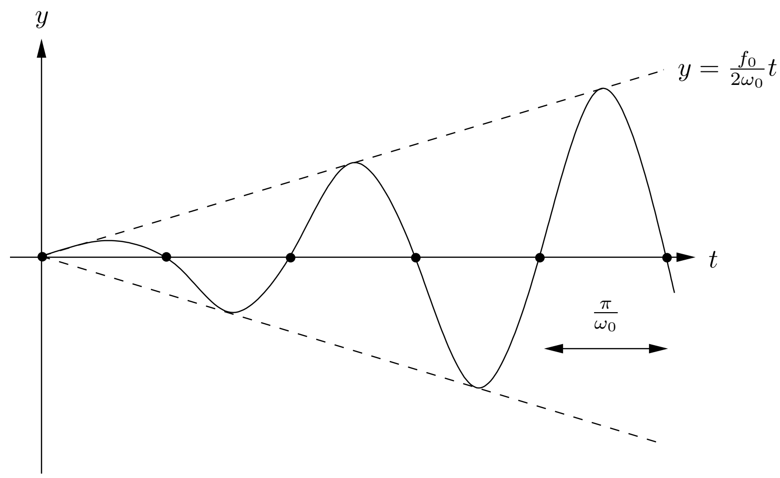


Figure 2.8: Undamped response when the driving frequency equals the natural frequency.

In the first case, we can use the trig identity

$$\cos A - \cos B = -2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right),$$

to write the solution (2.104) in the form

$$y = \left[\frac{2f_0}{\omega_0^2 - \omega^2}\sin\tfrac{1}{2}(\omega_0 - \omega)t\right]\sin\tfrac{1}{2}(\omega_0 + \omega)t. \tag{2.106}$$

If $|\ \omega_0 - \omega\ | \ll 1$ and $\omega_0 + \omega \gg |\ \omega_0 - \omega\ |$, then $\sin\frac{1}{2}(\omega_0 + \omega)t$ is a rapidly oscillating function compared to $\sin\frac{1}{2}(\omega_0 - \omega)t$. Thus, *the solution* (2.106) *represents a rapid oscillation with frequency* $\frac{1}{2}(\omega_0 + \omega)$, *but with a slowly varying sinusoidal amplitude with frequency* $\frac{1}{2}(\omega_0 - \omega)$ [see Figure 2.9]. This type of response, with a periodic variation of amplitude, exhibits what are called *beats*. This phenomenon can be heard when two tuning forks of nearly equal frequency are sounded simultaneously. In electronics, the periodic variation of amplitude with time is called *amplitude modulation*.

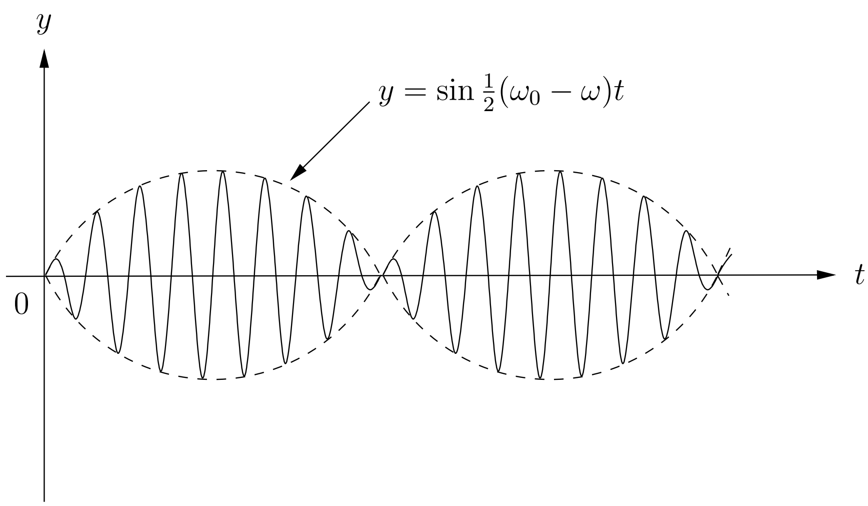


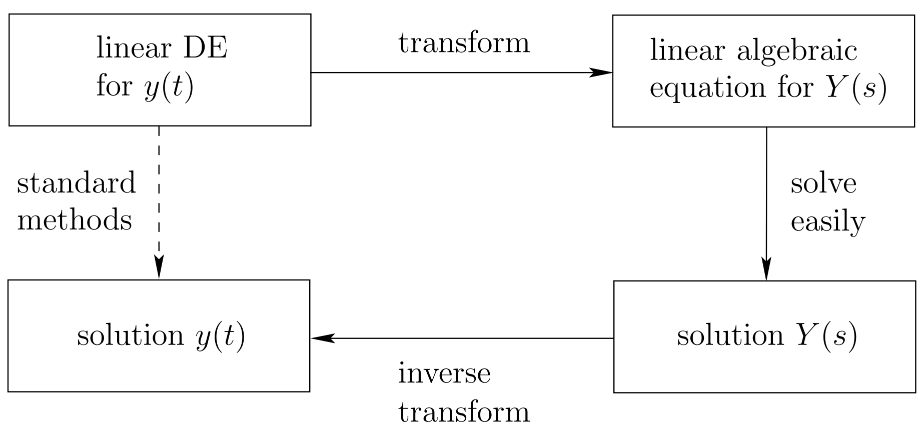
Figure 2.9: Graph of $y = \left[\sin \frac{1}{2}(\omega_0 - \omega)t\right] \sin \frac{1}{2}(\omega_0 + \omega)t$, showing the phenomenon of beats.

Chapter 3

The Laplace Transform and DEs

The idea behind the Laplace transform is this:

> transform a linear *differential* equation for $y(t)$ into a linear *algebraic* equation for $Y(s)$, thereby making it easier to solve.



The function $Y(s)$ is called the *Laplace transform* of the function $y(t)$.

Comment: In its simplest form, the method is restricted to *linear differential equations with constant coefficients*. It provides an alternative to the standard methods that we have developed so far. We'll mention its advantages as we proceed.

3.1 Elementary properties of the Laplace transform

3.1.1 Definition of the Laplace transform

The Laplace transform is one of several "transforms" that are used in engineering mathematics. They are defined using the integral.

Definition: Given a function $y(t)$ *the Laplace transform of* y is defined to be

$$Y(s) = \int_0^\infty e^{-st} y(t) dt, \tag{3.1}$$

for all values of s for which the improper integral converges.

Recall the definition of "converges":

The improper integral $\int_a^\infty g(t)dt$ *converges* means that $\lim_{r\to\infty} \int_a^r g(t)dt$ exists. □

Before discussing the significance of this definition we work out the simplest and most important example.

Example: Find the Laplace transform of $y(t) = e^{\alpha t}$, where α is a constant.

Solution: Referring to equation (3.1), consider

$$\begin{aligned} \int_0^r e^{-st} e^{\alpha t} dt &= \int_0^r e^{-(s-\alpha)t} dt, \\ &= -\frac{1}{s-\alpha} e^{-(s-\alpha)t}\Big|_0^r, \quad \text{assuming} \quad s \neq \alpha, \\ &= \frac{1}{s-\alpha}[1 - e^{-(s-\alpha)}]. \end{aligned} \tag{3.2}$$

If $s > \alpha$, then $\lim_{r\to\infty} e^{-(s-\alpha)r} = 0$, which by (3.2) implies that

$$\lim_{r\to\infty} \int_0^r e^{-st} e^{\alpha t} dt = \frac{1}{s-\alpha} \quad \text{(exists)}.$$

Thus, if $s > \alpha$, the Laplace transform of $y(t) = e^{\alpha t}$ is

$$Y(s) = \frac{1}{s-\alpha},$$

by the definition. □

We think of equation (3.1) as defining an *operator* $\mathcal{L}$ which acts on a function $y(t)$ to give a new function $Y(s)$. We write

$$\mathcal{L}[y] = Y,$$

or, if we wish to indicate the arguments,

$$\mathcal{L}[y(t)] = Y(s),$$

where $Y(s)$ is given by (3.1). We shall refer to $\mathcal{L}$ as the *Laplace transform operator.*

Referring to the Example, we can use the operator notation to write

$$\mathcal{L}[e^{\alpha t}] = \frac{1}{s-\alpha}, \quad s > \alpha. \tag{3.3}$$

This equation is read as "the Laplace transform of $e^{\alpha t}$ is $\frac{1}{s-\alpha}$".

The idea of an "operator" which acts on functions is not unfamiliar. One can think of the process of differentiation as defining an operator D that acts on functions, i.e. D acts on a differentiable function f to give its derivative:

$$D[f] = f'.$$

The operator D and $\mathcal{L}$ have one important property in common. The operator D satisfies

$$D[f_1 + f_2] = Df + D[f_2],$$

and

$$D[cf] = cD[f],$$

where c is a constant. These equations are simply the sum property and "multiplication by a constant" property of derivatives. These equations mean that D is a *linear operator*.[1]

Proposition: The Laplace transform operator $\mathcal{L}$ is a linear operator:

$$\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2],$$

and

$$\mathcal{L}[cy] = c\mathcal{L}[y],$$

where c is a constant.

Proof: These results follow from the linearity of the integral:

$$\int_a^b [y_1(t) + y_2(t)]dt = \int_a^b y_1(t)dt + \int_a^b y_2(t)dt,$$

and

$$\int_a^b cy(t)dt = c\int_a^b y(t)dt,$$

by taking limits to obtain the improper integrals. We skip the details. □

This proposition is used extensively when working with the Laplace transform.

Before continuing, we pause briefly to consider a theoretical question: what conditions on f are needed to ensure that $\mathcal{L}[f]$ exists?

Speaking intuitively, we can say that $\mathcal{L}[f]$ exists provided that

(i) $f(t)$ does not grow too rapidly as $t \to +\infty$,

and

(ii) the only discontinuities of f for $t \geq 0$ are jump discontinuities.

[1]This is the same concept as in linear algebra, where a 3×3 matrix A acting on vectors $\mathbf{x}$ in $\mathbb{R}^3$ (for example) satisfies $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2$, and $A(\lambda\mathbf{x}) = \lambda A\mathbf{x}$, thereby defining a *linear transformation.*

First, to describe the growth condition it is convenient to use the *big-O notation.* The definition is:

$$f(t) = O(g(t)) \quad \text{as} \quad t \to \infty,$$

means that there exist constants B and b such that

$$| f(t) | \leq B | g(t) | \quad \text{for all} \quad t \geq b.$$

The required growth condition is that

$$f(t) = O(e^{at}) \quad \text{as} \quad t \to \infty, \tag{3.4}$$

for *some constant* a. Intuitively (3.4) means that $f(t)$ does not grow more rapidly than e^{at} as $t \to \infty$.

Second, f has a *jump discontinuity at* c means that the 1-sided limits

$$\lim_{t \to c^+} f(t) \quad \text{and} \quad \lim_{t \to c^-} f(t)$$

exist and are unequal. The value of f at c is unimportant. The statement "*f is piecewise continuous on a finite interval I*" means that f is continuous on I except for a *finite* number of jump discontinuities.

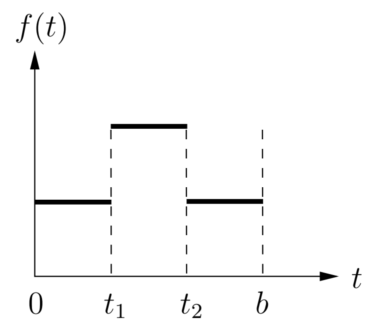


Figure 3.1: A piecewise continuous function with two jump discontinuities.

A function such as this (a pulse function) could act as the driving force of an oscillator.

Proposition: (Existence of the Laplace transform)

If H_1 : f is piecewise continuous on each interval $0 \leq t \leq r$,
H_2 : there is a constant a such that

$$f(t) = O(e^{at}) \quad \text{as} \quad t \to \infty,$$

then the Laplace transform $\mathcal{L}[f]$ exists for $s > a$.

Proof: Outline.

By H_1, $\int_0^r f(t)dt$ exists (just divide the interval into subintervals on which f is continuous).

By H_2, $| f(t) |\leq Be^{at}$ for $t \geq b$. Hence

$$| e^{-st}f(t) |\leq Be^{-(s-a)t}, \quad \text{for} \quad t \geq b.$$

If $s > a$, $\int_0^\infty Be^{-(s-a)t}dt$ converges. Hence by the Comparison Test for improper integrals, $\int_0^\infty e^{-st}f(t)dt$ converges i.e. $\mathcal{L}[f]$ exists by definition. □

Examples:

(i) $f(t) = (1+t)^t$ does not satisfy the growth condition H_2, and $\mathcal{L}[f]$ does not exist.

(ii) $f(t) = \frac{1}{t}$ is not piecewise continuous on any interval $[0, r]$, and $\mathcal{L}[f]$ does not exist. □

3.1.2 Calculating Laplace transforms

Our goal is to build up a table of Laplace transforms of elementary functions, which we can use to solve differential equations.

So far we have shown that

$$\mathcal{L}[e^{\alpha t}] = \frac{1}{s-\alpha}, \quad \text{for} \quad s > \alpha. \tag{3.5}$$

This formula includes the important special case of the constant function $f(t) = 1$, i.e. choose $\alpha = 0$:

$$\mathcal{L}[1] = \frac{1}{s}, \quad \text{for} \quad s > 0. \tag{3.6}$$

Another useful example is the Laplace transform of t^n.

Example: Show that

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad \text{for} \quad s > 0, \tag{3.7}$$

where $n \geq 0$ is an integer.

Solution: Outline (fill in the details as an exercise).

Use induction on n, with (3.6) giving the result for $n = 0$.

Apply integration by parts to $\int_0^b e^{-st}t^{k+1}dt$ to reduce the power of t by 1, and then let $b \to \infty$. Then the case $n = k$ will imply the case $n = k+1$. You will also need the result

$$\lim_{b\to\infty} \frac{t^{k+1}}{e^{sb}} = 0 \quad \text{for} \quad s > 0,$$

i.e. an exponential dominates a power as $b \to \infty$. □

The direct evaluation of Laplace transforms using the definition can be time-consuming, and so one seeks short cuts, for example

(i) by using complex functions,

(ii) by developing theorems to give new Laplace transforms from old ones, without extra calculation.

We first illustrate (i). The derivation leading to (3.5) is valid if α is complex, provided that we make one small change:

$$\lim_{r\to\infty} e^{-(s-\alpha)r} = 0 \quad \text{provided that} \quad s > Re(\alpha),$$

where $Re(\alpha)$ is the real part of α. We can now use Euler's formula

$$e^{ibt} = \cos bt + i \sin bt \tag{3.8}$$

and the linearity of $\mathcal{L}$ to calculate $\mathcal{L}[\cos bt]$ and $\mathcal{L}[\sin bt]$ directly from (3.5).

Example: Show that

$$\mathcal{L}[\cos bt] = \frac{s}{s^2+b^2}, \qquad \mathcal{L}[\sin bt] = \frac{b}{s^2+b^2}. \tag{3.9}$$

Solution: By linearity of $\mathcal{L}$ and (3.8):

$$\mathcal{L}[e^{ibt}] = \mathcal{L}[\cos bt] + i\mathcal{L}[\sin bt]. \tag{3.10}$$

The right hand side of (3.5), with $\alpha = ib$ is

$$\frac{1}{s-ib} = \frac{s+ib}{(s-ib)(s+ib)} = \frac{s+ib}{s^2+b^2}. \tag{3.11}$$

Choosing $\alpha = ib$ in (3.5) and substituting (3.10) and (3.11) gives

$$\mathcal{L}[\cos bt + i \sin bt] = \frac{s}{s^2+b^2} + i\frac{b}{s^2+b^2}.$$

Equating real and imaginary parts leads to (3.9). □

We now give a theorem which provides a quick way of calculating $\mathcal{L}[e^{ct} f(t)]$ if $\mathcal{L}[f(t)]$ is known.

First Shift Theorem:

If $\mathcal{L}[f(t)] = F(s)$ exists for $s > a \geq 0$, then

$$\mathcal{L}[e^{ct} f(t)] = F(s-c), \quad \text{for} \quad s > a + c, \tag{3.12}$$

where c is a constant.

Proof: Consider

$$\int_0^r e^{-st} e^{ct} f(t) dt = \int_0^r e^{-(s-c)t} f(t) dt. \tag{3.13}$$

By the hypothesis, $\lim_{r\to\infty} \int_0^r e^{-(s-c)t} f(t) dt$ exists for $s - c > a$, and equals $F(s-c)$. The result follows by letting $r \to \infty$ in (3.13). □

Example: Calculate the Laplace transform of $g(t) = te^{ct}$.

Solution: By (3.7), $\mathcal{L}[t] = \frac{1}{s^2}$ for $s > 0$. Thus by the Shift Theorem,

$$\mathcal{L}[te^{ct}] = \frac{1}{(s-c)^2} \quad \text{for} \quad s > c. \qquad \Box$$

Exercise: Calculate $\mathcal{L}[e^{-2t} \sin \pi t]$.

Answer: $\dfrac{\pi}{(s+2)^2 + \pi^2}$. □

3.1.3 The Laplace transform of a derivative

In using the Laplace transform to solve a linear DE,

$$y' + ky = f(t),$$

where k is a constant, we begin by applying the operator $\mathcal{L}$ to the DE:

$$\mathcal{L}[y' + ky] = \mathcal{L}[f].$$

By linearity of $\mathcal{L}$, we obtain

$$\mathcal{L}[y'] + k\mathcal{L}[y] = \mathcal{L}[f].$$

To proceed further we need to relate $\mathcal{L}[y']$ to $\mathcal{L}[y]$. The next proposition does just that.

Proposition 1:

If H_1 : $f(t) = O(e^{at})$ as $t \to \infty$, for some a,

H_2 : f' is piecewise continuous and f is continuous, on any interval $[0, r]$,

then $\mathcal{L}[f'(t)]$ exists, and

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0), \quad \text{for} \quad s > a. \tag{3.14}$$

Proof: For simplicity, we assume that f' is continuous.
Using integration by parts:

$$\int_0^r e^{-st} f'(t)dt = \left[e^{-sr} f(r) - f(0)\right] - \int_0^r (-s)e^{-st} f(t)dt. \qquad (3.15)$$

By H_1, $| e^{-sr} f(r) | \leq Be^{-(s-a)r}$. Thus, if $s > a$, $\lim_{r\to\infty} e^{-sr} f(r) = 0$.
The result now follows by letting $r \to \infty$ in (3.15). □

Proposition 1 extends in a natural way to the second derivative.

Proposition 2:

If H_1 : $f(t) = O(e^{at})$ and $f'(t) = O(e^{at})$ as $t \to \infty$,
H_2 : f'' is piecewise continuous and f', f are continuous, on any interval $[0, r]$,

then $\mathcal{L}[f''(t)]$ exists, and

$$\mathcal{L}[f''(t)] = s^2 \mathcal{L}[f(t)] - sf(0) - f'(0), \quad \text{for} \quad s > a. \qquad (3.16)$$

Proof: Apply Proposition 1, with f replaced by f', obtaining

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0).$$

Now apply Proposition 1 to the right side directly to get

$$\mathcal{L}[f''(t)] = s\left\{s\mathcal{L}[f(t)] - f(0)\right\} - f'(0). \qquad \square$$

3.1.4 The Inverse Laplace Transform

Referring to the diagram on the first page of this Chapter, the final (and most difficult) step in using the Laplace transform to solve a DE is this:

given a Laplace transform $Y(s)$, find the function $y(t)$.

One should first ask: given $Y(s)$, is there a *unique* $y(t)$?

The next proposition gives the answer YES.

Proposition: If y_1 and y_2 are continuous functions of order e^{at} as $t \to \infty$, then

$$y_1 \neq y_2 \quad \text{implies} \quad \mathcal{L}[y_1] \neq \mathcal{L}[y_2]. \qquad \square$$

Proof: The proof is beyond the scope of this course. See for example, Brauer & Nohel, page 391-2. □

This proposition states that the Laplace transform operator $\mathcal{L}$ is a *one-to-one operator*, and hence has an *inverse operator* $\mathcal{L}^{-1}$ which maps a Laplace transform $Y(s)$ onto the original function $y(t)$:

$$\mathcal{L}^{-1}[F(s)] = f(t) \quad \text{means that} \quad \mathcal{L}[f(t)] = F(s).$$

Equivalently we can write

$$\mathcal{L}^{-1}[\mathcal{L}[f(t)]] = f(t) \quad \text{and} \quad \mathcal{L}[\mathcal{L}^{-1}[F(s)]] = F(s).$$

We shall call $\mathcal{L}^{-1}$ the *inverse Laplace transform operator*, and shall call $f(t) = \mathcal{L}^{-1}[F(s)]$ the *inverse Laplace transform of* $F(s)$. Note that $\mathcal{L}^{-1}$ is also linear. $\square$

In elementary discussions, inverse Laplace transforms are found by referring to a *table of Laplace transforms* – calculating them directly is difficult (and sometimes impossible). Based on the results of Section 3.1.2 we can construct the following Table:

$f(t)$	$F(s) = \mathcal{L}[f(t)]$
$e^{\alpha t}$	$\dfrac{1}{s-\alpha}$
$\cos bt$	$\dfrac{s}{s^2+b^2}$
$\sin bt$	$\dfrac{b}{s^2+b^2}$
t^n	$\dfrac{n!}{s^{n+1}}$
$e^{ct}f(t)$	$F(s-c)$

For example, we can conclude directly from the Table that

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}, \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+\pi^2}\right] = \cos \pi t.$$

In the examples we shall encounter, the Laplace transform $F(s)$ will mostly be a *rational function*, i.e.

$$F(s) = \frac{p(s)}{q(s)},$$

where $p(s)$ and $q(s)$ are polynomial functions. It will be necessary to use *partial fraction methods* to write $F(s)$ as a sum of the simple terms appearing in the Table. The first shift theorem (summarized in the last line of the Table) will also be useful.

Example: Find $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+5s+6}\right]$.

Solution: Using partial fractions

$$\frac{s}{s^2+5s+6} = \frac{s}{(s+2)(s+3)} = \frac{3}{s+3} - \frac{2}{s+2}.$$

Using the Table and linearity of $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+5s+6}\right] = \mathcal{L}^{-1}\left[\frac{3}{s+3}\right] - \mathcal{L}^{-1}\left[\frac{2}{s+2}\right]$$
$$= 3e^{-3t} - 2e^{-2t}. \qquad \square$$

Example: Find $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+4s+5}\right]$.

Solution: Since the denominator does not factor, we complete the square:

$$\frac{s}{s^2+4s+5} = \frac{s}{(s+2)^2+1} = \frac{(s+2)}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}.$$

From the Table (using the Shift Theorem),

$$\mathcal{L}[e^{-2t}\cos t] = \frac{s+2}{(s+2)^2+1}, \quad \mathcal{L}[e^{-2t}\sin t] = \frac{1}{(s+2)^2+1}.$$

It thus follows that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+4s+5}\right] = e^{-2t}\cos t - 2e^{-2t}\sin t. \qquad \square$$

In view of this example it is convenient to restate the First Shift Theorem in terms of the inverse Laplace transform operator $\mathcal{L}^{-1}$.

First Shift Theorem (inverse form):

$$\text{If} \quad \mathcal{L}^{-1}[F(s)] = f(t), \quad \text{then} \quad \mathcal{L}^{-1}[F(s-c)] = e^{ct}f(t).$$

Exercise: Find (i) $\mathcal{L}^{-1}\left[\dfrac{s-3}{s^2-2s+5}\right]$ (ii) $\mathcal{L}^{-1}\left[\dfrac{s+4}{s^2+2s}\right]$

Answer: (i) $e^t(\cos 2t - \sin 2t)$ (ii) $2 - e^{-2t}$.

3.1.5 The Heaviside Step Function

A driving force $f(t)$ [or input function] that changes abruptly with time can be modelled by using a *discontinuous function*, and one that does not change smoothly can be modelled by a *non-differentiable function.*

When working with the Laplace transform of functions such as these it is convenient to introduce the *Heaviside* or *unit step* function $H(t)$, defined by

$$H(t) = \begin{cases} 0, & \text{if} \quad t < 0 \\ 1, & \text{if} \quad t \geq 0. \end{cases} \tag{3.17}$$

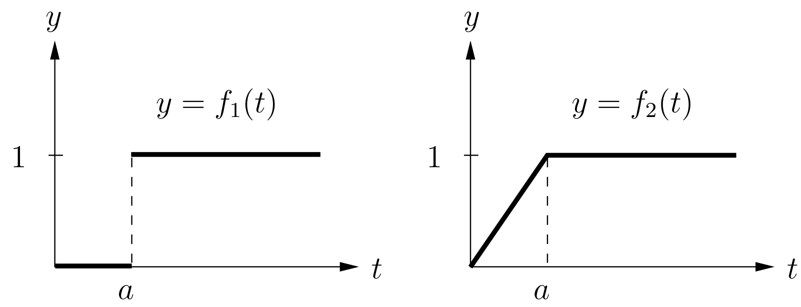


Figure 3.2: a step function $f_1(t)$ a ramp function $f_2(t)$.

Then the above step function $f_1(t)$ is given by

$$f_1(t) = H(t-a), \tag{3.18}$$

(translation to the right).

Example: Express the ramp function $f_2(t)$ above in terms of $H(t)$.

Solution: We have

$$\begin{aligned} f_2(t) &= \tfrac{1}{a}t, \quad H(t-a) = 0 \quad \text{for} \quad 0 \le t < a, \\ f_2(t) &= 1, \quad H(t-a) = 1, \quad \text{for} \quad t \ge a. \end{aligned}$$

Thus

$$f_2(t) = \frac{1}{a}t + \left(-\frac{1}{a}t + 1\right) H(t-a), \quad \text{for} \quad t \ge 0.$$

Thus the *ramp function* $f_2(t)$ can be written

$$f_2(t) = \tfrac{1}{a}[t - (t-a)H(t-a)]. \tag{3.19}$$

□

The next step is to calculate the Laplace transform of $H(t-a)$.

Example: Show that

$$\mathcal{L}[H(t-a)] = \frac{e^{-as}}{s}, \tag{3.20}$$

for $s > 0$, where $a > 0$ is a constant.

Solution: Consider, with $r > a$,

$$\begin{aligned} \int_0^r e^{-st} H(t-a)dt &= \int_a^r e^{-st} dt, \quad \text{since } H(t-a) = 0 \text{ if } t < a, \\ &= -\frac{1}{s}(e^{-sr} - e^{-sa}), \end{aligned}$$

by the Fundamental Theorem of Calculus. Thus, if $s > 0$,

$$\lim_{r\to\infty} \int_0^r e^{-st} H(t-a)dt$$

exists and equals $\frac{1}{s}e^{-sa}$. The result follows by definition of the Laplace transform. □

We now give a theorem which provides a quick way of calculating $\mathcal{L}[H(t-c)f(t-c)]$ if $\mathcal{L}[f(t)]$ is known.

Second Shift Theorem:

If $\mathcal{L}[f(t)] = F(s)$ exists for $s > a \geq 0$, and c is a positive constant, then

$$\mathcal{L}[H(t-c)f(t-c)] = e^{-cs}F(s), \quad s > a. \tag{3.21}$$

Proof: Consider

$$\begin{aligned} \int_0^r e^{-st}H(t-c)f(t-c)dt &= \int_c^r e^{-st}f(t-c)dt, \quad \text{by definition of} \quad H, \\ &= \int_0^{r-c} e^{-s(u+c)}f(u)du, \quad \text{by making a change of variable} \quad u = t-c, \\ &= e^{-cs}\int_0^{r-c} e^{-su}f(u)du. \end{aligned} \tag{3.22}$$

Since $\mathcal{L}[f(t)]$ exists, $\lim_{r\to\infty} \int_0^{r-c} e^{-su}f(u)du$ exists and equals $F(s)$ (note that $r-c \to \infty$, and the integration variable u can be relabelled t). The result follows by letting $r \to \infty$ in (3.22). □

Example: Calculate the Laplace transform of the ramp function (see Figure 3.2)

$$f_2(t) = \tfrac{1}{a}[t - (t-a)H(t-a)].$$

Solution: By linearity of $\mathcal{L}$,

$$\mathcal{L}[f_2(t)] = \tfrac{1}{a}\left\{\mathcal{L}[t] - \mathcal{L}[(t-a)H(t-a)]\right\}.$$

Since $\mathcal{L}[t] = \frac{1}{s^2}$ (see the Table in Section 3.1.4), the Second Shift Theorem gives

$$\mathcal{L}[f_2(t)] = \frac{1}{a}\left[\frac{1}{s^2} - \frac{e^{-as}}{s^2}\right] = \frac{1}{as^2}(1 - e^{-as}). \qquad \square$$

Another classic example is the *saw-tooth function*:

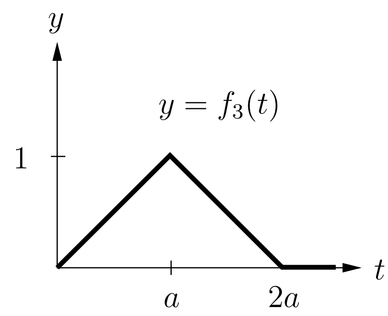


Figure 3.3: The saw-tooth function.

Exercise: Show that

$$f_3(t) = \tfrac{1}{a}[t - 2(t-a)H(t-a) + (t-2a)H(t-2a)].$$

Exercise: Show that

$$\mathcal{L}[f_3(t)] = \frac{1}{a}\left(\frac{1-e^{-as}}{s}\right)^2.$$

The Second Shift Theorem can be restated in terms of the inverse operator $\mathcal{L}^{-1}$, thereby providing a useful tool for calculating inverse Laplace transforms.

Second Shift Theorem (inverse form):

If $\mathcal{L}^{-1}[F(s)] = f(t)$, then $\mathcal{L}^{-1}[e^{-cs}F(s)] = H(t-c)f(t-c)$.

Example: Find $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s+3}\right]$.

Solution: We know $\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$, from the Table. Thus, by the Second Shift Theorem,

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s+3}\right] = H(t-2)e^{-3(t-2)} = \begin{cases} 0, & 0 \le t < 2 \\ e^{-3(t-2)}, & t \ge 2. \end{cases} \qquad \square$$

3.2 Solving DEs using the Laplace Transform

We have now developed the machinery needed to implement the algorithm outlined on the first page of this Chapter. In this Section we give examples to illustrate its use.

3.2.1 First order DEs

Example: Solve the first order initial value problem

$$y' + 3y = 13\sin(2t); \quad y(0) = y_0. \tag{3.23}$$

Solution: Apply the Laplace transform operator $\mathcal{L}$ to the DE and use linearity to obtain

$$\mathcal{L}[y'] + 3\mathcal{L}[y] = 13\mathcal{L}[\sin(2t)]. \tag{3.24}$$

By Proposition 1 in Section 3.1.3,

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0),$$

and by the Table in Section 3.1.4,

$$\mathcal{L}[\sin(2t)] = \frac{2}{s^2+4}.$$

Writing $\mathcal{L}[y(t)] = Y(s)$ as usual, and using the initial condition (3.23), equation (3.24) becomes

$$sY - y_0 + 3Y = \frac{26}{s^2+4}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{y_0}{s+3} + \frac{26}{(s+3)(s^2+4)}. \tag{3.25}$$

The next step is to calculate $y(t) = \mathcal{L}^{-1}[Y(s)]$, the inverse Laplace transform of $Y(s)$. Before applying $\mathcal{L}^{-1}$ to (3.25), we expand the second term in partial fractions (exercise):

$$\frac{26}{(s+3)(s^2+4)} = 2\left[\frac{1}{s+3} - \frac{s}{s^2+4} + \frac{3}{s^2+4}\right]. \tag{3.26}$$

Thus, applying $\mathcal{L}^{-1}$ to (3.25) and using (3.26) gives

$$\mathcal{L}^{-1}[Y(s)] = y_0\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] - 2\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] + 3\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right].$$

Referring to the Table, we get

$$y(t) = (y_0+2)e^{-3t} - 2\cos(2t) + 3\sin(2t).$$

as the solution of the initial value problem (3.23). □

Exercise: Solve the first order initial value problem

$$y' - y = 2e^t, \quad y(0) = y_0.$$

Answer: $y(t) = y_0 e^t + 2te^t$.

HINT: You will need to use the First Shift Theorem with $\mathcal{L}[t] = \frac{1}{s^2}$. □

Here's an example that involves arbitrary parameters.

Exercise: Solve the initial value problem:

$$y' + ky = kA\cos(\omega t); \quad y(0) = y_0.$$

Answer:

$$y(t) = \frac{kA}{k^2+\omega^2}\left[k(\cos\omega t - e^{-kt}) + \omega\sin\omega t\right] + y_0 e^{-kt}.$$

HINT: You will need the following partial fraction expansion:

$$\frac{s}{(s+k)(s^2+\omega^2)} = \frac{1}{k^2+\omega^2}\left[-\frac{k}{s+k} + \frac{ks}{s^2+\omega^2} + \frac{\omega^2}{s^2+\omega^2}\right].$$

Technical comment:

In applying $\mathcal{L}$ to the DE we used Proposition 1 in Section 3.1.3. You may ask: are the hypotheses of the Proposition satisfied in this application? We can be sure that H_2 is satisfied, since the solution of any constant coefficient linear DE with continuous input function is continuous and hence has a continuous derivative ($y' = -ky + f(t)$, both continuous). What about H_1: $y(t) = O(e^{at})$ as $t \to \infty$ for some a? We have no way of verifying this in advance, since $y(t)$ is the unknown function. So we proceed "on faith", and having found the solution, verify after the fact that the hypothesis is satisfied. $\square$

3.2.2 Second order DEs

Example: Solve the second order initial value problem:

$$y'' + 2y' + 2y = 3e^{-t}; \quad y(0) = y_0, \quad y'(0) = 0. \tag{3.27}$$

Solution: Apply the operator $\mathcal{L}$ to the DE and use linearity to obtain

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] + 2\mathcal{L}[y] = 3\mathcal{L}[e^{-t}]. \tag{3.28}$$

As in Section 3.1.3 the transforms of the derivatives are

$$\begin{aligned}\mathcal{L}[y'] &= sY(s) - y(0),\\ \mathcal{L}[y''] &= s^2Y(s) - sy(0) - y'(0),\end{aligned}$$

where $Y(s) = \mathcal{L}[y(t)]$, and the Table gives

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1}, \quad s > -1.$$

Thus (3.28) becomes

$$s^2Y - sy_0 + 2[sY - y_0] + 2Y = \frac{3}{s+1},$$

which is a *linear algebraic equation* for the unknown $Y(s)$. Collecting like terms gives

$$(s^2 + 2s + 2)Y(s) = (s+2)y_0 + \frac{3}{s+1}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{s+2}{s^2+2s+2} y_0 + \frac{3}{(s+1)(s^2+2s+2)}. \tag{3.29}$$

Since $s^2+2s+2 = (s+1)^2+1$, we express both terms in terms of $(s+1)$. First,

$$\frac{s+2}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}. \tag{3.30}$$

Second,

$$\frac{3}{(s+1)(s^2+2s+2)} = \frac{3}{s+1} - \frac{3(s+1)}{(s+1)^2+1}. \tag{3.31}$$

Aside: The simple way to get (3.31) is to observe that

$$\frac{1}{u(u^2+1)} = \frac{1}{u} - \frac{u}{u^2+1},$$

and choose $u = (s+1)$.

We now use the First Shift Theorem, knowing

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \cos t, \quad \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t.$$

It follows from (3.30) and (3.31) that

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{s+2}{s^2+2s+2}\right] &= e^{-t}\cos t + e^{-t}\sin t, \\ \mathcal{L}^{-1}\left[\frac{3}{(s+1)(s^2+2s+2)}\right] &= 3e^{-t} - 3e^{-t}\cos t. \end{aligned} \tag{3.32}$$

We now apply the operator $\mathcal{L}^{-1}$ to (3.29) and use (3.32) to obtain

$$y(t) = \mathcal{L}^{-1}[Y(s)] = y_0 e^{-t}(\cos t + \sin t) + 3e^{-t}(1 - \cos t),$$

as the solution of the initial value problem (3.27). □

Exercise: Solve

$$y'' + 2y' + y = t; \quad y'(0) = 1, \quad y(0) = 0.$$

Answer: $y = -2 + t + 2e^{-t}(1+t)$. □

Comment: The examples in this Section illustrate one advantage of the Laplace transform method, namely that the initial conditions are automatically incorporated.

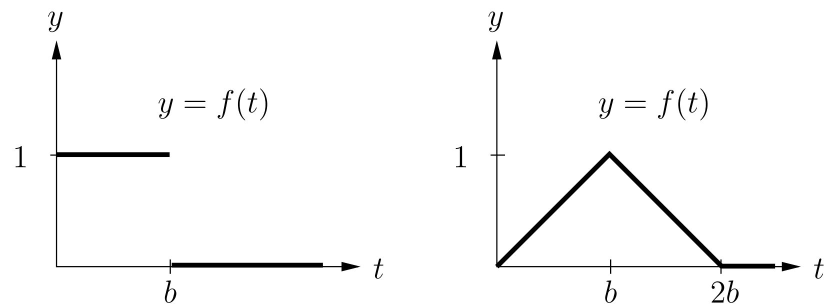


Figure 3.4: A step function and a saw-tooth function.

3.2.3 Discontinuous and Non-differentiable Inputs

A second advantage of the Laplace transform method is that non-smooth input functions e.g. step functions and saw-tooths, can be dealt with easily.

Consider a system (e.g. a mixing tank or electrical circuit) whose state $y(t)$ is governed by the DE

$$y' + y = f(t),$$

where $f(t)$ is the input function (assume that dimensionless variables have been introduced). If the input function is $f(t) = 0$, the solution is

$$y(t) = y(0)e^{-t},$$

showing that the system decays exponentially to its natural equilibrium state $y(t) = 0$. If the input function is a constant, say $f(t) = 1$, the solution is

$$y(t) = 1 + [y(0) - 1]e^{-t},$$

i.e. the system exponentially approaches a new equilibrium state $y(t) = 1$. The question we now ask is: what is the response $y(t)$ of the system if the input function is a step function:

$$f(t) = \begin{cases} 1, & 0 \le t < b \\ 0, & t \ge b, \end{cases} \tag{3.33}$$

i.e. a constant input of 1 for a finite time?

Example: Solve the initial value problem

$$y' + y = f(t); \quad y(0) = y_0, \tag{3.34}$$

where $f(t)$ is the step function (3.33).

Solution: We begin by writing the input function in terms of the Heaviside step function

$$f(t) = 1 - H(t - b). \tag{3.35}$$

Apply the operator $\mathcal{L}$ to the DE (3.34) with (3.35), using linearity to obtain

$$\mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[1] - \mathcal{L}[H(t - b)]. \tag{3.36}$$

We know from equation (3.20) that

$$\mathcal{L}[H(t-b)] = \frac{e^{-bs}}{s}, \quad s > 0$$

and from Section 3.1.3 that

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0).$$

Also, by equation (3.6)

$$\mathcal{L}[1] = \frac{1}{s}.$$

Thus, writing $\mathcal{L}[y] = Y(s)$ as usual, equation (3.36) becomes

$$sY - y_0 + Y = \frac{1}{s} - \frac{e^{-bs}}{s}.$$

Solving for $Y(s)$ leads to

$$Y(s) = \frac{y_0}{s+1} + \frac{1-e^{-bs}}{s(s+1)}. \tag{3.37}$$

In order to find the inverse transform, we first use the partial fraction expansion

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

to write (3.37) in the form

$$Y(s) = \frac{y_0 - 1}{s+1} + \frac{1}{s} - e^{-bs}\left(\frac{1}{s} - \frac{1}{s+1}\right). \tag{3.38}$$

We now use the Second Shift Theorem (Section 3.1.5). We know that

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = 1 - e^{-t},$$

from which follows

$$\mathcal{L}^{-1}\left[e^{-bs}\left(\frac{1}{s} - \frac{1}{s+1}\right)\right] = H(t-b)(1 - e^{-t+b}).$$

Thus, applying $\mathcal{L}^{-1}$ to (3.38) gives

$$y(t) = \mathcal{L}^{-1}[Y(s)] = (y_0 - 1)e^{-t} + 1 - H(t-b)(1 - e^{-t+b}). \tag{3.39}$$

as the solution of the initial value problem (3.34). □

Analysis of the solution:

Without further analysis the solution (3.39) does not give one much insight. The solution can be written in the form

$$y(t) = \begin{cases} 1 + (y_0 - 1)e^{-t}, & 0 \le t < b \\ (y_0 - 1 + e^b)e^{-t}, & t \ge b. \end{cases} \tag{3.40}$$

The derivative is

$$y'(t) = \begin{cases} -(y_0 - 1)e^{-t}, & 0 \leq t < b \\ -(y_0 - 1 + e^b)e^{-t}, & t \geq b. \end{cases} \tag{3.41}$$

Note that $y'(b)$ does not exist, but that $y(t)$ is continuous at $t = b$.

It follows from (3.40) that

$$\lim_{t\to\infty} y(t) = 0 \quad \text{for all values of} \quad y_0,$$

i.e. the system approaches its natural equilibrium state $y(t) = 0$ once the input is switched off at $t = b$. Equation (3.41) shows that the intermediate behaviour depends on y_0, and that there are two special values of y_0, namely

$$y_0 = 1 \qquad \text{and} \qquad y_0 = -e^b + 1.$$

By considering the sign of $y'(t)$, we reach the following conclusions.

- If $y_0 - 1 > 0$, then $y_0 - 1 + e^b > 0$ and $y(t)$ is *decreasing* for $t \geq 0$.
- If $y_0 - 1 + e^b < 0$, then $y_0 - 1 < 0$ and $y(t)$ is *increasing* for $t \geq 0$.
- If $-e^b + 1 < y_0 < 1$, then the response attains a maximum value

  $$y_{\max} = 1 + (y_0 - 1)e^{-b} > 0$$

  at $t = b$.

These results enable us to sketch the graphs in Figure 3.5.

Exercise: Solve $y' + y = f(t)$; $y(0) = y_0$, where $f(t)$ is given in Figure 3.6.
Answer: $y(t) = (y_0 - 1)e^{-t} + 1 - 2H(t-b)(1 - e^{-t+b}) + H(t-2b)(1 - e^{-t+2b})$ □

3.3 Convolution of functions

In this section we introduce the *convolution operation* and discuss the so-called *Convolution Theorem.*

3.3.1 Motivation and definition

In order to motivate the need for the convolution operation we consider the second order DE

$$y'' + a_1 y' + a_0 y = u(t), \tag{3.42}$$

and assume zero initial conditions,

$$y(0) = 0, \qquad y'(0) = 0.$$

Apply the Laplace transform operator $\mathcal{L}$ to the DE and use linearity and equations (3.14) and (3.16). One obtains

$$Y(s)(s^2 + a_1 s + a_0) = U(s)$$

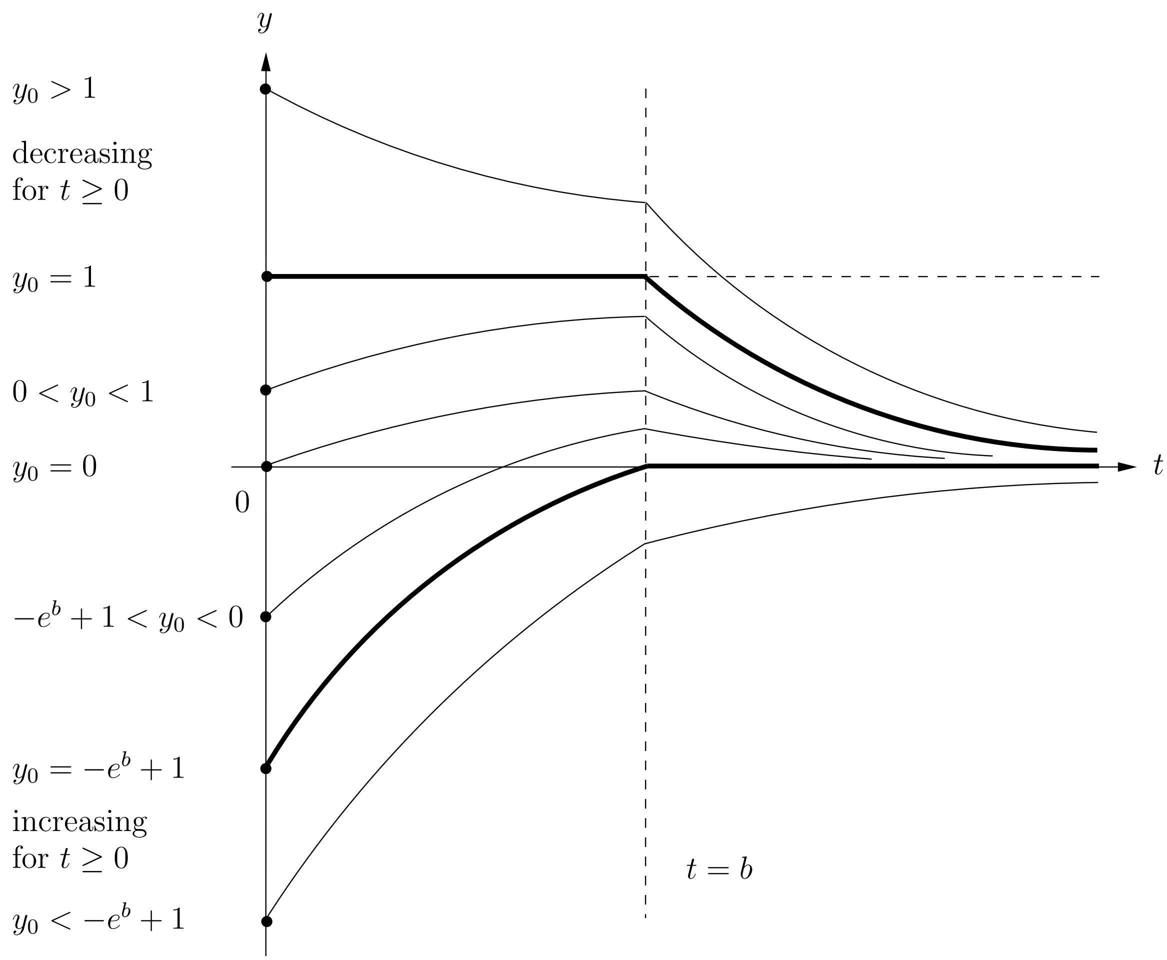


Figure 3.5: The family of solutions (3.40) with initial state y_0 as parameter. The exceptional solutions given by $y_0 = 1$ and $y_0 = -e^b + 1$ are shown in bold.

where $Y(s) = \mathcal{L}[y]$ and $U(s) = \mathcal{L}[u]$. It follows that

$$Y(s) = G(s)U(s), \tag{3.43}$$

where

$$G(s) = \frac{1}{s^2 + a_1 s + a_0}. \tag{3.44}$$

The solution of the DE is obtained by taking the inverse Laplace transform of $Y(s)$ in (3.43),

$$y(t) = \mathcal{L}^{-1}[G(s)U(s)]. \tag{3.45}$$

In order to proceed further, we have to answer the question: how is $\mathcal{L}^{-1}[G(s)U(s)]$ related to $\mathcal{L}^{-1}[G(s)]$ and $\mathcal{L}^{-1}[U(s)]$? An answer to this question will give a formula for the solution $y(t)$, since we know $\mathcal{L}^{-1}[U(s)] = u(t)$ and can calculate $\mathcal{L}^{-1}[G(s)]$.

The preceding question can be formulated in the following equivalent form:

$$\text{if} \qquad \mathcal{L}[f] = F(s) \quad \text{and} \quad \mathcal{L}[g] = G(s),$$

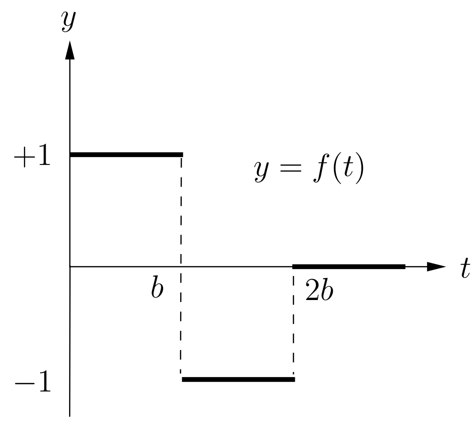


Figure 3.6:

what function $h(t)$ satisfies

$$\mathcal{L}[h] = F(s)G(s) \quad ?$$

Consideration of simple functions shows that $h(t) \neq f(t)g(t)$, as one might initially hope. For example, if

$$f(t) = 1, \qquad g(t) = t,$$

then

$$F(s) = \frac{1}{s}, \qquad G(s) = \frac{1}{s^2},$$

but

$$\mathcal{L}[fg] = \mathcal{L}[t] = \frac{1}{s^2} \neq F(s)G(s).$$

In other words

$$\mathcal{L}[fg] \neq \mathcal{L}[f]\mathcal{L}[g].$$

In order to obtain the correct "product formula" for $\mathcal{L}$, one has to define a new type of product of functions, namely the convolution operation.

Definition: Let f, g be piecewise continuous functions on any interval $0 \leq t \leq r$. The *convolution of* f *and* g, denoted by $f * g$, is defined by

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau, \tag{3.46}$$

for $t \geq 0$.

We illustrate this idea with a simple example.

Example 3.1:

Calculate the convolution $t * 1$, and show that

$$\mathcal{L}[t * 1] = \mathcal{L}[t]\mathcal{L}[1]. \tag{3.47}$$

Solution: By the definition (3.46),

$$\begin{aligned} t * 1 &= \int_0^t (t-\tau)(1)d\tau \\ &= \left(t\tau - \tfrac{1}{2}\tau^2\right)\Big|_{t=0}^{t=\tau} \\ &= \tfrac{1}{2}t^2, \end{aligned}$$

after simplifying. Using the formula

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}},$$

we have

$$\mathcal{L}[t*1] = \mathcal{L}\left[\tfrac{1}{2}t^2\right] = \frac{1}{s^3},$$

and

$$\mathcal{L}[t]\mathcal{L}[1] = \left(\frac{1}{s^2}\right)\left(\frac{1}{s}\right) = \frac{1}{s^3},$$

which gives the desired result (3.47).

This result is a special case of the Convolution Theorem, which we discuss in Section 3.3.2. □

Exercise 3.1:

Calculate the convolution $e^t * 1$, and verify directly that

$$\mathcal{L}[e^t * 1] = \mathcal{L}[e^t]\mathcal{L}[1].$$

Properties of the convolution $f * g$:

As suggested by the notation $f * g$, the convolution of f and g should be regarded as a product operation on functions. It is, of course, completely different from the usual product of f and g, defined by

$$(fg)(t) = f(t)g(t).$$

Nevertheless it does satisfy the same properties as the usual product:

i) $f * g = g * f$ (commutativity)

ii) $f * (g + h) = f * g + f * h$ (distributivity)

iii) $f * (g * h) = (f * g) * h$ (associativity).

Here the operation $+$ is the usual addition of functions. We leave the proofs of i)-iii) as exercises.

3.3.2 The Convolution Theorem

In this Section we discuss the main theorem concerning the convolution operation.

Theorem 3.1 (Convolution Theorem):
If $\mathcal{L}[f] = F(s)$ and $\mathcal{L}[g] = G(s)$ exist for $Re(s) > a$, then

$$\mathcal{L}[f * g] = \mathcal{L}[f]\mathcal{L}[g]. \tag{3.48}$$

Discussion:
Thinking of the Laplace transform $\mathcal{L}$ as an operator that maps a function $f(t)$ on the half-line $0 \leq t < \infty$ (the time domain) onto a function $F(s)$ on the complex s-plane (the frequency domain), one can express the convolution theorem symbolically as follows:

$$\begin{array}{ll} \text{if} & f(t) \xrightarrow{\mathcal{L}} F(s) \quad \text{and} \quad g(t) \xrightarrow{\mathcal{L}} G(s), \\ \text{then} & (f * g)(t) \xrightarrow{\mathcal{L}} F(s)G(s). \end{array}$$

One can describe the theorem colloquially by saying that "convolution in the time domain corresponds to multiplication in the frequency domain", giving another example of the way in which an operation in the frequency domain is simpler than the corresponding operation in the time domain.

The convolution theorem can also be expressed in terms of the inverse Laplace transform.

Inverse form of the Convolution Theorem:

$$\mathcal{L}^{-1}[FG] = \mathcal{L}^{-1}[F] * \mathcal{L}^{-1}[G]. \tag{3.49}$$

We now give the proof of the Convolution Theorem.

Proof: By definition of the convolution operation and of the Laplace transform,

$$\begin{aligned} \mathcal{L}[f * g] &= \int_0^\infty e^{-st} \left[\int_0^t f(t-\tau)g(\tau)d\tau \right] dt \\ &= \int_0^\infty \left[\int_0^t e^{-st} f(t-\tau)g(\tau)d\tau \right] dt. \end{aligned}$$

Reversing the order of integration[2] as indicated in Figure 3.7 gives

$$\mathcal{L}[f * g] = \int_0^\infty \left[\int_\tau^\infty e^{-st} f(t-\tau)g(\tau)dt \right] d\tau.$$

Making the change of variable $t = \tau + r$ in the bracketed integral leads to

[2]Proving that the limits of integration can be reversed for improper double integrals requires a discussion of uniform convergence for improper integrals, and is beyond the scope of this course. We refer to Churchill, Operational Mathematics.

$$
\begin{aligned}
\mathcal{L}[f * g] &= \int_0^\infty \left[\int_0^\infty e^{-s(\tau+r)} f(r) g(\tau) dr\right] d\tau \\
&= \int_0^\infty e^{-s\tau} g(\tau) \left[\int_0^\infty e^{-sr} f(r) dr\right] d\tau \\
&= \mathcal{L}[f]\mathcal{L}[g],
\end{aligned}
$$

the last step following by definition of the Laplace transform. □

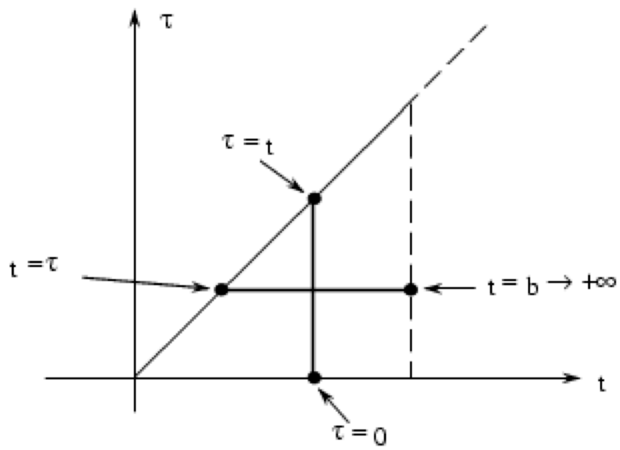


Figure 3.7: The inequalities $0 \leq \tau \leq t$ and $0 \leq t < +\infty$ describe the same region as the inequalities $\tau \leq t < +\infty$ and $0 \leq \tau < +\infty$.

We now return to the problem of solving the DE (3.42), i.e.

$$y'' + a_1 y' + a_0 y = u(t),$$

with zero initial conditions, that we considered as motivation in Section 3.2.1. We obtained the solution in the form (3.45), i.e.

$$y(t) = \mathcal{L}^{-1}[G(s)U(s)],$$

where

$$G(s) = \frac{1}{s^2 + a_1 s + a_0}, \tag{3.50}$$

and $U(s) = \mathcal{L}[u(t)]$. It now follows immediately from the inverse form of the Convolution Theorem (3.49) that

$$y(t) = (g * u)(t), \tag{3.51}$$

where

$$g(t) = \mathcal{L}^{-1}[G(s)]. \tag{3.52}$$

Using the definition (3.46) of the convolution, (3.51) assumes the form

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau. \tag{3.53}$$

We now summarize the result.

Summary:

The unique solution of the initial value problem

$$y'' + a_1 y' + a_0 y = u(t),$$

$$y(0) = 0, \quad y'(0) = 0,$$

is

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau,$$

where

$$g(t) = \mathcal{L}^{-1}[G(s)],$$

and

$$G(s) = \frac{1}{s^2 + a_1 s + a_0}. \qquad \square$$

The form of $g(t)$ depends on the constants a_1 and a_0, as shown in the following example.

It is worth noting that $G(s)$ can be written in the form

$$G(s) = \frac{1}{p(s)}, \tag{3.54}$$

where

$$p(\lambda) = \lambda^2 + a_1\lambda + a_0, \tag{3.55}$$

is the characteristic polynomial of the given DE.

Example 3.2:

Express the solution of the initial value problem

$$y'' + \omega^2 y = u(t),$$

$$y(0) = 0, \quad y'(0) = 0,$$

where the input function $u(t)$ is (piecewise) continuous, as a convolution.

Solution:

We know from (3.43) that the Laplace transform of $y(t)$ has the form

$$Y(s) = G(s)U(s),$$

where $U(s) = \mathcal{L}[u(t)]$ and

$$G(s) = \frac{1}{s^2 + \omega^2}.$$

Using the fact that

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2},$$

it follows that

$$g(t) = \mathcal{L}^{-1}[G(s)] = \frac{1}{\omega} \sin \omega t.$$

By the inverse form of the Convolution Theorem, the solution of the initial value problem is

$$y(t) = (g * u)(t),$$

i.e.

$$y(t) = \frac{1}{\omega} \int_0^t \sin \omega (t - \tau) u(\tau) d\tau. \qquad \square$$

3.4 Application to linear time-invariant systems: The Transfer Function

A *linear time-invariant system* (LTI) is a system whose evolution in time is described by a linear DE with constant coefficients.

In Section 3.2.1 we showed that for a scalar DE

$$y'' + a_1 y' + a_0 y = u(t),$$

with initial conditions

$$y(0) = 0, \quad y'(0) = 0,$$

the Laplace transform $Y(s) = \mathcal{L}[y(t)]$ of the solution, i.e., the response of the physical system, is given by

$$Y(s) = G(s)U(s), \tag{3.56}$$

where $U(s) = \mathcal{L}[u(t)]$ is the Laplace transform of the input, and

$$G(s) = \frac{1}{s^2 + a_1 s + a_0}. \tag{3.57}$$

The solution itself is given as a convolution

$$y(t) = (g * u)(t),$$

where

$$g(t) = \mathcal{L}^{-1}[G(s)],$$

(see (3.51)). Written in full using the definition of convolution the solution is

$$y(t) = \int_0^t g(t - \tau) u(\tau) d\tau. \tag{3.58}$$

Equation (3.58) gives the dependence of the response on the input in the t-domain (the *time-domain*), while (3.56) gives this dependence in the s-domain (called the *frequency domain*). The key point is that this dependence is particularly simple in the s-domain, i.e.

multiplication by the function $G(s)$, which is directly determined by the coefficients of the DE through equation (3.57). This function is called the *transfer function* of the physical system.

More generally, the right hand side of the DE will be a linear combination of $u(t)$ and its derivatives; the transfer function is defined using (3.56) as

$$G(s) = \frac{Y(s)}{U(s)}$$

which will be a rational function (see problems 23 and 24 in problem set 3).

The importance of the transfer function, in addition to the simplicity of the relation (3.56), lies in the fact that knowing $G(s)$ one can obtain information about the behaviour of the physical system. An important example is to obtain the response of the system to a sinusoidal input

$$u(t) = Re[\alpha e^{i\omega t}], \tag{3.59}$$

of angular frequency ω and amplitude α. We consider a trial solution

$$y(t) = Re[\mathcal{A}e^{i\omega t}], \tag{3.60}$$

where $\mathcal{A}$ is a complex number which will depend on the frequency ω. On substituting the complex input $\alpha e^{i\omega t}$ and complex trial solution $\mathcal{A}e^{i\omega t}$ into the DE, we obtain

$$\mathcal{A}(\omega)\left[(i\omega)^2 + a_1(i\omega) + a_0\right] = \alpha,$$

after cancelling a common factor $e^{i\omega t}$. On replacing s by $i\omega$ in (3.57) we see that the preceding equation can be written in the form

$$\mathcal{A}(\omega) = G(i\omega)\alpha,$$

where

$$G(i\omega) = \frac{1}{(i\omega)^2 + a_1(i\omega) + a_0}. \tag{3.61}$$

We can express $G(i\omega)$ in terms of its magnitude and argument

$$G(i\omega) = \rho(\omega)e^{i\delta(\omega)}. \tag{3.62}$$

Substituting (3.62) into (3.60) gives the solution

$$y(t) = \alpha\rho(\omega)\cos\left[\omega t + \delta(\omega)\right]. \tag{3.63}$$

This equation, with (3.62) and (3.61), shows how the amplitude $\rho(\omega)$ and phase shift $\delta(\omega)$ of the response depends on the frequency ω.

The classic example is the DE

$$y'' + 2\lambda y' + \omega_0^2 y = u(t), \tag{3.64}$$

where λ and ω are positive constants, which can represent a mechanical oscillator (see page 44) or an electrical RLC circuit (page 45). The transfer function is thus

$$G(s) = \frac{1}{s^2 + 2\lambda s + \omega_0^2}. \tag{3.65}$$

It is customary to plot $20\log_{10}\rho$ versus $\log_{10}\omega$, and δ versus $\log_{10}\omega$. The quantity

$$dB = 20\log_{10}\rho$$

gives a decibel measure of the amplitude. These graphs are shown in Figure 3.8.

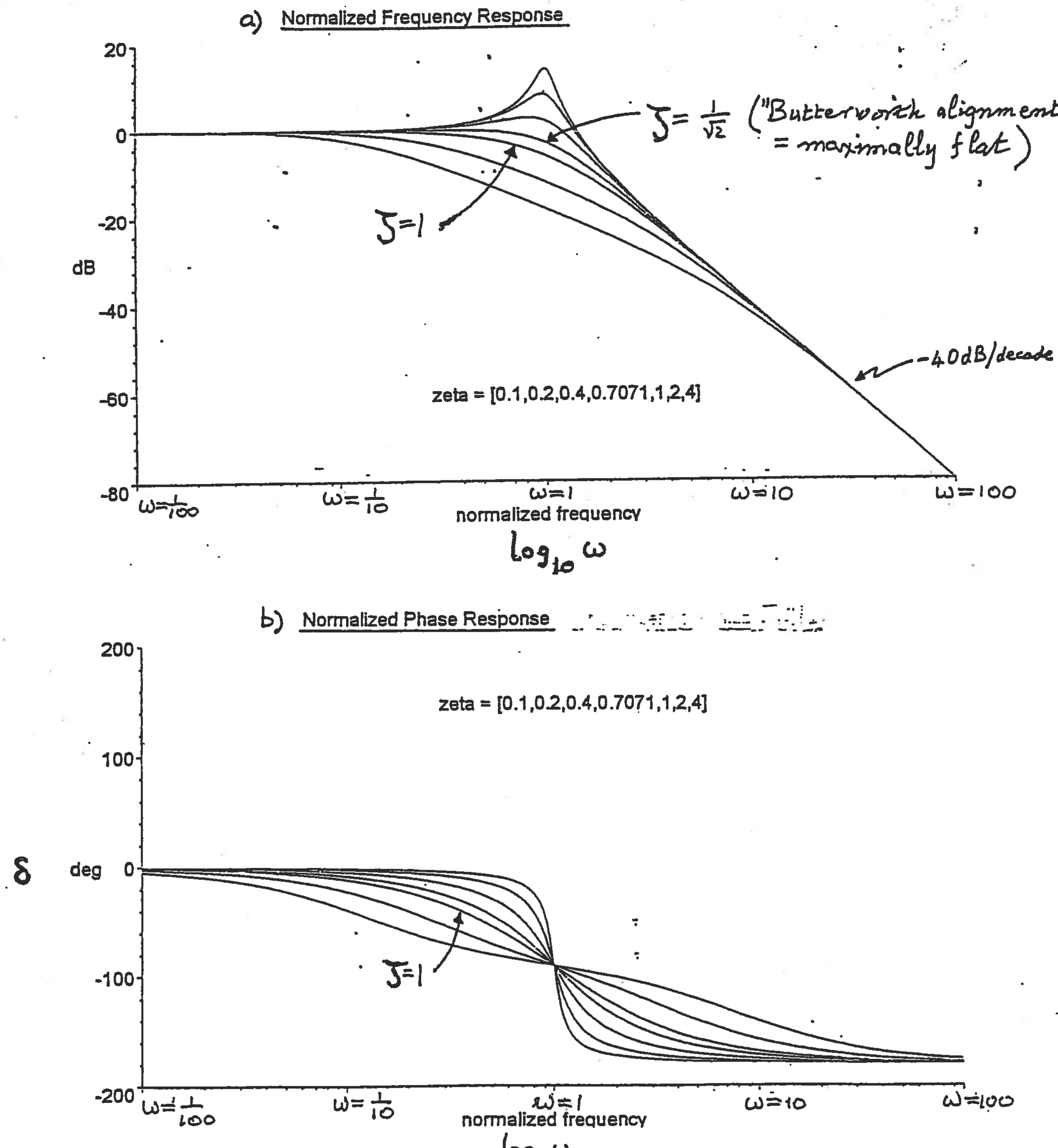


Figure 3.8: Frequency response showing $dB = 20\log_{10}\rho$ versus $\log_{10}\omega$ and phase response showing δ versus $\log_{10}\omega$.

3.5 Response to an impulsive input

In this section we give an introduction to the notion of an *impulsive force*, which can be thought as a very large force that acts over a very short time interval, for example the impact of a club on a golf ball. More generally, one can think of an *impulsive input* to any linear time-invariant system, for example, an electrical circuit. Our goal is to be able to determine the response of the system to an impulsive input.

3.5.1 Impulsive forces of finite duration

Suppose that an impulsive force acts, during the time interval $t_1 \leq t \leq t_2$ on a particle of mass m in linear motion. Newton's Second Law reads

$$\frac{d}{dt}(mv) = f(t),$$

which when integrated from t_1 to t_2 yields

$$mv(t_2) - mv(t_1) = \int_{t_1}^{t_2} f(t)dt.$$

Thus the principal effect of the impulsive force $f(t)$ is a sudden change in momentum, given by the integral

$$\int_{t_1}^{t_2} f(t)dt$$

called the *impulse of the force over the interval* $t_1 \leq t \leq t_2$.

As a mathematical model of an impulsive force of unit impulse acting at time $t = 0$ we take the function $\delta_\varepsilon(t)$, defined by

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & \text{if} \quad 0 \leq t < \varepsilon \\ 0, & \text{otherwise.} \end{cases} \tag{3.66}$$

The translate $\delta_\varepsilon(t-a)$ represents an impulsive force acting at time $t = a$. The graphs of these functions are shown in Figure 3.9.

Thinking of $\delta_\varepsilon(t-a)$ is a one-parameter family of functions labelled by $\varepsilon > 0$, we ask whether the limit $\lim\limits_{\varepsilon \to 0^+} \delta_\varepsilon(t-a)$ defines a function. Referring to Figure 3.9, we see that

$$\lim_{\varepsilon \to 0^+} \delta_\varepsilon(t-a) = \begin{cases} 0 & \text{if} \quad t \neq a \\ +\infty & \text{if} \quad t = a. \end{cases} \tag{3.67}$$

Thus the limit does not define a function. However the limit of the integral of the impulse function is well-defined, since[3]

$$\int_0^\infty \delta_\varepsilon(t-a)dt = 1$$

for all $\varepsilon > 0$ and $a \geq 0$.

[3] From Figure 3.9 the area under the graph of $y = \delta_\varepsilon(t-a)$ equals 1, for any $\varepsilon > 0$.

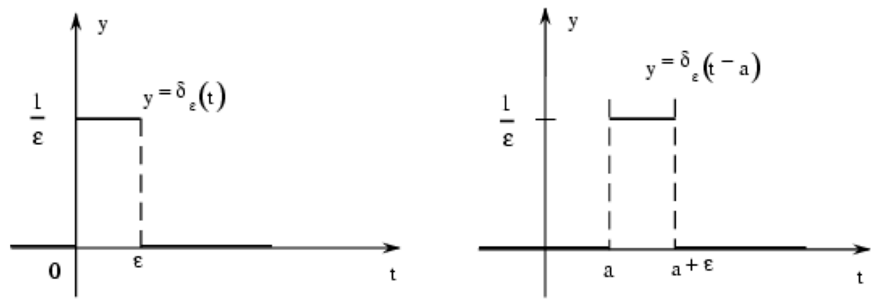


Figure 3.9: Graphs of the impulse functions $\delta_\varepsilon(t)$ and $\delta_\varepsilon(t-a)$.

More generally we have the following result.

Proposition 3.2:
If g is continuous on some neighbourhood of a, then

$$\lim_{\varepsilon \to 0^+} \int_0^\infty g(t)\delta_\varepsilon(t-a)dt = g(a), \tag{3.68}$$

where $\delta_\epsilon(t)$ is the impulse function (3.66).

Proof: By (3.66)

$$\int_0^\infty g(t)\delta_\varepsilon(t-a)dt = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} g(t)dt.$$

For ε sufficiently close to zero, we can apply the mean value theorem for integrals, obtaining

$$\frac{1}{\varepsilon}\int_a^{a+\varepsilon} g(t)dt = g(\bar{t}),$$

where $a < \bar{t} < a + \varepsilon$. Since g is continuous at a,

$$\lim_{\varepsilon \to 0^+} g(\bar{t}) = g(a),$$

and the result follows. □

We now solve a problem involving an impulsive force in order to motivate the future developments.

Example 3.3:
Find the zero state response of an undamped mass-spring system to a constant impulsive force of duration ε and total impulse p acting at time $t = 0$. Show the limit of the response exists as $\varepsilon \to 0^+$.

Solution:

The equation of motion is

$$my'' + ky = p\delta_\varepsilon(t),$$

with

$$y(t) = 0 \quad \text{and} \quad y'(t) = 0 \quad \text{for} \quad t \le 0.$$

Dividing by m gives

$$y'' + \omega^2 y = \frac{p}{m}\delta_\varepsilon(t). \tag{3.69}$$

Using the result of Example 3.2, with input

$$u_\varepsilon(t) = \frac{p}{m}\delta_\varepsilon(t),$$

we get[4]

$$y_\varepsilon(t) = \frac{p}{m\omega}\int_0^t \sin\,\omega(t-\tau)\delta_\varepsilon(\tau)d\tau. \tag{3.70}$$

By (3.66)

$$y_\varepsilon(t) = \frac{p}{m\omega\varepsilon}\begin{cases} \displaystyle\int_0^\varepsilon \sin\omega(t-\tau)d\tau, & \text{if} \quad t \ge \varepsilon \\ \displaystyle\int_0^t \sin\omega(t-\tau)d\tau, & \text{if} \quad 0 \le t < \varepsilon. \end{cases}$$

Evaluating the integrals gives

$$y_\varepsilon(t) = \frac{p}{m\omega^2\varepsilon}\begin{cases} \cos\omega(t-\varepsilon) - \cos\omega t, & \text{if} \quad t \ge \varepsilon \\ 1 - \cos\omega t, & \text{if} \quad 0 \le t < \varepsilon. \end{cases} \tag{3.71}$$

The first and second derivatives are

$$y'_\varepsilon(t) = \frac{p}{m\omega\varepsilon}\begin{cases} \sin\omega t - \sin\omega(t-\varepsilon), & \text{if} \quad t \ge \varepsilon \\ \sin\omega t, & \text{if} \quad 0 \le t < \varepsilon \end{cases} \tag{3.72}$$

and

$$y''_\varepsilon(t) = \frac{p}{m\varepsilon}\begin{cases} \cos\omega t - \cos\omega(t-\varepsilon), & \text{if} \quad t > \varepsilon \\ \cos\omega t, & \text{if} \quad 0 < t < \varepsilon. \end{cases} \tag{3.73}$$

We now show that $\lim_{\varepsilon\to 0^+} y_\varepsilon(t)$ exists for any $t > 0$. Given $t > 0$ we choose ε close enough to zero to ensure that $0 < \varepsilon < t$. Then the first line in (3.71) applies. By l'Hopital's theorem,

$$\lim_{\varepsilon\to 0^+} \frac{\cos\omega(t-\varepsilon) - \cos\omega t}{\omega\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{\omega\sin\omega(t-\varepsilon)}{\omega} = \sin\omega t.$$

It follows that[5]

$$\lim_{\varepsilon\to 0^+} y_\varepsilon(t) = \frac{p}{m\omega}\sin\omega t,$$

[4]We denote the solution by $y_\varepsilon(t)$ to emphasize the dependence on ε.

[5]This result can be obtained more quickly by applying Proposition 3.2 to equation (3.70).

for any $t > 0$. We thus regard the function

$$y(t) = \frac{p}{m\omega} \sin \omega t, \quad t > 0, \tag{3.74}$$

with

$$y(t) = 0, \qquad t < 0, \tag{3.75}$$

as the response of the mass-spring system to an (instantaneous) impulse of magnitude p/m per unit mass.

The graphs of $y_\varepsilon(t)$ and $y(t)$ and their first two derivatives are shown in Figure 3.10. We note that $y_\varepsilon(t)$ is of class C^1, i.e. $y'_\varepsilon(t)$ is continuous at $t = 0$, while $y(t)$ is only continuous: observe that $y'(t)$ has a jump discontinuity at $t = 0$. The idealized impulse (i.e. $\varepsilon \to 0^+$) transfers momentum instantaneously to the mass at time $t = 0$. The function $y(t)$ does not describe the physical behaviour of the acceleration at $t = 0$, which is given by the limit of $y''_\varepsilon(t)$ as $\varepsilon \to 0^+$. Evaluating $y''_\varepsilon(t)$ at the centre of the ε-interval and letting $\varepsilon \to 0^+$ gives

$$\lim_{\varepsilon \to 0^+} y''_\varepsilon \left(\frac{\varepsilon}{2}\right) = \lim_{\varepsilon \to 0^+} \left(\frac{p}{m}\right) \frac{\cos \frac{1}{2}\omega\varepsilon}{\varepsilon} = +\infty.$$

This singular behaviour is indicated by the vertical arrow at $t = 0$ in the graph of $y''(t)$ in Figure 3.10.

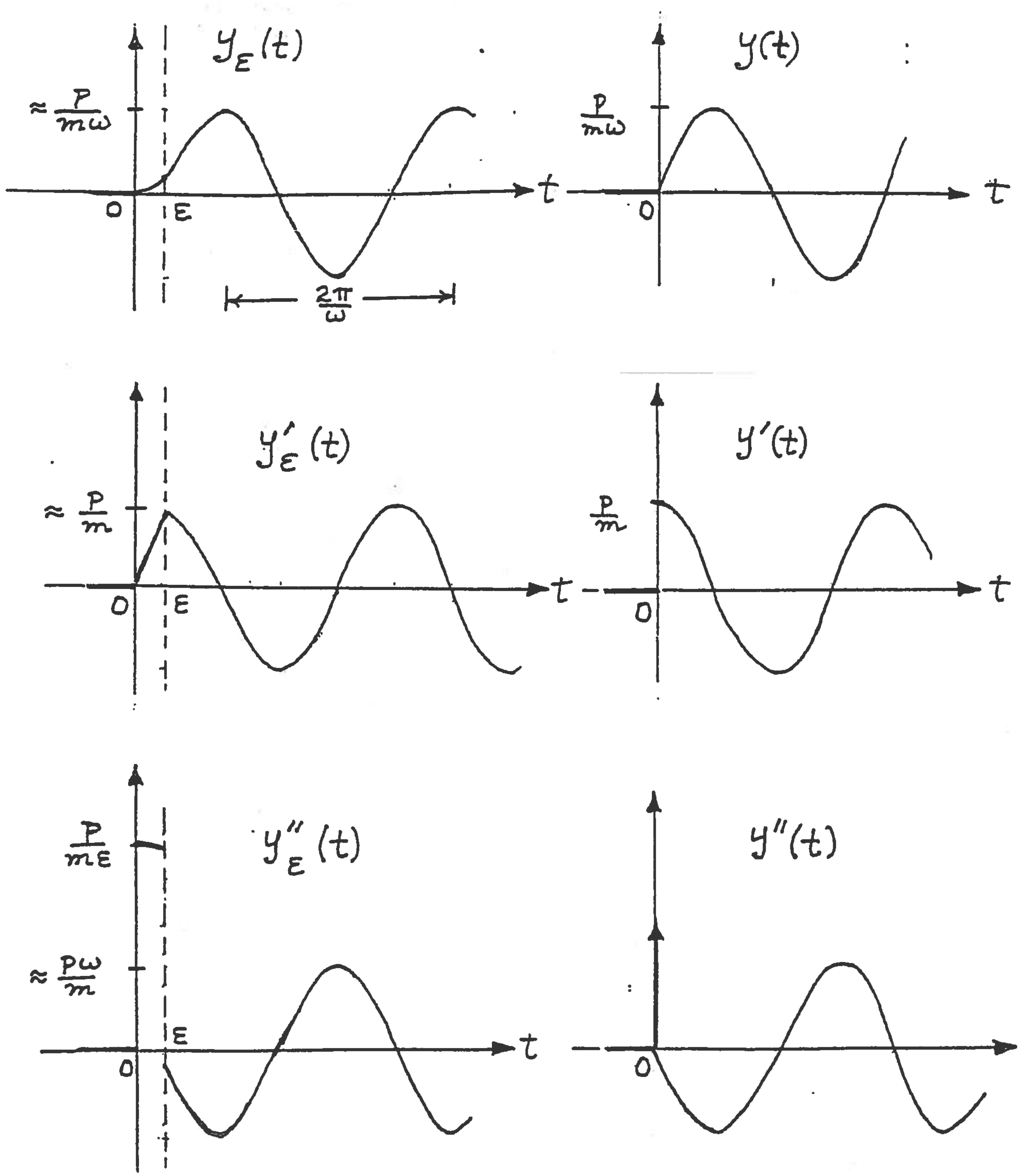


Figure 3.10: Response of an undamped mass-spring system to an impulsive force of impulse p/m per unit mass, a) for finite duration $\varepsilon > 0$ and b) in the limit as $\varepsilon \to 0$.

3.5.2 The Dirac delta symbol

In this section we develop the formal procedure that is used for calculating the response of a linear time-invariant system to an impulsive input, in the limit as $\varepsilon \to 0^+$, where ε is the duration of the impulse.

We have seen that the family of impulse functions $\delta_\varepsilon(t-a)$, defined by (3.66), satisfy (3.67), i.e.

$$\lim_{\varepsilon \to 0^+} \delta_\varepsilon(t-a) = \begin{cases} 0 & \text{if} \quad t \neq a \\ +\infty & \text{if} \quad t = a. \end{cases}$$

Although this limit does not exist it is convenient to represent it by the symbol

$$\delta(t-a),$$

called the *Dirac delta symbol*. Using this notation we write (3.68) symbolically as

$$\int_0^\infty g(t)\delta(t-a)dt = g(a), \tag{3.76}$$

where g is any continuous function, and $a \geq 0$. In particular, for $g(t) = 1$,

$$\int_0^\infty \delta(t-a)dt = 1,$$

for $a \geq 0$.

Physically we think of $\delta(t)$ as representing an idealized impulsive force with unit impulse and zero duration, acting at time $t = 0$. More generally, one can think of $\delta(t-a)$ as an impulsive input of unit magnitude, acting at time $t = a$. Thinking in these terms, we write the DE (3.69) in the limit $\varepsilon \to 0^+$ symbolically as

$$y'' + \omega^2 y = \frac{p}{m}\delta(t). \tag{3.77}$$

It is natural to ask whether one can solve this equation to obtain (3.74) by using the Laplace transform. Equation (3.76) with $g(t) = e^{-st}$ reads

$$\int_0^\infty e^{-st}\delta(t-a)dt = e^{-sa},$$

which motivates the following *definition of the Laplace transform of the Dirac delta symbol*:

$$\mathcal{L}[\delta(t-a)] = e^{-as}, \tag{3.78}$$

for $a \geq 0$. In particular, for $a = 0$,

$$\mathcal{L}[\delta(t)] = 1. \tag{3.79}$$

We now rework Example 3.3 using the Dirac delta symbol.

Example 3.4:

Find the zero state response of an undamped mass-spring system to an impulse of magnitude p/m per unit mass, acting at time $t = 0$.

Solution:

The equation of motion is (3.77), i.e.

$$y'' + \omega^2 y = \frac{p}{m}\delta(t),$$

with

$$y(0) = 0 = y'(0). \tag{3.80}$$

Take the Laplace transform of the DE using (3.16), (3.79) and (3.80) to obtain

$$s^2 Y(s) + \omega^2 Y(s) = \frac{p}{m},$$

where $Y(s) = \mathcal{L}[y(t)]$. Solving for $Y(s)$ gives

$$Y(s) = \left(\frac{p}{m}\right)\frac{1}{s^2+\omega^2}.$$

Taking the inverse Laplace transform and recalling

$$\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2} \tag{3.81}$$

gives

$$y(t) = \frac{p}{m\omega}\sin\omega t,$$

for $t > 0$, the same expression as found by solving the problem with $\varepsilon > 0$ and then taking the limit as $\varepsilon \to 0^+$.

Example 3.5:

An undamped mass-spring system of natural frequency ω is initially at rest. At each time $t = 0, \frac{\pi}{\omega}, \frac{2\pi}{\omega}, \ldots$ the mass is struck with a hammer, which imparts an impulse of magnitude p per unit mass in the positive direction. Determine the resulting motion.

Solution: The DE is

$$y'' + \omega^2 y = p\sum_{n=0}^{\infty}\delta\left(t - \frac{n\pi}{\omega}\right), \tag{3.82}$$

with initial conditions

$$y(0) = 0, \quad y'(0) = 0.$$

Applying the Laplace transform as in Example 3.6 and using (3.78) gives

$$s^2 Y(s) + \omega^2 Y(s) = p\sum_{n=0}^{\infty} e^{-\frac{n\pi s}{\omega}}$$

and hence

$$Y(s) = p\sum_{n=0}^{\infty}\frac{e^{-\frac{n\pi s}{\omega}}}{s^2+\omega^2}. \tag{3.83}$$

By the Second Shift Theorem and equation (3.81),

$$\mathcal{L}^{-1}\left[\frac{e^{-\frac{n\pi s}{\omega}}}{s^2+\omega^2}\right] = \frac{1}{\omega}H\left(t-\frac{n\pi}{\omega}\right)\sin(\omega t - n\pi).$$

Thus applying $\mathcal{L}^{-1}$ to (3.83) gives the response in the form

$$y(t) = \left(\frac{p}{\omega}\right)\sum_{n=0}^{\infty} H\left(t-\frac{n\pi}{\omega}\right)\sin(\omega t - n\pi).$$

Since $\sin(\omega t - n\pi) = (-1)^n \sin\omega t$ and $H\left(t-\frac{n\pi}{\omega}\right) = 0$ for $t < \frac{n\pi}{\omega}$, it follows that if $\frac{n\pi}{\omega} < t < \frac{(n+1)\pi}{\omega}$, then

$$y(t) = \frac{p}{\omega}\left[\sin\omega t - \sin\omega t + \sin\omega t + \cdots + (-1)^n \sin\omega t\right],$$

i.e.

$$y(t) = \begin{cases} \frac{p}{\omega}\sin\omega t, & \text{if} \quad n \quad \text{is even} \\ 0, & \text{if} \quad n \quad \text{is odd.} \end{cases}$$

The graph of the response $y(t)$ is shown in Figure 3.11. The physical interpretation is as follows: the first blow starts the mass moving in the positive direction; just as it returns to the origin the second blow stops it dead; it remains at rest until the third blow sets it in motion again, and so on.

□

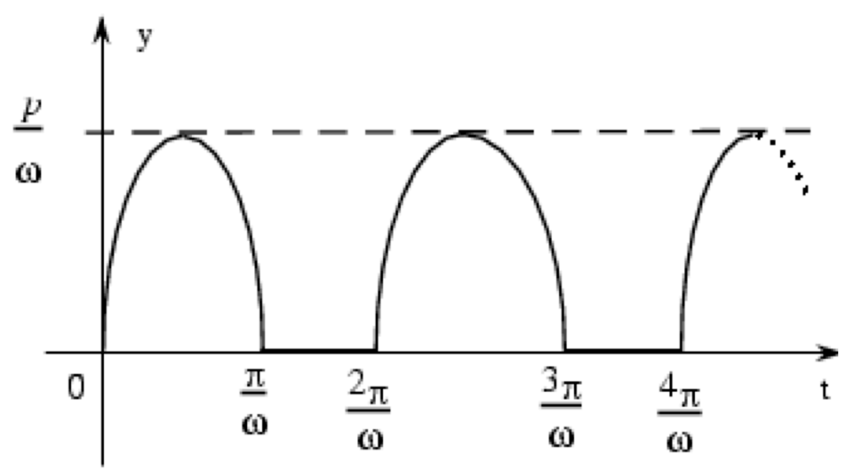


Figure 3.11: Response $y(t)$ from example 3.5.

Warning:

It should be kept in mind that the calculations in Section 3.5.2 are of a formal nature, since the Dirac delta symbol $\delta(t-a)$ is not a well-defined mathematical quantity: the limit (3.67) does not exist and hence $\delta(t-a)$ *is not a function*. Thus, statements involving $\delta(t-a)$, have to be interpreted as limits. In particular

$$\text{“}\mathcal{L}[\delta(t)] = 1\text{”} \quad \text{means} \quad \lim_{\varepsilon\to 0^+} \mathcal{L}[\delta_\varepsilon(t)] = 1,$$

$$\text{“}\int_0^\infty g(t)\delta(t-a)dt = g(a)\text{”} \quad \text{means} \quad \lim_{\varepsilon\to 0^+} \int_0^\infty g(t)\delta_\varepsilon(t-a)dt = g(a),$$

and

$$\text{“}y(t) \quad \text{is a solution of the initial value problem}$$
$$y'' + \omega^2 y = \delta(t), \qquad y(0) = 0, \qquad y'(0) = 0\text{”},$$

means

$$y(t) = \lim_{\varepsilon\to 0^+} y_\varepsilon(t), \quad \text{where} \quad y_\varepsilon(t) \quad \text{is a solution of}$$
$$y'' + \omega^2 y = \delta_\varepsilon(t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

The final results of the calculations are, however, mathematically valid, since they can be derived rigorously by using the pulse function $\delta_\varepsilon(t-a)$ and then taking the limit as $\varepsilon \to 0^+$ once the solution of the DE has been found. We illustrated this process in Examples 3.3 and 3.4, first giving the rigorous method with $\varepsilon > 0$, and then giving the quick formal solution.

We finally note that the Dirac delta symbol has been placed on a sound mathematical footing[6] by introducing the class of “generalized functions” or “distributions”. In this setting one can write statements such as

$$H'(t-a) = \delta(t-a),$$

formally capturing the intuitive idea that the unit step function has an “infinite rate of change” at the step.

3.5.3 The unit impulse response

We have seen that in the frequency domain the response $Y(s)$ is related to the input $U(s)$ in a particularly simple way, namely, multiplication by the transfer function $G(s)$:

$$Y(s) = G(s)U(s). \tag{3.84}$$

We have seen in Section 3.5.2 that it is useful to consider an *idealized unit impulse*, denoted by $\delta(t)$, as the input, since the response is mathematically simpler than the response to an impulse of duration ε. Using the definition

$$\mathcal{L}[\delta(t)] = 1,$$

we get $U(s) = 1$, and so (3.84) reduces to

$$Y(s) = G(s).$$

[6]By Laurent Schwarz, in the 1950's.

Thus the response in the time domain to the idealized unit impulse $\delta(t)$ is

$$y(t) = \mathcal{L}^{-1}[G(s)] = g(t),$$

for $t > 0$. The function $g(t)$ is thus called the *unit impulse response* of the linear time-invariant system.

In the time domain, the response $y(t)$ to an input $u(t)$ is given by

$$y(t) = (g * u)(t) = \int_0^t g(t-\tau)u(\tau)d\tau. \tag{3.85}$$

Thus (3.85) shows that *the unit impulse response* $g(t)$ *determines the response to an arbitrary input* $u(t)$ via the convolution operation.

Example 3.6: Consider the undamped mass-spring system with equation of motion

$$y'' + \omega^2 y = u(t)$$

with $y(0) = 0 = y'(0)$. Following example 3.4, we see that the response to a unit impulse $u(t) = \delta(t)$ is

$$y_{\text{impulse}}(t) = g(t) = \frac{1}{\omega}\sin\omega t$$

(just set $\frac{p}{m} = 1$). Find the **step response**, i.e. the response to a unit step (Heaviside) input $u(t) = H(t)$.

Solution: By (3.85), the response is

$$\begin{aligned}
y_{\text{step}}(t) &= \int_0^t g(t-\tau)u(\tau)d\tau \\
&= \int_0^t \frac{1}{\omega}\sin\omega(t-\tau)H(\tau)d\tau \\
&= \frac{1}{\omega}\int_0^t \sin\omega(t-\tau)d\tau \\
&= \frac{1}{\omega^2}\cos\omega(t-\tau)\Big|_0^t \\
&= \frac{1}{\omega^2}(1-\cos\omega t).
\end{aligned}$$

Chapter 4

Linear Vector Differential Equations

In this Chapter we generalize the concept of differential equations to the case where the unknown is a vector function in $\mathbb{R}^2$. We begin by discussing two familiar physical systems from a different point of view, in order to motivate the idea.

4.1 Introduction

4.1.1 Coupled Mixing Tanks

Consider a system of two coupled mixing tanks each of volume V, with flow rates as shown, and constant inflow concentration c. Let $m_1(t)$ and $m_2(t)$ denote the mass of chemical in the two tanks at time t, respectively. Then the mass balance equation (1.6) applied to each tank separately leads to the two DEs

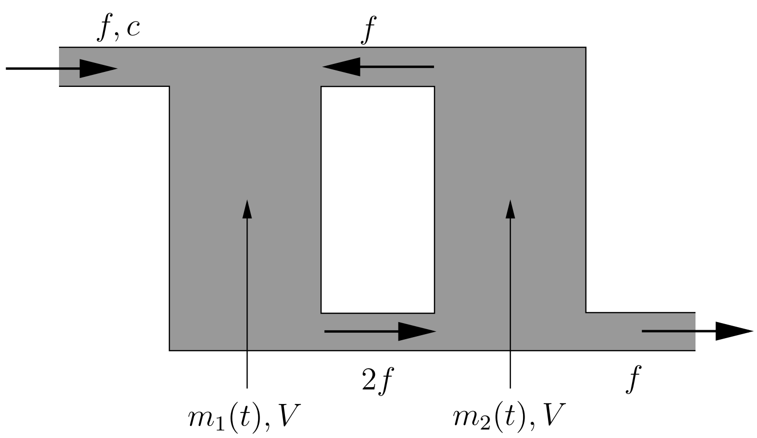


Figure 4.1: A system of two coupled mixing tanks.

$$m_1' = -2\frac{f}{V}m_1 + \frac{f}{V}m_2 + cf$$
$$m_2' = 2\frac{f}{V}m_1 - 2\frac{f}{V}m_2,$$

where $'$ denotes differentiation with respect to t. Fill in the details as an exercise, referring to Section 1.1.2, if necessary. We describe the *state of this system* at time t by the vector

$$\mathbf{x}(t) = \begin{pmatrix} m_1(t) \\ m_2(t) \end{pmatrix} \in \mathbb{R}^2.$$

The two scalar DEs can be written as one vector DE

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}' = \begin{pmatrix} -2\frac{f}{V} & \frac{f}{V} \\ 2\frac{f}{V} & -2\frac{f}{V} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} cf \\ 0 \end{pmatrix}, \tag{4.1}$$

using the 2×2 coefficient matrix. Using the state vector $\mathbf{x}$ we write (4.1) in vector notation as

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}, \tag{4.2}$$

where

$$A = \begin{pmatrix} -2\frac{f}{V} & \frac{f}{V} \\ 2\frac{f}{V} & -2\frac{f}{V} \end{pmatrix}, \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} cf \\ 0 \end{pmatrix}.$$

We refer to (4.2) as a *first order vector DE in* $\mathbb{R}^2$. □

Comment: The notation used in equation (4.2) makes sense in $\mathbb{R}^n$. For example, one can imagine a system of 3 coupled mixing tanks leading to a vector DE in $\mathbb{R}^3$.

4.1.2 The mechanical oscillator

The motion of a damped mass-spring system with applied force $mf(t)$ is described by the second order DE

$$y'' + 2\lambda y' + \omega_0^2 y = f(t), \tag{4.3}$$

Note that the damping parameter is $\lambda = \frac{c}{2m}$ and the natural frequency is $\omega_0^2 = \frac{k}{m}$ where k is the spring constant and c is the damping constant (see section 2.3.1).

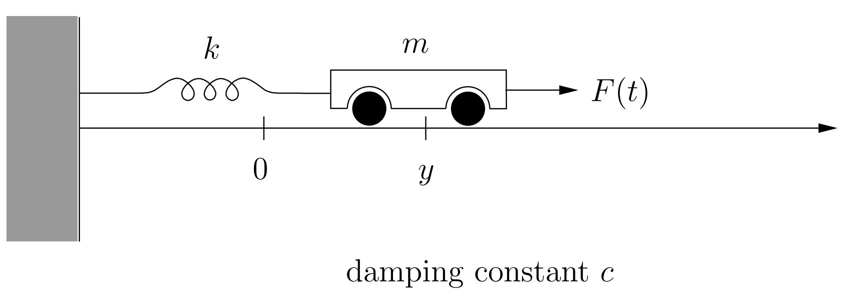


Figure 4.2: A damped mass-spring system.

To describe the state of the system at an instant of time it is not enough to give the *displacement* y: one has also to give the *velocity* $\dfrac{dy}{dt}$.

So we introduce the state vector

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \in \mathbb{R}^2,$$

with components

$$x_1 = y \quad \text{and} \quad x_2 = y', \tag{4.4}$$

where $'$ denotes differentiation with respect to t.

But what vector DE does the state vector satisfy? Well, from (4.4),

$$x_1' = y' = x_2,$$

and from (4.3) and (4.4)

$$\begin{aligned} x_2' = y'' &= -2\lambda y' - \omega_0^2 y + f(t) \\ &= -2\lambda x_2 - \omega_0^2 x_1 + f(t), \end{aligned}$$

Collecting the results, we have

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\omega_0^2 x_1 - 2\lambda x_2 + f(t). \end{aligned}$$

In vector form this reads

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}, \tag{4.5}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\lambda \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} 0 \\ f \end{pmatrix}. \qquad \square$$

Comment: One can obtain vector DEs in higher dimensions in this context. The system shown in figure 5.3 will have a state vector

$$\mathbf{x} = \begin{pmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{pmatrix} \in \mathbb{R}^4.$$

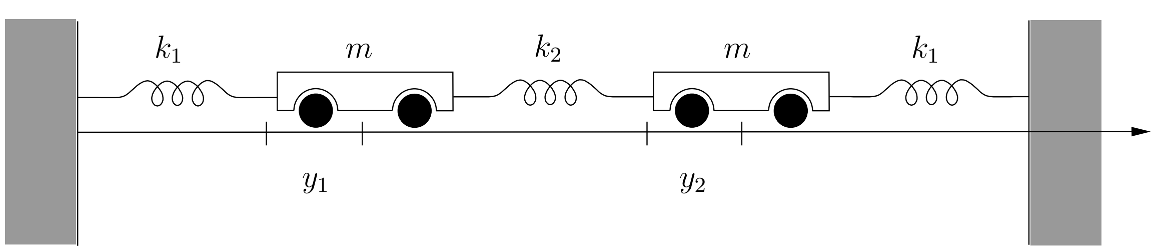


Figure 4.3: A two-mass oscillator.

Exercise: The procedure leading from the second order linear DE (4.3) to the linear vector DE (4.5) can be used to write *any* second order linear DE as a vector DE. Show that

$$y'' + py' + qy = 0 \tag{4.6}$$

is equivalent to

$$\mathbf{x}' = A\mathbf{x}, \quad \text{with} \quad \mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}. \tag{4.7}$$

4.1.3 Overview

From a mathematical point of view the object of study in this chapter is a *linear vector DE* of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}(t),$$

or more concisely,

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t). \tag{4.8}$$

Here

$$\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{R}^2 \tag{4.9}$$

is the *unknown vector-valued function*,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{4.10}$$

is the 2×2 *coefficient matrix*, whose entries are constants, and

$$\mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \tag{4.11}$$

is a given vector-valued function. The *initial condition* is of the form

$$\mathbf{x}(0) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{4.12}$$

The DE (4.8) should be thought of as describing the *evolution in time* of a physical system (e.g. coupled mixing tanks or mechanical oscillator). The unknown vector-valued function $\mathbf{x}$ is the *state vector* of the system. The constant coefficient matrix A describes the *internal characteristics* of the system (e.g. flow rate, damping), and the vector-valued function $\mathbf{f}(t)$ describes the *external input* to the system.

The system can be represented symbolically in a so-called *block diagram* (see figure 5.4).

The goal is to determine the state $\mathbf{x}(t)$ of the system at time t (the "output"), given the input $\mathbf{f}(t)$ and initial state $\mathbf{x}(0) = \mathbf{a}$. In this Chapter we shall develop algorithms to solve this problem.

Finally we note that it is sometimes helpful to write a vector DE (4.8) in *component form*. Using (4.9)-(4.11), (4.8) can be written as

$$\begin{cases} x_1' &= a_{11}x_1 + a_{12}x_2 + f_1(t) \\ x_2' &= a_{21}x_1 + a_{22}x_2 + f_2(t), \end{cases} \tag{4.13}$$

which is referred to as a *system of linear DEs*. Indeed the terms "linear vector DE" and "system of linear DEs" have the same meaning.

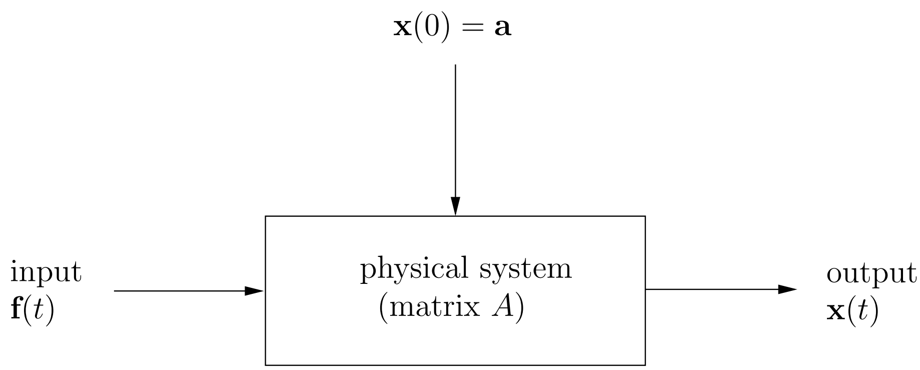


Figure 4.4: Block diagram for the physical system described by the linear vector DE $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$.

4.1.4 Linearity and Superposition

We shall begin by considering the case of zero external input, in which case the DE (4.7) reduces to

$$\mathbf{x}' = A\mathbf{x}, \tag{4.14}$$

referred to as a *homogeneous* linear vector DE (the zero function $\mathbf{x}(t) = \mathbf{0}$ is a solution).

As in the case of a homogeneous linear DE for a scalar (see Section 2.2.1), the Principle of Superposition holds for a homogeneous linear vector DE.

Proposition: If $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are solutions of

$$\mathbf{x}' = A\mathbf{x},$$

then

$$c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$$

is also a solution for any constant scalars c_1 and c_2.

Proof: We are given that

$$\mathbf{x}^{(1)\prime} = A\mathbf{x}^{(1)} \quad \text{and} \quad \mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)}.$$

Multiply the first by c_1, the second by c_2, and add, using the matrix property

$$A(c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}) = c_1 A\mathbf{x}^{(1)} + c_2 A\mathbf{x}^{(2)},$$

to get

$$(c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)})' = A(c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}),$$

as required. □

For a linear vector DE (4.14) in $\mathbb{R}^2$, the general solution must contain *two* arbitrary constants because the initial condition (4.12) contains two arbitrary constants, i.e. the components of the vector $\mathbf{a}$. We can thus state:

the *general solution of the homogeneous linear vector DE in* $\mathbb{R}^2$,

$$\mathbf{x}' = A\mathbf{x},$$

is of the form

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t), \tag{4.15}$$

where $\mathbf{x}^{(1)}$ *and* $\mathbf{x}^{(2)}$ *are two linearly independent solutions of the DE, and* c_1, c_2 *are arbitrary constants.*

4.2 Solving vector DEs using eigenvectors

4.2.1 The method

Consider a *homogeneous* linear vector DE in $\mathbb{R}^2$:

$$\mathbf{x}' = A\mathbf{x}. \tag{4.16}$$

As discussed in Section 5.1.4 we need to obtain two linearly independent solutions. We consider a trial function containing an *exponential* (surprise!)

$$\mathbf{x} = e^{\lambda t}\mathbf{v}, \tag{4.17}$$

where $\mathbf{v} \in \mathbb{R}^2$ is a constant vector, and λ is a constant scalar (which may be complex).

The derivative of (4.17) is

$$\mathbf{x}' = \lambda e^{\lambda t}\mathbf{v}.$$

Substituting in (4.16) gives

$$\lambda e^{\lambda t}\mathbf{v} = A(e^{\lambda t}\mathbf{v}) = e^{\lambda t}A\mathbf{v}.$$

Multiply by the scalar $e^{-\lambda t}$ obtaining

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{4.18}$$

This equation states that the scalar λ in the trial function (4.17) is an *eigenvalue* of the coefficient matrix A, and $\mathbf{v}$ is an associated *eigenvector*.

The eigenvalues can be found by rewriting (4.18) in the form

$$(A - \lambda I)\mathbf{v} = \mathbf{0},$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the 2×2 identity matrix. This equation will have a non-zero solution for $\mathbf{v}$ iff the matrix $A - \lambda I$ is non-invertible i.e., the inverse matrix $(A - \lambda I)^{-1}$ does not exist. This is the case iff the determinant of $A - \lambda I$ is zero. We let

$$h(\lambda) = \det(A - \lambda I). \tag{4.19}$$

Then the eigenvalues of A are the roots of the equation

$$h(\lambda) = 0, \tag{4.20}$$

which is called the *characteristic equation* of A. The function $h(\lambda)$ defined by (4.19) is in fact a polynomial of degree two:

$$h(\lambda) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}.$$

Here we used the usual formula for the determinant of a 2×2 matrix. The function $h(\lambda)$ is called the *characteristic polynomial* of the matrix A.

If λ is an eigenvalue of A (i.e. a solution of (4.20)) then (4.18) will have a non-zero solution for $\mathbf{v}$, and (4.17) will be a solution of the DE (4.16). There are three cases:

A) Unequal real eigenvalues

B) Complex eigenvalues

C) Equal real eigenvalues,

which we illustrate with examples in the rest of this Section. In case A) we immediately get two linearly independent solutions of the DE (4.16) (since eigenvectors associated with distinct eigenvalues are linearly independent). In case B) we have to take real and imaginary parts of the solutions, while case C) requires special treatment.

4.2.2 Unequal real eigenvalues

Example 1: Find the general solution of the vector DE

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{pmatrix} -4 & 1 \\ -2 & -1 \end{pmatrix}. \tag{4.21}$$

Solution: We consider a trial function

$$\mathbf{x} = e^{\lambda t}\mathbf{v}. \tag{4.22}$$

Substituting (4.22) in (4.21) leads to

$$(A - \lambda I)\mathbf{v} = 0. \tag{4.23}$$

Thus λ must satisfy

$$h(\lambda) = 0,$$

where the characteristic polynomial is

$$\begin{aligned} h(\lambda) = \det(A - \lambda I) &= \det \begin{pmatrix} -4 - \lambda & 1 \\ -2 & -1 - \lambda \end{pmatrix} \\ &= (-4 - \lambda)(-1 - \lambda) - (-2)(1) \\ &= \lambda^2 + 5\lambda + 6 \\ &= (\lambda + 2)(\lambda + 3). \end{aligned}$$

Thus the eigenvalues of A are

$$\lambda = -2 \quad \text{and} \quad -3.$$

First, considering $\lambda = -2$, equation (4.23) becomes

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

leading to

$$\begin{aligned} -2v_1 + v_2 &= 0 \\ -2v_1 + v_2 &= 0, \end{aligned}$$

which requires that $v_2 = 2v_1$ (the two equations are identical). In other words, any vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix}$$

with $v_1 \neq 0$, is an eigenvector of A associated with the eigenvalue $\lambda = -2$. Choosing $v_1 = 1$, it follows from (4.22) that

$$\mathbf{x} = e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{4.24}$$

is a solution of the DE (4.21).

Second, considering $\lambda = -3$, equation (4.23) becomes

$$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which implies $v_1 = v_2$. In other words, any vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix},$$

with $v_1 \neq 0$, is an eigenvector of A with eigenvalue $\lambda = -3$. Choosing $v_1 = 1$ it follows from (4.22) that

$$\mathbf{x} = e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.25}$$

is a solution of the DE (4.21).

Finally, since the vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are linearly independent, the solutions (4.24) and (4.25) are linearly independent. Thus by Section 5.1.4 (see equation (4.15)) the general solution of the DE (4.21) is

$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{4.26}$$

where c_1 and c_2 are arbitrary constants. □

Initial conditions:

The constants c_1 and c_2 are determined by the initial condition

$$\mathbf{x}(0) = \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Setting $t = 0$ in (4.26) gives

$$\mathbf{a} = \begin{pmatrix} c_1 + c_2 \\ 2c_1 + c_2 \end{pmatrix},$$

which can be solved to give

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_2 - a_1 \\ 2a_1 - a_2 \end{pmatrix}. \tag{4.27}$$

Two special cases will be important later, namely $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. From (4.26) and (4.27) we obtain:

$$\mathbf{x} = \begin{pmatrix} -e^{-2t} + 2e^{-3t} \\ -2e^{-2t} + 2e^{-3t} \end{pmatrix}, \qquad \text{with} \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.28}$$

and

$$\mathbf{x} = \begin{pmatrix} e^{-2t} - e^{-3t} \\ 2e^{-2t} - e^{-3t} \end{pmatrix}, \qquad \text{with} \qquad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.29}$$

□

4.2.3 Complex eigenvalues

Example 2: Find the general solution of the vector DE

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix}. \tag{4.30}$$

Solution: We consider a trial function

$$\mathbf{x} = e^{\lambda t}\mathbf{v}. \tag{4.31}$$

As in Example 1, λ must be an eigenvalue of A i.e. λ must be a solution of

$$h(\lambda) = \det(A - \lambda I) = 0,$$

and $\mathbf{v}$ must be an eigenvector, i.e. a non-zero solution of

$$(A - \lambda I)\mathbf{v} = \mathbf{0}. \tag{4.32}$$

The characteristic equation is

$$h(\lambda) = \det \begin{pmatrix} 1-\lambda & 5 \\ -1 & -3-\lambda \end{pmatrix} = (\lambda+1)^2 + 1,$$

after simplifying and completing the square. Setting $h(\lambda) = 0$, we obtain the eigenvalues

$$\lambda = -1+i, \quad -1-i.$$

Choosing $\lambda = -1+i$, equation (4.32) becomes

$$\begin{pmatrix} 2-i & 5 \\ -1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

leading to

$$\begin{aligned} (2-i)v_1 + 5v_2 &= 0, \\ -v_1 - (2+i)v_2 &= 0. \end{aligned}$$

These two equations are essentially the same, since multiplying the second by $-(2-i)$ yields the first. Thus

$$v_2 = -\tfrac{1}{5}(2-i)v_1,$$

and choosing $v_1 = 5$ gives

$$\mathbf{v} = \begin{pmatrix} 5 \\ -2+i \end{pmatrix}$$

as a solution of equation (4.32) in the case $\lambda = -1+i$.

Then, using (4.31), we obtain

$$\mathbf{x} = e^{(-1+i)t} \begin{pmatrix} 5 \\ -2+i \end{pmatrix} \tag{4.33}$$

as a *complex* solution of the DE (4.30). The real and imaginary parts of this solution are themselves solutions of (4.30). Using Euler's formula we decompose (4.33) into its real and imaginary parts:

$$\begin{aligned} \mathbf{x} &= e^{-t}(\cos t + i \sin t) \left[\begin{pmatrix} 5 \\ -2 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\ &= e^{-t} \left[\left\{ \cos t \begin{pmatrix} 5 \\ -2 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} + i \left\{ \sin t \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \right] \\ &= e^{-t} \begin{pmatrix} 5\cos t \\ -2\cos t - \sin t \end{pmatrix} + i e^{-t} \begin{pmatrix} 5 \sin t \\ \cos t - 2 \sin t \end{pmatrix}. \end{aligned}$$

These two solutions are linearly independent since one is not a multiple of the other. Thus, by equation (4.15), the general solution of the DE (4.30) is

$$\mathbf{x} = e^{-t} \left[c_1 \begin{pmatrix} 5\cos t \\ -2\cos t - \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5 \sin t \\ \cos t - 2 \sin t \end{pmatrix} \right], \tag{4.34}$$

where c_1 and c_2 are arbitrary constants. □

4.2.4 Equal real eigenvalues

Example 3: Find the general solution of the vector DE

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}. \tag{4.35}$$

Solution: We consider a trial function

$$\mathbf{x} = e^{\lambda t}\mathbf{v}. \tag{4.36}$$

As in Example 1, λ must be a solution of

$$h(\lambda) = \det(A - \lambda I) = 0,$$

and $\mathbf{v}$ must be a non-zero solution of

$$(A - \lambda I)\mathbf{v} = \mathbf{0}. \tag{4.37}$$

The characteristic equation is

$$h(\lambda) = \det\begin{pmatrix} -3-\lambda & 4 \\ -1 & 1-\lambda \end{pmatrix} = (\lambda+1)^2,$$

after simplifying, and so we have equal eigenvalues

$$\lambda_1 = \lambda_2 = -1.$$

Equation (4.37) becomes

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

giving $-v_1 + 2v_2 = 0$. Choosing $v_1 = 2$ gives

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{4.38}$$

and so by (4.36),

$$\mathbf{x} = e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{4.39}$$

is a solution of the DE (4.35).

We are now faced with the problem of finding a second linearly independent solution. In the past we have found that in such situations "multiplying by t" was a good thing to do. So we consider

$$\mathbf{x} = te^{-t}\mathbf{v}$$

as a new trial function ("plan A"). When substituted in the DE (4.35), this choice leads to a contradiction. We presumably need "more constants" in the trial function. So we try "plan B", a trial function of the form

$$\mathbf{x} = te^{-t}\mathbf{v} + e^{-t}\mathbf{w}. \tag{4.40}$$

Differentiating and simplifying gives

$$\mathbf{x}' = e^{-t}[-t\mathbf{v} + (\mathbf{v} - \mathbf{w})].$$

Substitute in the DE (4.35) and multiply by e^t to obtain

$$-t\mathbf{v} + (\mathbf{v} - \mathbf{w}) = \mathbf{tAv} + \mathbf{Aw},$$

which must hold for all t. Equating coefficients gives

$$A\mathbf{v} = -\mathbf{v} \quad \text{and} \quad A\mathbf{w} = \mathbf{v} - \mathbf{w},$$

i.e.

$$(A + I)\mathbf{v} = 0 \quad \text{and} \quad (A + I)\mathbf{w} = \mathbf{v}.$$

We can use (4.38) as a solution of the first equation, i.e. $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Then the second equation becomes

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solving this linear system gives $w_1 = -1$, $w_2 = 0$. Thus, the trial function (4.40) gives the solution

$$\mathbf{x} = e^{-t} \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right]. \tag{4.41}$$

of the DE (4.35).

The solutions (4.39) and (4.41) are linearly independent by inspection. Thus by equation (4.15) the general solution of the DE (4.35) is

$$\mathbf{x} = c_1 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\},$$

which can be rearranged as

$$\mathbf{x} = e^{-t} \left[(c_1 + tc_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right]. \qquad \square$$

Exercises:

1) Solve $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$.

   Answer: $\quad \mathbf{x} = c_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

2) Solve $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$.

   Answer: $\quad \mathbf{x} = e^{-t} \left[c_1 \begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix} \right]$

3) Solve $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{x}$.

   Answer: $\quad \mathbf{x} = e^{3t} \left[(c_1 + tc_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]. \qquad \square$

4.3 Orbits of a vector DE in state space

In this Section we show how to sketch the solutions of the homogeneous,linear vector DE

$$\mathbf{x}' = A\mathbf{x}$$

as a family of curves in the state space $\mathbb{R}^2$. These curves are called the *orbits of the DE.* The goal is to use this picture of the solutions (sometimes called the "phase portrait") to understand the behaviour of the underlying physical system (e.g. oscillator or coupled mixing tanks):

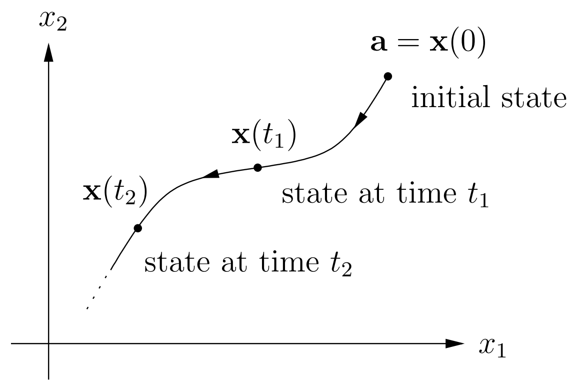


Figure 4.5: An orbit of the DE $\mathbf{x}' = A\mathbf{x}$.

Oscillator:

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

Coupled mixing tanks:

$$\mathbf{x} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

The evolution of a physical system described by the DE $\mathbf{x}' = A\mathbf{x}$ is thus represented by a *moving point* $\mathbf{x}(t)$ *in the state space* $\mathbb{R}^2$, i.e. as time passes, the point $\mathbf{x}(t)$ moves along the orbit, as shown in Figure 5.5. From the shape of the orbit one can draw various conclusions e.g. are the individual components x_1 and x_2 increasing or decreasing, do they attain a maximum or minimum etc.?

We have seen that the second order oscillator DE with zero driving force, i.e.

$$y'' + 2\lambda y' + \omega_0^2 y = 0$$

can be written as a first order vector DE

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \in \mathbb{R}^2, \tag{4.42}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\lambda \end{pmatrix}, \tag{4.43}$$

and λ is the damping parameter. We are going to sketch the orbits for two values of λ, one for which the system is overdamped $\left(\lambda = \frac{5}{4}\omega_0 > \omega_0\right)$ and one for which the system is underdamped $\left(\lambda = \frac{3}{5}\omega_0 < \omega_0\right)$.

4.3.1 Phase portraits: unequal real eigenvalues

Example 1: Give a qualitative sketch of the orbits of the vector DE

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{pmatrix} \tag{4.44}$$

in the state space $\mathbb{R}^2$.

Solution: It is best to use the form of the solution obtained using the eigenvector method. One obtains (exercise):

$$\mathbf{x} = c_1 e^{-\frac{1}{2}t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \tag{4.45}$$

where c_1 and c_2 are arbitrary constants. This equation determines a solution for any initial state $\mathbf{x}(0) = \mathbf{a}$, and thus determines an orbit through each point of $\mathbb{R}^2$. To sketch the orbits we rely on three properties of the family of solutions.

1) Exceptional solutions:

There are *three* exceptional solutions in the family (4.45). First, the equilibrium solution

$$\mathbf{x} = \mathbf{0},$$

given by $c_1 = 0 = c_2$. Second, the solution given by $c_2 = 0$, $c_1 \neq 0$, i.e.

$$\mathbf{x} = c_1 e^{-\frac{1}{2}t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{4.46}$$

Third, the solution given by $c_1 = 0$, $c_2 \neq 0$, i.e.

$$\mathbf{x} = c_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{4.47}$$

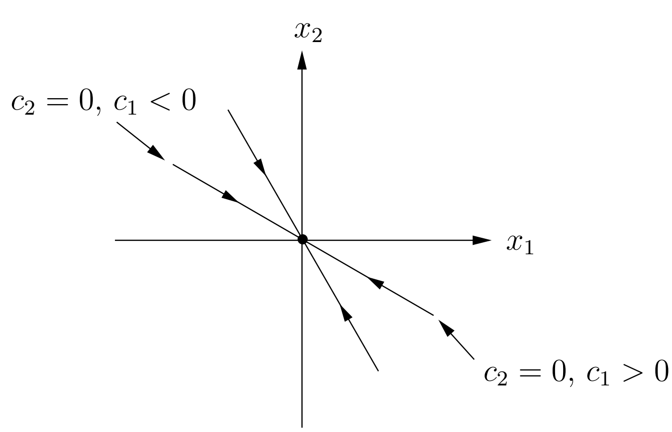


Figure 4.6: Three exceptional solutions.

The solutions (4.46) and (4.47) are represented by straight lines in the state space: eliminating t gives $x_2 = -\frac{1}{2}x_1$ for (4.46) and $x_2 = -2x_1$ for (4.47).

2) Asymptotic behaviour as $t \to +\infty$:

It follows immediately from (4.45) that

$$\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{0},$$

for all values of the constants c_1 and c_2. This result means that *every orbit approaches the equilibrium orbit* $\mathbf{x} = \mathbf{0}$ *as* $t \to +\infty$. Moreover, since e^{-2t} is small compared to $e^{-\frac{1}{2}t}$ as $t \to +\infty$, we can write

$$\mathbf{x} \approx c_1 e^{-\frac{1}{2}t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{as} \quad t \to +\infty,$$

i.e. any solution with $c_1 \neq 0$ is approximated by the exceptional solution (4.46) as $t \to \infty$. This result means that all orbits with $c_1 \neq 0$ approach the origin along the line $x_2 = -\frac{1}{2}x_1$ i.e. this line "attracts" other orbits as they approach the origin. This property of the orbits is shown in Figure 5.8.

3) Sign of the slope of the orbits:

In component form the DE (4.44) reads

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - \tfrac{5}{2}x_2. \end{aligned}$$

Keeping in mind that the vector $\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$ is tangent to the orbit $\mathbf{x} = \mathbf{x}(t)$, and that their slope is $\frac{dx_2}{dx_1} = \frac{x_2'}{x_1'}$, we conclude that

(i) the tangent line to an orbit is *vertical* whenever an orbit crosses the line $x_2 = 0$ (i.e. $x_1' = 0$), called the *vertical isocline*,

(ii) the tangent line to an orbit is *horizontal* whenever an orbit crosses the line $x_2 = -\frac{2}{5}x_1$ (i.e. $x_2' = 0$), called the *horizontal isocline*, and

(iii) the slope of an orbit is *positive* when it lies in the region defined by

$$-\tfrac{2}{5}x_1 < x_2 < 0,$$

or

$$0 < x_2 < -\tfrac{2}{5}x_1,$$

i.e. the shaded region in Figure 5.7.

The information obtained under 1)–3) leads to Figure 5.8.

Comment: The phase portrait for any DE $x' = Ax$ with unequal negative real eigenvalues will have the general form shown in Figure 5.8. The specific shape will depend on the

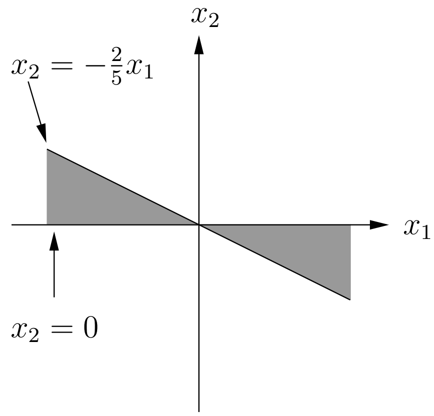


Figure 4.7: Region in which the slope of an orbit is positive.

location of the attracting orbit and of the horizontal and vertical isoclines. If the matrix A is diagonal, the horizontal and vertical isoclines coincide with the exceptional solutions giving the simple picture shown in Figure 5.9.

Physical interpretation of the phase portrait:

In example 1, the state vector $\mathbf{x}$ is given by $\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$, where y is the displacement of the trolley and y' is its velocity. Thus, when an orbit cuts the horizontal axis ($x_2 = 0$) the velocity is zero and when an orbit cuts the vertical axis ($x_1 = 0$) the displacement is zero. In Figure 5.8, the dark orbit represents the situation in which the trolley is given a negative velocity ($x_2 < 0$) initially, after which it comes momentarily to rest ($x_2 = 0$) as it reaches maximum (negative) displacement; then, returning to the equilibrium position, it reaches maximum velocity then slows down to a state of rest.

One can see from Figure 5.8 that in order for "overshoot" to occur the initial state must lie in a restricted region in state space, namely the region between the vertical axis and the line $x_2 = -2x_1$, i.e.

$$\begin{array}{llll} y > 0 & \text{and} & y' < -2y, & \text{or} \\ y < 0 & \text{and} & y' > -2y. & \end{array} \qquad \square$$

4.3.2 Phase portraits: complex eigenvalues

We begin by considering an example with complex eigenvalues that is simple algebraically.

Example 2: Give a qualitative analysis of the orbits of the DE

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}. \tag{4.48}$$

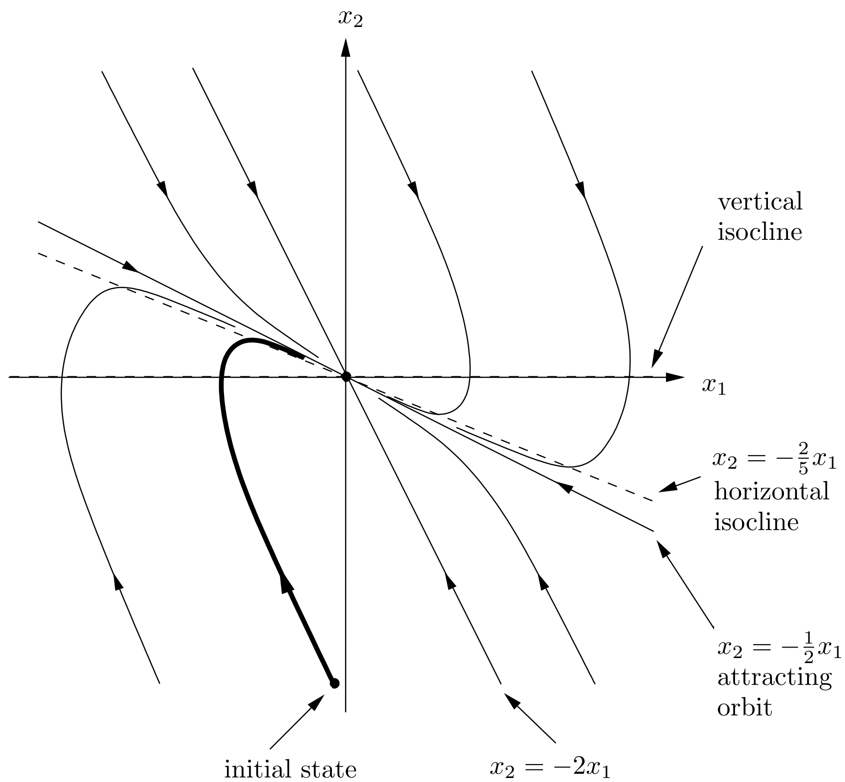


Figure 4.8: Orbits of the oscillator DE (4.44): the overdamped case $\left(\lambda = \frac{5}{4}\omega_0 > \omega_0\right)$. □

Solution: We use the eigenvalue method to get the general solution

$$\begin{aligned}\mathbf{x} &= e^{-t}\left[c_1\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + c_2\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}\right]\\ &= e^{-t}\begin{pmatrix}c_1\cos t + c_2\sin t\\ -c_1\sin t + c_2\cos t\end{pmatrix},\end{aligned}$$

(exercise). We can write each component as a single cosine or sine respectively:

$$\mathbf{x} = be^{-t}\begin{pmatrix}\cos(t-\delta)\\ -\sin(t-\delta)\end{pmatrix}, \tag{4.49}$$

where

$$b = \sqrt{c_1^2 + c_2^2}, \quad \cos\delta = \frac{c_1}{b}, \quad \sin\delta = \frac{c_2}{b}.$$

In this case, the only exceptional solution is the equilibrium solution $\mathbf{x} = \mathbf{0}$ (i.e. $b = 0$), and all other solutions are asymptotic to it as $t \to +\infty$:

$$\lim_{t\to+\infty}\mathbf{x}(t) = \mathbf{0}.$$

Because of the cos and sin in (4.49), the orbits spiral around in a clockwise manner as they approach the origin. [Recall that $\mathbf{x} = \begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}$ describes a circle traversed clockwise.] The

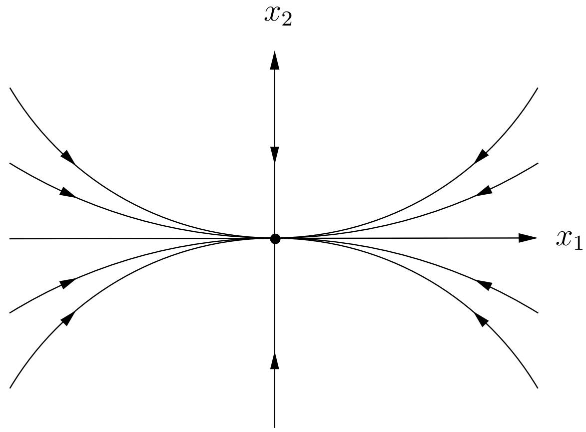


Figure 4.9: Orbits of the DE $\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\mathbf{x}$.

shape of the spirals is determined by the horizontal and vertical isoclines, which are obtained by writing the DE (4.48) in component form

$$\begin{aligned} x_1' &= -x_1 + x_2 \\ x_2' &= -x_1 - x_2. \end{aligned}$$

The horizontal isocline ($x_2' = 0$) is $x_2 = -x_1$ and the vertical isocline ($x_1' = 0$) is $x_2 = x_1$. The orbits are shown in Figure 5.10.

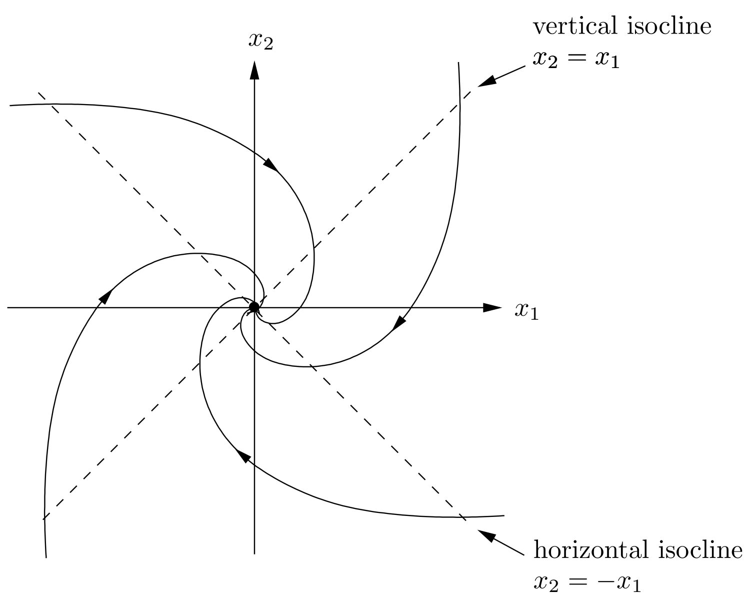


Figure 4.10: Orbits of a first order linear vector DE with complex eigenvalues: the horizontal and vertical isoclines are orthogonal.

Comment: The phase portrait for any DE $\mathbf{x}' = A\mathbf{x}$ having complex eigenvalues with negative real parts, is qualitatively the same as Figure 5.10 – a family of spirals focusing on the origin. The specific shape will depend on the horizontal and vertical isoclines, which in general will not be orthogonal, unlike the above special case. □

We now consider the example that is a special case of the oscillator DE (4.42)-(4.43).

Example 3: Give a qualitative sketch of orbits of the DE

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{6}{5} \end{pmatrix}.$$

Solution: We use the eigenvalue method to find the general solution:

$$\mathbf{x} = e^{-\frac{3}{5}t}\left[\left(c_1 \cos \tfrac{4}{5}t + c_2 \sin \tfrac{4}{5}t\right)\begin{pmatrix} 1 \\ -\frac{3}{5} \end{pmatrix} + \left(-c_1 \sin \tfrac{4}{5}t + c_2 \cos \tfrac{4}{5}t\right)\begin{pmatrix} 0 \\ \frac{4}{5} \end{pmatrix}\right],$$

(exercise).

Since the real part of the eigenvalues is negative $\left(Re(\lambda) = -\frac{3}{5}\right)$, the solution contains a decaying exponential, and so the orbits spiral into the origin. The shape of the spiral is determined by the isoclines. In component form the DE is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - \tfrac{6}{5}x_2. \end{aligned}$$

Thus the horizontal isocline ($x_2' = 0$) is $x_2 = -\frac{5}{6}x_1$, and the vertical isocline ($x_1' = 0$) is $x_2 = 0$. The orbits are shown in Figure 5.11.

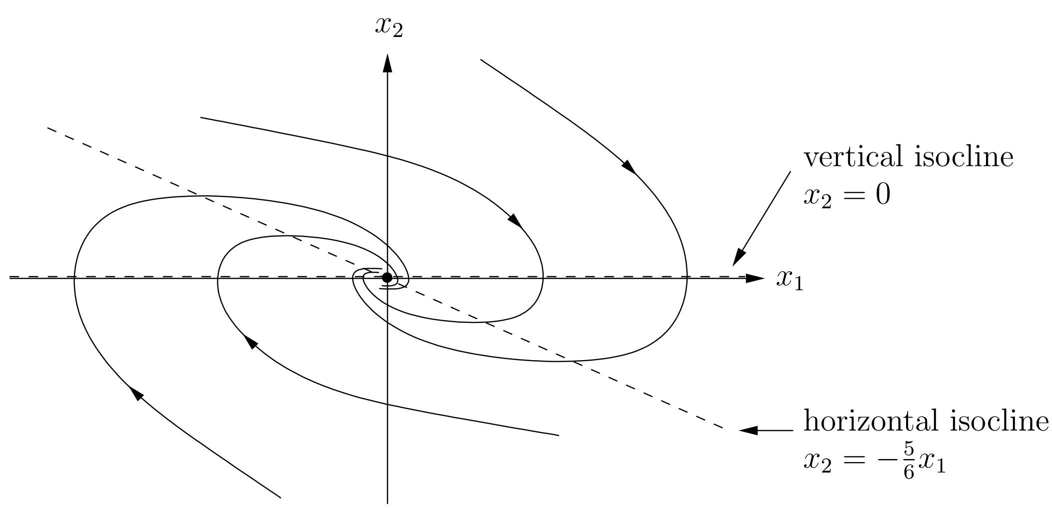


Figure 4.11: Orbits of a first order linear vector DE with complex eigenvalues: the horizontal and vertical isoclines are not orthogonal.

4.3.3 Long term evolution and stability

The examples of homogeneous vector DEs $\mathbf{x}' = A\mathbf{x}$ in this section have the property that all solutions satisfy

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}. \tag{4.50}$$

Thus, for all initial states the system approaches the equilibrium state $\mathbf{x} = \mathbf{0}$ in the long term. In this case we say that the equilibrium state is *asymptotically stable*.

One can predict whether or not the system is asymptotically stable without finding the solutions explicitly, simply by considering the eigenvalues of the coefficient matrix A. The solutions are linear combinations of the trial functions

$$\mathbf{x} = e^{\lambda t}\mathbf{v}. \tag{4.51}$$

We have seen that λ has to be an eigenvalue of the matrix A. If all eigenvalues of A satisfy

$$Re(\lambda) < 0,$$

then every solution (4.51) will contain a decaying exponential and hence all solutions will satisfy (4.50). We summarize this result in the following Proposition.

Proposition: If all eigenvalues of the matrix A satisfy

$$Re(\lambda) < 0,$$

then all solutions of the DE $\mathbf{x}' = A\mathbf{x}$ satisfy

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0},$$

i.e. the equilibrium state $\mathbf{x} = \mathbf{0}$ is asymptotically stable. $\square$

4.4 Solving inhomogeneous linear vector DEs

In this Section we show how to solve the inhomogeneous vector DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t). \tag{4.52}$$

This DE should be thought of as describing the state $\mathbf{x}(t)$ of a linear physical system, with input function $\mathbf{f}(t)$ (see Section 5.1.3).

As with any linear DE, the general solution will be of the form

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t),$$

where $\mathbf{x}_h(t)$ is the general solution of the homogeneous DE $\mathbf{x}' = A\mathbf{x}$, and $\mathbf{x}_p(t)$ is a particular solution of the inhomogeneous DE (4.52)

Three methods are available to find $\mathbf{x}_p(t)$:

(i) the method of undetermined coefficients (a generalization of the method used for second order scalar DEs),

(ii) the Laplace transform method, and

(iii) the method of "variation of parameters", which makes use of the two linearly independent solutions in $\mathbf{x}_h(t)$ which are used to form the *fundamental matrix*, $\Phi(t)$.

Methods (i) and (ii) are based on techniques which you have already learned, but limited as regards the possible forms of the input function $\mathbf{f}(t)$. Method (iii) is generally applicable, and has a simple formulation in terms of the fundamental matrix $\Phi(t)$.

We now describe each method.

4.4.1 The method of undetermined coefficients

As before, the method of undetermined coefficients will consist of making educated guesses about the form of a particular solution.

Example 1: Find the general solution of $x' = Ax + f(t)$ where $A = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}$ and $f(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}$.

Using the eigenvalue method, one can show that the homogeneous solution is

$$\mathbf{x}_h(t) = c_1 \begin{pmatrix} \cos t + 3\sin t \\ 2\sin t \end{pmatrix} + c_2 \begin{pmatrix} -5\sin t \\ \cos t - 3\sin t \end{pmatrix}.$$

Now we see that the input has the form $t\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. We will use the trial function

$$\mathbf{x} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

where the a's and b's are arbitrary constants.

Putting this into the DE we get

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix} \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right) + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Which gives us

$$\begin{aligned} b_1 &= 3a_1 - 5a_2 + 3b_1 t - 5b_2 t \\ b_2 &= 2a_1 - 3a_2 + 2b_1 t - 3b_2 t + t \end{aligned}$$

Equating coefficients we get the four equations in four unknowns:

$$\begin{aligned} 3a_1 - 5a_2 - b_1 &= 0 \qquad & 3b_1 - 5b_2 &= 0 \\ 2a_1 - 3a_2 - b_2 &= 0 \qquad & 2b_1 - 3b_2 &= -1 \end{aligned}$$

Solving the two equations on the right we get $b_1 = -5$ and $b_2 = -3$. Hence the other equations become

$$3a_1 - 5a_2 = -5, \qquad 2a_1 - 3a_2 = -3,$$

which gives us $a_1 = 0$ and $a_2 = 1$. Therefore the particular solution is

$$\mathbf{x}_p(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -5 \\ -3 \end{pmatrix}.$$

Hence the general solution is $\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$.

Example 2: Solve the system $\mathbf{x}' = \begin{pmatrix} 4 & \frac{1}{3} \\ 9 & 6 \end{pmatrix} \mathbf{x} + e^t \begin{pmatrix} -3 \\ 10 \end{pmatrix}$.

First, the homogeneous solution (exercise) is

$$\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 1 \\ 9 \end{pmatrix}.$$

Since $\mathbf{f}(t)$ has the form $e^{kt} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and k is not an eigenvalue we will use $\mathbf{x}(t) = e^t \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ as the trial function. Putting this into the DE we get

$$e^t \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 4 & \frac{1}{3} \\ 9 & 6 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + e^t \begin{pmatrix} -3 \\ 10 \end{pmatrix}.$$

Hence we have

$$a_1 = 4a_1 + \frac{1}{3}a_2 - 3 \Rightarrow 3a_1 + \frac{1}{3}a_2 = 3$$
$$a_2 = 9a_1 + 6a_2 + 10 \Rightarrow 9a_1 + 5a_2 = -10$$

Solving this gives us $a_1 = \frac{55}{36}$ and $a_2 = -\frac{19}{4}$. Thus the general solution is

$$\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 1 \\ 9 \end{pmatrix} + e^t \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix}.$$

Comment: As before, the tricky part of using the method of undetermined coefficients is in choosing the correct trial function. Although it is quite often the same as choosing the trial function for 2nd order ODEs, it is not always the same. For example, if our input is of the form $e^{kt} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ where k is an eigenvalue of multiplicity 1 (a single root of the characteristic equation), for a 2nd order DE we would choose the trial function $y = Ate^{kt}$, but for a vector DE we need to choose the trial function

$$\mathbf{x}(t) = e^{kt} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + te^{kt} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

4.4.2 The Laplace transform method

This method may be used to solve both *homogeneous* and *inhomogeneous* vector DEs; for simplicity we start with the former, and then show that the method also applies to the latter.

The Laplace transform method for homogeneous DEs

Consider a *homogeneous* linear vector DE

$$\mathbf{x}' = A\mathbf{x}, \tag{4.53}$$

with initial condition

$$\mathbf{x}(0) = \mathbf{a}. \tag{4.54}$$

We wish to apply the Laplace transform operator $\mathcal{L}$ to this DE. In order to do this we first have to define the Laplace transform of a vector-valued function $\mathbf{x}(t)$.

Definition: The *Laplace transform of a vector-valued function* $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is defined by

$$\mathcal{L}[\mathbf{x}(t)] = \begin{pmatrix} \mathcal{L}[x_1(t)] \\ \mathcal{L}[x_2(t)] \end{pmatrix},$$

provided that the Laplace transform of each component function exists. □

NOTE: The Laplace transform of $\mathbf{x}(t)$ is itself a vector-valued function, which we denote by $\mathbf{X}(s)$, i.e.

$$\mathbf{X}(s) = \mathcal{L}[\mathbf{x}(t)].$$

We now need to relate $\mathcal{L}[\mathbf{x}']$ and $\mathcal{L}[A\mathbf{x}]$ to $\mathcal{L}[\mathbf{x}]$. These matters are taken care of in the following propositions.

Proposition 1: Given a vector-valued function with derivative $\mathbf{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$. If $\mathcal{L}[x_1'(t)]$ and $\mathcal{L}[x_2'(t)]$ exist, then

$$\mathcal{L}[\mathbf{x}'(t)] = s\mathcal{L}[\mathbf{x}(t)] - \mathbf{x}(0).$$

Proof: By the definition,

$$\begin{aligned}
\mathcal{L}[\mathbf{x}'(t)] &= \begin{pmatrix} \mathcal{L}[x_1'(t)] \\ \mathcal{L}[x_2'(t)] \end{pmatrix}, \\
&= \begin{pmatrix} s\mathcal{L}[x_1(t)] - x_1(0) \\ s\mathcal{L}[x_2(t)] - x_2(0) \end{pmatrix}, \quad \text{(Laplace transform of the derivative of a scalar function)} \\
&= s\begin{pmatrix} \mathcal{L}[x_1(t)] \\ \mathcal{L}[x_2(t)] \end{pmatrix} - \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}, \quad \text{(using standard operations on vectors)} \\
&= s\mathcal{L}[\mathbf{x}(t)] - \mathbf{x}(0),
\end{aligned}$$

again using the definition. $\square$

Proposition 2: If A is a constant matrix, and $\mathcal{L}[\mathbf{x}(t)]$ exists, then

$$\mathcal{L}[A\mathbf{x}(t)] = A\mathcal{L}[\mathbf{x}(t)].$$

Proof: Consider

$$\begin{aligned}
\mathcal{L}[A\mathbf{x}(t)] &= \mathcal{L}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}\right], \\
&= \mathcal{L}\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}, \quad \text{(a matrix acting on a vector)} \\
&= \begin{pmatrix} \mathcal{L}[a_{11}x_1 + a_{12}x_2] \\ \mathcal{L}[a_{21}x_1 + a_{22}x_2] \end{pmatrix}, \quad \text{(by the definition)} \\
&= \begin{pmatrix} a_{11}\mathcal{L}[x_1] + a_{12}\mathcal{L}[x_2] \\ a_{21}\mathcal{L}[x_1] + a_{22}\mathcal{L}[x_2] \end{pmatrix}, \quad \text{(since } \mathcal{L} \text{ is a linear operator)} \\
&= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \mathcal{L}[x_1] \\ \mathcal{L}[x_2] \end{pmatrix} \quad \text{(a matrix acting on a vector)} \\
&= A\mathcal{L}[\mathbf{x}(t)], \quad \text{by the definition.} \quad \square
\end{aligned}$$

We can now apply $\mathcal{L}$ to the DE (4.53):

$$\mathcal{L}[\mathbf{x}'] = \mathcal{L}[A\mathbf{x}].$$

By Propositions 1 and 2 this becomes

$$s\mathcal{L}[\mathbf{x}(t)] - \mathbf{x}(0) = A\mathcal{L}[\mathbf{x}(t)].$$

Writing $\mathcal{L}[\mathbf{x}(t)] = \mathbf{X}(s)$, and using equation (4.54), we get

$$s\mathbf{X}(s) - \mathbf{a} = A\mathbf{X}(s),$$

which can be rearranged to read

$$(sI - A)\mathbf{X}(s) = \mathbf{a}, \tag{4.55}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix. Equation (4.55) is a system of linear algebraic equations with coefficient matrix $(sI - A)$. Provided that $sI - A$ is invertible, the solution of (4.55) is

$$\mathbf{X}(s) = (sI - A)^{-1}\mathbf{a}. \tag{4.56}$$

Having obtained $\mathbf{X}(s)$, the solution $\mathbf{x}(t)$ of the DE is obtained by taking the inverse Laplace transform:

$$\mathbf{x}(t) = \mathcal{L}^{-1}[\mathbf{X}(s)],$$

i.e. apply $\mathcal{L}^{-1}$ to the components of $\mathbf{X}(s)$. $\square$

Comment: Finding the inverse of a 2×2 matrix to obtain the solution (4.56) is easy:

$$\text{if} \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{then} \quad B^{-1} = \frac{1}{\det(B)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{4.57}$$

where $\det(B) = ad - bc$.

This can be verified by showing that $BB^{-1} = I$. $\square$

We now illustrate the method with an example.

Example 1: Solve the vector DE

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} -4 & 1 \\ -2 & -1 \end{pmatrix}, \tag{4.58}$$

with initial condition

$$\mathbf{x}(0) = \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Solution: Apply $\mathcal{L}$ to the DE (4.58) and use Propositions 1 and 2 to obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s)$$

where we have written $\mathcal{L}[\mathbf{x}(t)] = \mathbf{X}(s)$, as usual. Rearrange and use the initial condition to get

$$(sI - A)\mathbf{X}(s) = \mathbf{a},$$

i.e.

$$\begin{pmatrix} s+4 & -1 \\ 2 & s+1 \end{pmatrix} \mathbf{X}(s) = \mathbf{a}. \tag{4.59}$$

To find the inverse matrix we note

$$\det \begin{pmatrix} s+4 & -1 \\ 2 & s+1 \end{pmatrix} = (s+2)(s+3),$$

after simplifying. Thus by equation (4.57),

$$\begin{pmatrix} s+4 & -1 \\ 2 & s+1 \end{pmatrix}^{-1} = \frac{1}{(s+2)(s+3)} \begin{pmatrix} s+1 & 1 \\ -2 & s+4 \end{pmatrix},$$

and so the solution of (4.59) is

$$\mathbf{X}(s) = \frac{1}{(s+2)(s+3)} \begin{pmatrix} s+1 & 1 \\ -2 & s+4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{4.60}$$

for $s \neq -2, -3$. In component form

$$X_1(s) = \frac{s+1}{(s+2)(s+3)} a_1 + \frac{1}{(s+2)(s+3)} a_2,$$
$$X_2(s) = \frac{-2}{(s+2)(s+3)} a_1 + \frac{s+4}{(s+2)(s+3)} a_2.$$

Performing the partial fraction expansions yields

$$X_1(s) = \left(\frac{2}{s+3} - \frac{1}{s+2}\right) a_1 + \left(\frac{1}{s+2} - \frac{1}{s+3}\right) a_2,$$
$$X_2(s) = 2\left(\frac{1}{s+3} - \frac{1}{s+2}\right) a_1 + \left(\frac{2}{s+2} - \frac{1}{s+3}\right) a_2.$$

We can calculate the inverse Laplace transforms using $\mathcal{L}^{-1}\left[\frac{1}{s-\alpha}\right] = e^{\alpha t}$. This gives

$$\begin{aligned} x_1(t) &= \mathcal{L}^{-1}[X_1(s)] = (2e^{-3t} - e^{-2t})a_1 + (e^{-2t} - e^{-3t})a_2, \\ x_2(t) &= \mathcal{L}^{-1}[X_2(s)] = 2(e^{-3t} - e^{-2t})a_1 + (2e^{-2t} - e^{-3t})a_2. \end{aligned} \tag{4.61}$$

These equations can be written in vector form as

$$\mathbf{x}(t) = \begin{pmatrix} 2e^{-3t} - e^{-2t} & e^{-2t} - e^{-3t} \\ 2(e^{-3t} - e^{-2t}) & 2e^{-2t} - e^{-3t} \end{pmatrix} \mathbf{a}, \tag{4.62}$$

giving the solution of the vector DE (4.58) which satisfies the initial condition $\mathbf{x}(0) = \mathbf{a}$. □

Exercise: Use the Laplace transform method to solve the vector DE

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix},$$

with initial condition $\mathbf{x}(0) = \mathbf{a}$. This is example 2 in Section 5.2.3.

Answer: $\mathbf{x}(t) = e^{-t} \begin{pmatrix} \cos t + 2\sin t & 5\sin t \\ -\sin t & \cos t - 2\sin t \end{pmatrix} \mathbf{a}.$

HINT: $\dfrac{s+3}{s^2+2s+2} = \dfrac{(s+1)+2}{(s+1)^2+1}.$ □

In the Laplace transform method for homogeneous DEs, the Laplace transform of the solution $\mathbf{x}(t)$, denoted $\mathbf{X}(s)$, is given by

$$\mathbf{X}(s) = (sI - A)^{-1}\mathbf{a} \tag{4.63}$$

The solution $\mathbf{x}(t) = \mathcal{L}^{-1}[\mathbf{X}(s)]$ is obtained by applying $\mathcal{L}^{-1}$ to (4.63). In the example in this section we wrote (4.63) in component form and applied $\mathcal{L}^{-1}$ to each component. One can apply $\mathcal{L}^{-1}$ directly to (4.63) obtaining

$$\mathbf{x}(t) = \mathcal{L}^{-1}[(sI - A)^{-1}]\mathbf{a},$$

where the Laplace transform of the matrix is obtained by applying $\mathcal{L}^{-1}$ to each entry. In the example,

$$\mathbf{X}(s) = \begin{pmatrix} \frac{2}{s+3} - \frac{1}{s+2} & \frac{1}{s+2} - \frac{1}{s+3} \\ \frac{2}{s+3} - \frac{2}{s+2} & \frac{2}{s+2} - \frac{1}{s+3} \end{pmatrix} \mathbf{a}.$$

Thus

$$\mathbf{x}(t) = \mathcal{L}^{-1}\mathbf{X}(s) = \begin{pmatrix} \mathcal{L}^{-1}[\quad] & \mathcal{L}^{-1}[\quad] \\ \mathcal{L}^{-1}[\quad] & \mathcal{L}^{-1}[\quad] \end{pmatrix} \mathbf{a}$$

$$= \begin{pmatrix} 2e^{-3t} - e^{-2t} & e^{-2t} - e^{-3t} \\ 2e^{-3t} - 2e^{-2t} & 2e^{-2t} - e^{-3t} \end{pmatrix} \mathbf{a},$$

in agreement with equation (4.62). □

The Laplace transform method for inhomogeneous DEs

The method generalizes easily for an inhomogeneous DE, as illustrated in the following example.

Example 2: Solve the vector DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \quad A = \begin{pmatrix} -4 & 1 \\ -2 & -1 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} t \\ 2 \end{pmatrix}$$

with initial condition

$$\mathbf{x}(0) = \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Solution: Apply $\mathcal{L}$ to the DE (4.58) to obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s) + \mathbf{F}(s)$$

Rearrange, using the initial condition and the fact that $\mathbf{F}(s) = \begin{pmatrix} \frac{1}{s^2} \\ \frac{2}{s} \end{pmatrix}$ to get

$$(sI - A)\mathbf{X}(s) = \mathbf{a} + \begin{pmatrix} \frac{1}{s^2} \\ \frac{2}{s} \end{pmatrix},$$

i.e.

$$\mathbf{X}(s) = (sI - A)^{-1}\mathbf{a} + (sI - A)^{-1} \begin{pmatrix} \frac{1}{s^2} \\ \frac{2}{s} \end{pmatrix}$$

Hence, the solution is

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\mathbf{a}\right] + \mathcal{L}^{-1}\left[(sI - A)^{-1} \begin{pmatrix} \frac{1}{s^2} \\ \frac{2}{s} \end{pmatrix}\right] \tag{4.64}$$

The first term was evaluated in example 1 of this section. To evaluate the second term, we note that

$$(sI - A)^{-1} = \frac{1}{(s+2)(s+3)} \begin{pmatrix} s+1 & 1 \\ -2 & s+4 \end{pmatrix},$$

again from example 1, and therefore

$$(sI-A)^{-1}\begin{pmatrix}\frac{1}{s^2}\\ \frac{2}{s}\end{pmatrix} = \begin{pmatrix}\frac{s+1}{s^2(s+2)(s+3)}+\frac{2}{s(s+2)(s+3)}\\ \frac{-2}{s^2(s+2)(s+3)}+\frac{2(s+4)}{s(s+2)(s+3)}\end{pmatrix}$$
$$= \begin{pmatrix}\frac{13/36}{s}+\frac{1/6}{s^2}-\frac{5/4}{s+2}+\frac{8/9}{s+3}\\ \frac{29/18}{s}-\frac{1/3}{s^2}-\frac{5/2}{s+2}+\frac{8/9}{s+3}\end{pmatrix}$$

after using a partial fraction expansion and simplifying. Hence the solution (4.64) becomes

$$\mathbf{x}(t) = \begin{pmatrix}2e^{-3t}-e^{-2t} & e^{-2t}-e^{-3t}\\ 2(e^{-3t}-e^{-2t}) & 2e^{-2t}-e^{-3t}\end{pmatrix}\mathbf{a} + \begin{pmatrix}\frac{13}{36}+\frac{1}{6}t-\frac{5}{4}e^{-2t}+\frac{8}{9}e^{-3t}\\ \frac{29}{18}-\frac{1}{3}t-\frac{5}{2}e^{-2t}+\frac{8}{9}e^{-3t}\end{pmatrix}.$$

We now summarize the Laplace transform method schematically, writing $\mathcal{L}[\mathbf{x}(t)] = \mathbf{X}(s)$ and $\mathcal{L}[\mathbf{f}(t)] = \mathbf{F}(s)$, as usual.

$$\begin{cases}\mathbf{x}' = A\mathbf{x}+\mathbf{f}(t)\\ \mathbf{x}(0)=\mathbf{a}\end{cases} \xrightarrow[\text{and rearrange}]{\text{apply } \mathcal{L}} \mathbf{X}(s) = (sI-A)^{-1}(\mathbf{a}+\mathbf{F}(s)) \xrightarrow{\text{apply } \mathcal{L}^{-1}} \mathbf{x}(t)$$

4.4.3 The method of Variation of parameters

The Fundamental Matrix

In example 1 from the last section, in which we solved a homogeneous vector DE, the solution (4.62) had the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{a}, \tag{4.65}$$

where $\Phi(t)$ is the 2×2 matrix in (4.62). This form of the solution shows directly how the state $\mathbf{x}(t)$ at time t depends on the initial state $\mathbf{x}(0) = \mathbf{a}$. The matrix $\Phi(t)$ is called *the fundamental matrix of the DE* (4.58).

A different form of the solution was obtained using the eigenvalue method for the same example (see Example 1 in Section 5.2.2). The solution was obtained in the form (see equation (4.26))

$$\mathbf{x} = c_1e^{-2t}\begin{pmatrix}1\\2\end{pmatrix} + c_2e^{-3t}\begin{pmatrix}1\\1\end{pmatrix}. \tag{4.66}$$

This form of the solution is simpler algebraically and shows clearly that there are two distinct rates of decay i.e. e^{-2t} and e^{-3t}.

How are the two forms of the solution related? By equation (4.62) the columns of the fundamental matrix are

$$\mathbf{\Phi}^1(t) = \begin{pmatrix}2e^{-3t}-e^{-2t}\\ 2(e^{-3t}-e^{-2t})\end{pmatrix}, \quad \mathbf{\Phi}^2(t) = \begin{pmatrix}e^{-2t}-e^{-3t}\\ 2e^{-2t}-e^{-3t}\end{pmatrix}. \tag{4.67}$$

Each column vector is a solution of the DE. The column $\mathbf{\Phi}^1(t)$ is the solution corresponding to the initial condition $\mathbf{a} = \begin{pmatrix}1\\0\end{pmatrix}$, and the column $\mathbf{\Phi}^2(t)$ is the solution corresponding to the initial condition $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. [Choose $a_1 = 1$, $a_2 = 0$ and $a_1 = 0, a_2 = 1$ in (4.61) to get these solutions.] If we have the solution in eigenvector form (4.66), we can construct the fundamental matrix $\Phi(t)$ by finding the two special solutions (4.67): impose the initial conditions $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ successively and determine the constants c_1 and c_2 in (4.66) (this was done in equations (4.28) and (4.29) in Section 5.2.2). □

Properties of the Fundamental Matrix

In the previous section we showed that the general solution of the homogeneous DE $\mathbf{x}' = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \mathbf{a}$ has the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{a}, \tag{4.68}$$

where $\Phi(t)$ is a 2×2 time-dependent matrix called the fundamental matrix of the DE. It follows by setting $t = 0$ in (4.68) that $\Phi(0)\mathbf{a} = \mathbf{a}$. Since $\mathbf{a}$ is arbitrary, this equation implies that

$$\Phi(0) = I, \tag{4.69}$$

where I is the 2×2 identity matrix.

We need two additional properties of $\Phi(t)$, the first of which follows directly from the fact that (4.68) is a solution of the homogeneous DE

$$\mathbf{x}' = A\mathbf{x}. \tag{4.70}$$

Proposition 1: The fundamental matrix $\Phi(t)$ of the DE $\mathbf{x}' = A\mathbf{x}$ satisfies

$$\Phi'(t) = A\Phi(t). \tag{4.71}$$

Proof: Since (4.68) is a solution of (4.70) we have

$$[\Phi(t)\mathbf{a}]' = A[\Phi(t)\mathbf{a}],$$

which yields

$$[\Phi'(t) - A\Phi(t)]\mathbf{a} = \mathbf{0},$$

after rearranging. Since this holds for all $\mathbf{a} \in \mathbb{R}^2$, it follows that

$$\Phi'(t) - A\Phi(t) = 0,$$

where "0" denotes the zero matrix, which gives the required result. □

We shall refer to (4.71) as *the derivative property of* $\Phi(t)$.

The second property of $\Phi(t)$ follows from the geometric interpretation of equation (4.68). One can think of $\Phi(t)$ as an operator that transforms an initial state $\mathbf{x}(0) = \mathbf{a}$ into the state $\mathbf{x}(t) = \Phi(t)\mathbf{a}$ at time t.

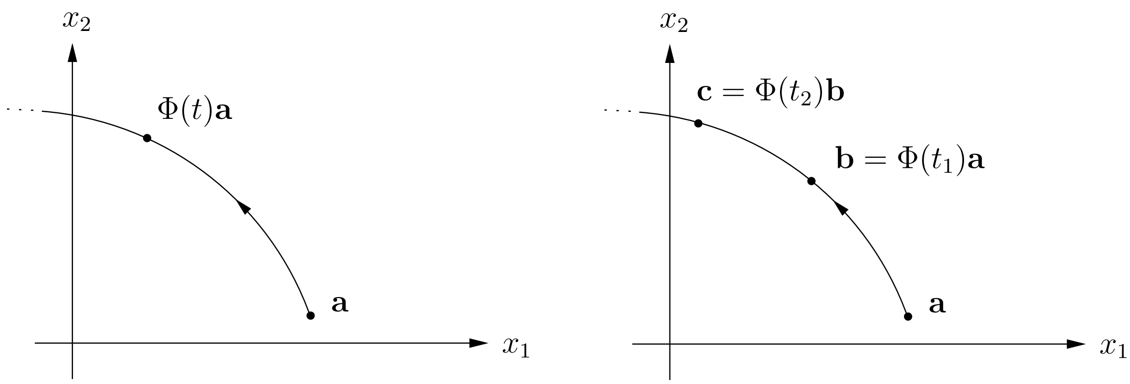


Figure 4.12: The action of the fundamental matrix.

Consider the situation shown in Figure 5.12. At time $t = 0$ the system is in state $\mathbf{a}$, and after a time t_1 it evolves into state $\mathbf{b}$, given by

$$\mathbf{b} = \Phi(t_1)\mathbf{a}. \tag{4.72}$$

After a further time t_2, it evolves into state $\mathbf{c}$, so that

$$\mathbf{c} = \Phi(t_2)\mathbf{b}, \tag{4.73}$$

thinking of $\mathbf{b}$ as the initial state. On the other hand, in a time $t_2 + t_1$ the system will evolve from $\mathbf{a}$ to $\mathbf{c}$ so that

$$\mathbf{c} = \Phi(t_2 + t_1)\mathbf{a}. \tag{4.74}$$

Substituting (4.72) in (4.73) and comparing with (4.74) gives

$$\Phi(t_2 + t_1)\mathbf{a} = \Phi(t_2)\Phi(t_1)\mathbf{a}.$$

Since $\mathbf{a}$ is an arbitrary vector, it follows that

$$\Phi(t_2 + t_1) = \Phi(t_2)\Phi(t_1),$$

a matrix equation.

We summarize this result and the result (4.69) in the following Proposition.

Proposition 2: The fundamental matrix $\Phi(t)$ of the DE $\mathbf{x}' = A\mathbf{x}$ satisfies

$$\Phi(0) = I,$$

and

$$\Phi(t_2 + t_1) = \Phi(t_2)\Phi(t_1), \tag{4.75}$$

for all $t_1, t_2 \in \mathbb{R}$. $\square$

The property we need is an immediate consequence of this Proposition. Choosing $t_2 = -t_1$ and writing $t_1 = t$, equation (4.75) with (4.69) gives

$$\Phi(-t)\Phi(t) = I.$$

In other words, $\Phi(t)$ is invertible for any t, and

$$[\Phi(t)]^{-1} = \Phi(-t). \tag{4.76}$$

We shall refer to (4.76) as *the inverse property of the fundamental matrix*. This result is quite remarkable. It states that to find the inverse matrix of a fundamental matrix $\Phi(t)$, one simply replaces t by $-t$.

The method of variation of parameters

Given the DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \tag{4.77}$$

let $\Phi(t)$ be the fundamental matrix of the related homogeneous DE $\mathbf{x}' = A\mathbf{x}$. We know that $\mathbf{x}(t) = \Phi(t)\mathbf{a}$ is a solution of $\mathbf{x}' = A\mathbf{x}$ for any $\mathbf{a} \in \mathbb{R}^2$. This suggests that in order to find a particular solution of (4.77) we consider a trial function of the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{v}(t), \tag{4.78}$$

where $\mathbf{v}(t)$ is an arbitrary vector-valued function. In other words, we replace the constant vector $\mathbf{a}$ (whose components are two arbitrary parameters) by a time-dependent vector $\mathbf{v}(t)$ (whose components are two arbitrary scalar functions), i.e. we "vary the parameters". This choice of trial function is thus called the *method of variation of parameters.*

Differentiate (4.78) with respect to t using the Product Rule and the derivative property (4.71) of $\Phi(t)$ to obtain

$$\begin{aligned}\mathbf{x}'(t) &= \Phi'(t)\mathbf{v}(t) + \Phi(t)\mathbf{v}'(t)\\ &= A\Phi(t)\mathbf{v}(t) + \Phi(t)\mathbf{v}'(t)\\ &= A\mathbf{x}(t) + \Phi(t)\mathbf{v}'(t),\end{aligned}$$

by (4.78). Equating this expression for $\mathbf{x}'$ with $\mathbf{x}'$ in (4.77) gives

$$\Phi(t)\mathbf{v}'(t) = \mathbf{f}(t).$$

We now multiply by the matrix $\Phi(-t)$ and use the inverse property (4.76) to get

$$\mathbf{v}'(t) = \Phi(-t)\mathbf{f}(t). \tag{4.79}$$

Equations (4.78) *and* (4.79) *constitute the method of variation of parameters.* Given $\mathbf{f}(t)$, and having calculated $\Phi(t)$, one obtains $\mathbf{v}(t)$ by taking the antiderivative of (4.79). Then (4.78) gives a particular solution of the DE (4.77). □

Comment: When taking the antiderivative of (4.79), a constant of integration, which is a constant vector, arises. If one simply wants a particular solution one can choose this vector to be zero. Alternately, one can choose this vector so that

$$\mathbf{v}(0) = \mathbf{0}.$$

Then the particular solution

$$\mathbf{x}_p(t) = \Phi(t)\mathbf{v}(t)$$

satisfies

$$\mathbf{x}_p(0) = \mathbf{0}. \tag{4.80}$$

With this choice of particular solution, the general solution $\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$ can be written

$$\mathbf{x}(t) = \Phi(t)\mathbf{a} + \mathbf{x}_p(t). \tag{4.81}$$

Equation (4.80) ensures that

$$\begin{aligned}\mathbf{x}(0) &= \Phi(0)\mathbf{a} + \mathbf{x}_p(0)\\ &= I\mathbf{a} + \mathbf{0} = \mathbf{a},\end{aligned}$$

as required. □

Example: Solve the DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(0) = \mathbf{a}, \tag{4.82}$$

with

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

The fundamental matrix of $x' = Ax$ is

$$\Phi(t) = e^t \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \tag{4.83}$$

Solution: The solution is of the form (4.81), i.e.

$$\mathbf{x}(t) = \Phi(t)\mathbf{a} + \mathbf{x}_p(t). \tag{4.84}$$

Substituting a trial function

$$\mathbf{x}_p(t) = \Phi(t)\mathbf{v}(t) \tag{4.85}$$

in the DE leads to (by equation (4.79))

$$\begin{aligned}\mathbf{v}'(t) &= \Phi(-t)\mathbf{f}(t)\\ &= e^{-t}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}\\ &= e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\end{aligned}$$

Taking the antiderivative yields

$$\mathbf{v}(t) = -e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{c}. \tag{4.86}$$

Setting $t = 0$ gives $\mathbf{v}(0) = -\begin{pmatrix}1\\0\end{pmatrix} + \mathbf{c}$. We want $\mathbf{v}(0) = 0$, and so we choose $\mathbf{c} = \begin{pmatrix}1\\0\end{pmatrix}$. Thus (4.86) becomes

$$\mathbf{v}(t) = (1 - e^{-t})\begin{pmatrix}1\\0\end{pmatrix}. \tag{4.87}$$

By (4.83), (4.85) and (4.87) the particular solution is

$$\mathbf{x}_p(t) = (e^t - 1)\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = (e^t - 1)\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix},$$

and by (4.84) the full solution is

$$\mathbf{x}(t) = e^t\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\mathbf{a} + (e^t - 1)\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}. \qquad \square$$

Exercise: Solve the DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(0) = \mathbf{a},$$

with

$$A = \begin{pmatrix}-3 & 2\\ 2 & -3\end{pmatrix} \qquad \mathbf{f} = e^{-3t}\begin{pmatrix}1\\1\end{pmatrix}.$$

The fundamental matrix is

$$\Phi(t) = \tfrac{1}{2}\begin{pmatrix}e^{-t} + e^{-5t} & e^{-t} - e^{-5t}\\ e^{-t} - e^{5t} & e^{-t} + e^{-5t}\end{pmatrix}.$$

Answer: $\mathbf{x}(t) = \Phi(t)\mathbf{a} + \frac{1}{2}(e^{-t} - e^{-3t})\begin{pmatrix}1\\1\end{pmatrix}$. $\square$

4.4.4 The fundamental matrix and the matrix exponential

In this Section we give a different representation of the fundamental matrix $\Phi(t)$ of the time-invariant linear DE

$$\mathbf{x}' = A\mathbf{x}. \tag{4.88}$$

Specifically, we define the "exponential of an $n \times n$ matrix A", denoted e^A or $\exp(A)$ and show that

$$\Phi(t) = e^{tA}.$$

The notion of e^A seems bizarre initially but the idea begins to make sense once one thinks in terms of the Taylor series of the exponential function e^t:

$$e^t = 1 + t + \tfrac{1}{2!}t^2 + \cdots + \tfrac{1}{k!}t^k + \cdots$$

which converges for all $t \in \mathbb{R}$. Given an $n \times n$ matrix A, we can form the matrix series

$$I + A + \tfrac{1}{2!}A^2 + \cdots + \tfrac{1}{k!}A^k + \cdots,$$

the powers A^k being formed by successive matrix multiplication, and the sum operation being addition of matrices. We can use Σ-notation to write this series in the concise form

$$\sum_{k=0}^{\infty} \frac{1}{k!} A^k,$$

where $A^0 = I$, the identity matrix. This series is essentially n^2 numerical series, one for each matrix entry, and it can be proved that given any matrix A, these n^2 series converge. Thus the matrix series has a finite sum.

We thus define *the exponential* e^A *of an* $n \times n$ *matrix* A as the sum of this series:

$$e^A := I + A + \tfrac{1}{2!}A^2 + \cdots + \tfrac{1}{k!}A^k + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k. \tag{4.89}$$

Example 1:

The following matrix exponentials are obtained by direct calculation of the powers of the matrix A and the use of standard Taylor series

i) if $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $e^A = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$,

ii) if $A = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$, then $e^A = \begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix}$,

iii) if $A = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$, then $e^A = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}$,

iv) if $A = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$, then $e^A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.

We list some properties of the matrix exponential.

Property i):

$$e^0 = I, \tag{4.90}$$

where 0 is the zero matrix.

The exponential of a sum property does not hold in general, i.e.

$$e^{A+B} \neq e^A e^B, \tag{4.91}$$

but property ii) below can be proved, using the definition (4.89).

Property ii):

If A, B are real $n \times n$ matrices and $AB = BA$, then

$$e^{A+B} = e^A e^B. \tag{4.92}$$

Choosing $B = -A$ in Property ii) immediately gives property iii).

Property iii):

If A is an $n \times n$ real matrix, then e^A is invertible and

$$(e^A)^{-1} = e^{-A}. \qquad \square \tag{4.93}$$

Exercise 1:

Verify that $e^{A+B} \neq e^A e^B$ for the matrices $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. $\square$

We now consider a matrix-valued function

$$M(t) = \{M_{ij}(t)\},$$

for all $t \in \mathbb{R}$, i.e. each entry of the $n \times n$ matrix $M(t)$ is a function of t. If these functions are differentiable then we can calculate the derivative of $M(t)$ according to

$$M'(t) = \{M'_{ij}(t)\},$$

i.e. just differentiate each entry. We apply this notion to the matrix-valued function

$$e^{tA} = I + tA + \tfrac{1}{2!}t^2a^2 + \cdots + \tfrac{1}{k!}t^kA^k + \cdots = I + \sum_1^\infty \tfrac{1}{k!}t^kA^k.$$

Each entry is a power series in t and so we can differentiate term by term, obtaining

$$\begin{aligned} \frac{d}{dt}(e^{tA}) &= \sum_1^\infty \tfrac{1}{(k-1)!}t^{k-1}A^k \\ &= \sum_0^\infty \tfrac{1}{k!}t^kA^{k+1} \quad \text{(replace } k-1 \text{ by } k) \\ &= A\left(\sum_0^\infty \tfrac{1}{k!}t^kA^k\right) \\ &= Ae^{tA}. \end{aligned}$$

Summarizing, we have

$$\frac{d}{dt}(e^{tA}) = Ae^{tA} = e^{tA}A \tag{4.94}$$

for any constant matrix A (the order of the matrices A and e^{tA} is immaterial).

We are now ready to present the main result.

Theorem 4.1

The unique solution of the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{a}, \tag{4.95}$$

is

$$\mathbf{x}(t) = e^{tA}\mathbf{a}. \tag{4.96}$$

Proof: It is simply a matter of verifying that the function (4.96) satisfies the DE and initial condition in (4.95). First, by (4.94),

$$\mathbf{x}'(t) = Ae^{tA}\mathbf{a} = A\mathbf{x}(t).$$

Second, by (4.90)

$$\mathbf{x}(0) = e^{0}\mathbf{a} = \mathbf{a}. \qquad \square$$

By (4.68) the solution of the initial value problem (4.95) can also be written

$$\mathbf{x}(t) = \Phi(t)\mathbf{a}.$$

It now follows by comparing this equation with (4.96) that

$$\Phi(t) = e^{tA}, \tag{4.97}$$

our stated goal.

Discussion:

At first sight, Theorem 4.1 appears to provide an algorithm for solving the DE $\mathbf{x}' = A\mathbf{x}$ that is more direct than the eigenvector method; just calculate the matrix exponential e^{tA}, obtaining all solutions in the form (4.96). There is a catch: it is not feasible to calculate e^{tA} directly, for an arbitrary matrix A, even in the 2×2 case. There is a systematic way to calculate a matrix exponential e^{B}, and that is to use a similarity transformation

$$\tilde{B} = P^{-1}BP, \tag{4.98}$$

to write B in a simple form $\tilde{B}$, a so-called *canonical form*,[1] such that $e^{\tilde{B}}$ can be calculated directly. Knowing P, one can then use the following Proposition to find e^{B}. However, one has to find the eigenvectors of B in order to construct the matrix P.

Proposition:

If

$$\tilde{B} = P^{-1}BP, \quad \text{then} \quad e^{B} = Pe^{\tilde{B}}P^{-1}.$$

Proof (outline): Use the series (4.89) for the matrix exponential, and show that

$$\tilde{B}^k = P^{-1}B^kP. \qquad \square$$

The next example shows how to calculate the matrix exponential for one of the canonical forms.

[1]For example, any symmetric matrix can be transformed to diagonal form.

Example 2:

If A is a 2×2 matrix with a repeated real eigenvector λ but only one independent eigenvalue, then the canonical form is

$$\tilde{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{4.99}$$

Calculate $e^{t\tilde{A}}$.

Solution: We write

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The two matrices on the right commute, and so by Property ii)

$$e^{t\tilde{A}} = e^{\begin{pmatrix} \lambda t & 0 \\ 0 & \lambda t \end{pmatrix}} e^{\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}},$$

leading to (see Example 1 i) and iv))

$$e^{t\tilde{A}} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}. \tag{4.100}$$

Comment:

It is easy to calculate the exponential of the matrix $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ since $M^2 = 0$, so that the series (4.89) becomes a finite sum:

$$e^M = I + M.$$

Any matrix that satisfies $M^k = 0$ for some positive integer $k > 1$ is called *nilpotent*. The exponential of any such matrix is a finite sum, as above. □

Exercise 2:

The canonical form for a 2×2 matrix with a pair of complex eigenvalues $\lambda_\pm = \alpha \pm i\beta$ is

$$\tilde{A} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{4.101}$$

Show that

$$e^{t\tilde{A}} = \begin{pmatrix} e^{\alpha t}\cos\beta t & e^{\alpha t}\sin\beta t \\ -e^{\alpha t}\sin\beta t & e^{\alpha t}\cos\beta t \end{pmatrix}. \tag{4.102}$$

In conclusion, calculating e^{tA} provides an alternate method for solving $\mathbf{x}' = A\mathbf{x}$, which may be more efficient than the eigenvector method if the matrix A has a simple form. On the purely theoretical side, e^{tA} provides a useful notation for the fundamental matrix $\Phi(t)$, in that it explicitly indicates the dependence on the matrix A, and makes obvious the properties of $\Phi(t)$, i.e.

$$\Phi(0) = I \quad \text{follows from} \quad e^0 = I$$

$$\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2) \quad \text{follows from} \quad e^{t_1 A + t_2 A} = e^{t_1 A} e^{t_2 A}$$

$$[\Phi(t)]^{-1} = \Phi(-t) \quad \text{follows from} \quad (e^{tA})^{-1} = e^{-tA}.$$

Appendix A

Series Solutions of DEs

In this Chapter we give an introduction to the method of solving second order linear homogeneous DEs using power series.

We consider DEs of the form

$$y'' + P(x)y' + Q(x)y = 0,$$

where $P(x)$ and $Q(x)$ are polynomial functions. The standard method, which we develop in this Section, can be used to find two linearly independent solutions in the form of power series

$$y = \sum_{n=0}^{\infty} a_n x^n. \tag{A.1}$$

The first example illustrates the method in the simplest setting, and the solutions obtained are well-known.

Example 1:

Find two linearly independent power series solutions of the DE

$$y'' + y = 0. \tag{A.2}$$

Solution: We assume a solution of the form (A.1), where the series has an interval of convergence $|x| < R$ (R may be infinite). We can calculate the derivatives of y using term-by-term differentiation of the series:

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

The next step (an essential one!) is to redefine the summation variable n in the series for y'' so that the n^{th} term is a multiple of x^n:

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n. \tag{A.3}$$

Now substitute (A.1) and (A.3) into (A.2), and combine the series, obtaining

$$\sum_{n=0}^{\infty} \big[(n+2)(n+1)a_{n+2} + a_n\big]x^n = 0.$$

Since this equation must hold for all x satisfying $|x| < R$, it follows that the coefficient of x^n must be zero:

$$(n+2)(n+1)a_{n+2} + a_n = 0,$$

or equivalently,

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)}, \tag{A.4}$$

for $n = 0, 1, 2, \ldots$. This equation is called a *recursion formula*, and serves to determine the coefficients in the power series (A.1), as follows. The idea is that one can choose a_0 arbitrarily, and then (A.4) successively determines $a_2, a_4, a_6, \ldots$ in terms of a_0. Choosing $n = 0, 2, 4, 6, \ldots$ gives

$$a_2 = -\frac{a_0}{1 \cdot 2}, \qquad a_4 = -\frac{a_2}{3 \cdot 4} = \frac{a_0}{4!}, \qquad a_6 = -\frac{a_4}{5 \cdot 6} = -\frac{a_0}{6!}.$$

In general,

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!},$$

for $k = 1, 2, 3, \ldots$. Similarly, choosing $n = 1, 3, 5, \ldots$ leads to

$$a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!},$$

for $k = 1, 2, 3, \ldots$, with a_1 arbitrary. The series (A.1) can be grouped into even and odd powers in the form

$$y = \sum_{k=0}^{\infty} a_{2k}x^{2k} + \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1},$$

which leads to

$$y = a_0 \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}}_{y_1(x)} + a_1 \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}}_{y_2(x)}. \tag{A.5}$$

This equation gives the general solution of the DE (A.2) as a linear combination of two linearly independent[1] solutions $y_1(x)$ and $y_2(x)$. It can be verified using the Ratio Test that both series converge for all $x \in \mathbb{R}$. □

Comment:

In this example we can recognize the power series as familiar Taylor series, giving the general solution in the form

$$y = a_0 \cos x + a_1 \sin x. \qquad \square$$

[1]The solutions are linearly independent since $y_1(0) = 1$ and $y_2(0) = 0$.

We now discuss a second example,with only the key steps given, leaving the reader to fill in the details.

Example 2:

Find two linearly independent power series solutions of the DE

$$y'' - 2xy' + y = 0. \tag{A.6}$$

Solution outline:

The starting point is the general power series (A.1). It is essential to write each term in the DE as a power series with the n$^{\text{th}}$ term being a multiple of x^n:

$$y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n,$$

$$-2xy' = \sum_{n=0}^{\infty}(-2n)a_n x^n.$$

The recurrence relation is

$$a_{n+2} = \frac{2n-1}{(n+1)(n+2)}a_n.$$

for $n = 0, 1, 2, \ldots$. Solving the recursion formula gives

$$a_{2k} = \frac{(-1)(3)(7)\cdots(4k-5)}{(2k)!}a_0,$$
$$a_{2k+1} = \frac{(1)(5)(9)\cdots(4k-3)}{(2k+1)!}a_1.$$

Two linearly independent solutions are obtained by choosing $a_0 = 1$, $a_1 = 0$ and $a_0 = 0$, $a_1 = 1$:

$$y_1(x) = 1 + \sum_{k=1}^{\infty}\frac{(-1)(3)(7)\cdots(4k-5)}{(2k)!}x^{2k},$$
$$y_2(x) = x + \sum_{k=1}^{\infty}\frac{(1)(5)(9)\cdots(4k-3)}{(2k+1)!}x^{2k+1}.$$

It can be verified using the ratio test that both series converge for all $x \in \mathbb{R}$. $\square$

Theoretical justification:

In a general theoretical analysis of the standard method, one assumes that the coefficient functions $P(x)$ and $Q(x)$ (not necessarily polynomials) are analytic at $x = 0$, i.e. they have power series expansions that converge on the interval $|\, x \,| < R$ (the radius of convergence R may be infinite). It can then be proved that the DE

$$y'' + P(x)y' + Q(x)y = 0$$

has two linearly independent power series solutions that converge on the same interval $| x |< R$. We refer to Simmons (see Theorem A on page 155) for a readable proof of this result. It follows from the general theorem that *if $P(x)$ and $Q(x)$ are polynomials, then the resulting series converge for all $x \in \mathbb{R}$.*

The Method of Frobenius

If $P(x)$ or $Q(x)$ is not analytic at $x = 0$, then the method of this chapter breaks down. For example, the DE

$$xy'' + y = 0 \qquad \text{i.e.} \qquad y'' + \frac{1}{x}y = 0$$

cannot be solved using this method (trying it would lead to a contradiction).

To solve DEs of this kind, one uses the *Method of Frobenius*; the basic idea is to use the trial function

$$y = x^r \sum_{n=0}^{\infty} a_n x^n \tag{A.7}$$

where r is a non-integer, real constant. Then one obtains a power series in non-integer powers of x. Here is an example involving a first-order DE for simplicity.

Example 3: Consider the DE $xy' + (1-x)y = 0$, which may be written $y' + \frac{1-x}{x}y = 0$. We see that the coefficient function is not analytic at $x = 0$. First we try to solve this with the basic method to see what happens.

Assume a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Thus

$$xy' = \sum_{n=1}^{\infty} n a_n x^n.$$

Substituting into the DE we obtain

$$\sum_{n=1}^{\infty} n a_n x^n + (1-x) \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} + \sum_{n=-1}^{\infty} a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1}$$

Thus we have $a_0 = 0$

$$(n+1)a_{n+1} + a_{n+1} - a_n = 0 \Rightarrow a_{n+1} = \frac{1}{n+2} a_n.$$

But then $a_n = 0$ for all n, which gives only the trivial solution. To fix this problem, we will instead use the trial function

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Where r is some constant which will be determined. As before we find

$$y' = \sum_{n=1}^{\infty}(n+r)a_n x^{n+r-1},$$

and the DE becomes

$$\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1}.$$

Therefore we get

$$\sum_{n=-1}^{\infty}(n+r+1)a_{n+1}x^{n+r+1} + \sum_{n=-1}^{\infty} a_{n+1}x^{n+r+1} - \sum_{n=0}^{\infty} a_n x^{n+r+1},$$

and so we now find that

$$ra_0 + a_0 = 0 \Rightarrow (r+1)a_0 = 0$$

and

$$a_{n+1} = \frac{1}{n+r+2}a_n.$$

Solving the first equation gives $r = -1$ (otherwise $a_0 = 0$ which leads to the trivial solution). Hence we get $a_{n+1} = \frac{1}{n+1}a_n$ and so the solution is

$$y = \sum_{n=0}^{\infty} a_n x^{n-1} = a_0 \sum_{n=0}^{\infty} \frac{1}{n!}x^{n-1}.$$

Remark: Using this method to solve a second order DE yields two values for r, each corresponding to a series solution. If these solutions are linearly independent, the general solution is a linear combination of these two solutions. However, for the DE $xy'' + y = 0$, both values of r produce the same solution (try it!), so the method of Frobenius does not produce the general solution to the DE, and the method needs to be extended (it turns out that a logarithmic term must appear).

This method is discussed in most introductory differential equations texts, for example Simmons, page 188.

Appendix B

Boundary Value Problems

Up until now most, if not all, of the problems we have dealt with would be considered *initial value* problems where the arbitrary constants of integration were determined using conditions specified at a single value of the independent variable (e.g. $y(0) = y_0$ and $y'(0) = v_0$).

Many physical problems in engineering and science require conditions be specified at multiple values of the independent variable. For example we might have $y(0) = a$ and $y(L) = b$, or we could even specify how the derivative behaves at two different input points, e.g. $y'(0) = m_1$ and $y'(L) = m_2$. There are even cases where we specify more general conditions such as $y(0) < \infty$ (i.e. the solution can't blow up at 0) or $\lim_{x\to\infty} y(x) = 0$. These types of conditions alongside an appropriate differential equation are called *boundary value* problems.

For now the boundary will generally refer to two points of the independent variable (usually thought of as two spatial points) but as we move to higher dimensions it will come to represent more realistic physical boundaries such as the walls of a container or the entire surface of a three dimensional object.

For the sake of introduction much of this section will omit any theoretical matters and will focus on examples where BVPs arise as well as some purely mathematical examples with solutions. It should be noted that many of the previous methods for solving DEs are still applicable; What ultimately changes is how we determine the constants of integration.

B.1 Examples giving rise to boundary conditions

B.1.1 Fluid flow

In the study of fluid mechanics boundary conditions play a crucial role in determining the flow of the fluid. In many cases there is physical object within the region that the fluid will "wrap" around (e.g. a tree in a river, air currents flowing around a tall building) but there are also cases of the boundary being the driving force of the motion (e.g. the wing of an airplane, a loudspeaker).

In general, the equations of fluid mechanics (called the Navier-Stokes equations) are quite involved and very difficult to solve. In specialized scenarios however we can remove many of the dependencies and obtain certain analytic results.

One such common scenario arises when we consider *laminar* flow. This is sometimes described as "smooth" flow as opposed to "turbulent" flow (imagine the way milk looks when its pouring out of a bag compared to how steam spins and rotates as it rises above hot tea).

To simplify even further, consider the image below:

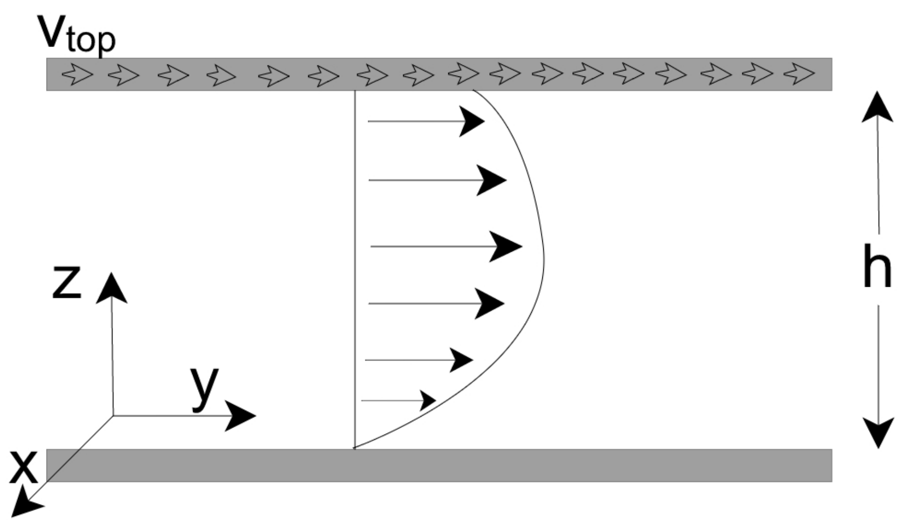


Figure B.1: An example of laminar flow

Here we have a fluid moving horizontally between two infinite parallel plates where the upper plate is moving in the positive y direction with a speed of v_{top} while the bottom plate is stationary.

In this setup, it turns out that the velocity only depends on the height z and is independent of time as well as position y (and, less surprisingly, is also independent of x). If we let v represent the velocity in the y direction the equation governing this situation is given by

$$\mu \frac{d^2 v}{dz^2} = p_g$$

with

- μ = fluid viscosity (units of $ML^{-1}T^{-1}$)
- p_g = pressure gradient (in the y direction - it is constant in this setup)

This equation, together with the boundary conditions $v(0) = 0$ and $v(h) = v_{top}$ form a boundary value problem.

Variations on the value of v_{top} as well as the shape in which the fluid flows (circular vs planar) give rise to other boundary value problems. We will consider these in a later section.

B.1.2 Steady-state temperature

Generally speaking temperature distribution, u, of a body is dependent on three spatial variables as well as time, $u = u(x, y, z, t)$. In the case of a geometrically "simple" object the system can be described by one spatial variable as well as time. For example, consider a rod of uniform cross-section such as the one shown in the picture below.

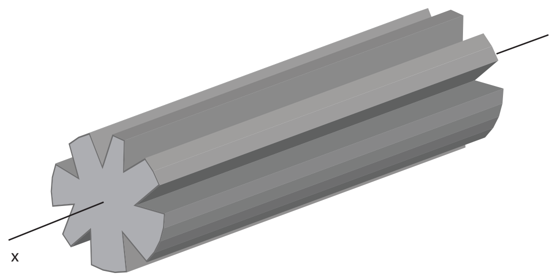


Figure B.2: Uniform rod with constant thermal properties

The temperature, u, at position x can be modelled by

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} - \frac{Ph}{A}(u(x,t) - U_a)$$

where the constants (which we are assuming do not depend on space or time) are:

- c = specific heat of the rod
- ρ = mass density of the rod
- K_0 = thermal conductivity of the rod
- P = perimeter of a cross-section
- A = area of a cross-section
- h = heat transfer coefficient
- U_a = ambient temperature

A further simplification can be made once we assume we are at a steady-state, i.e. $\frac{\partial u}{\partial t} = 0$. With this we end up with

$$0 = K_0\frac{d^2 u}{dx^2} - \frac{Ph}{A}(u(x) - U_a)$$

which is now an ordinary differential equation. In order to solve this we require boundary conditions. There are various scenarios that we can encounter. Some of these include:

- **Fixed endpoint temperatures**: $u(0) = U_0, u(L) = U_L$
- **Insulated endpoints**: $\left.\frac{du}{dx}\right|_{x=0} = \left.\frac{du}{dx}\right|_{x=L} = 0$
- **Convective endpoint, i.e. Newton's Law of Cooling**:

$$\left.\frac{du}{dx}\right|_{x=0} = \frac{h}{K_0}(u(0) - U_a) \quad \text{(i.e.left endpoint } x = 0 \text{ open to the "air")}$$

$$\left.\frac{du}{dx}\right|_{x=L} = -\frac{h}{K_0}(u(L) - U_a) \quad \text{(i.e. right endpoint } x = L \text{ open to the "air")}$$

  We are assuming the heat transfer coefficient, h, is the same at both ends as well as surrounding the rod.

Note that anyone of these (along with many others) can be mixed and matched at each end point. For example, we can have one end insulated and another set to a fixed temperature.

B.1.3 Hanging wire/cable

Using some basic trigonometry and a balancing of forces it can be shown that the equation describing the height, y, of a cable of linear density ρ_c suspending a load of linear density ρ_ℓ is given by

$$\frac{d^2y}{dx^2} = \frac{\rho_\ell g}{T} + \frac{\rho_c g}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where we have the following (assumed) constants:

- ρ_ℓ is the linear density of the load, e.g. road.
- ρ_c is the linear density of the cable
- g is gravitational acceleration
- T is the tension at the lowest point of the cable

Normally we would then prescribe conditions on the height of the cable at two different points, for example $y(0) = h$ and $y(L) = H$.

B.2 Solving BVPs

Generally speaking solving a BVP is nearly identical to solving an IVP. The only change happens when we have to find the arbitrary constants. That is, our prescribed "boundary" values will be used to obtain the values of any constants that were introduced.

For the relatively simple equations that we can deal with at this point this will likely be the extent of it. However in more complicated problems the boundary conditions are sometimes used to simplify the differential equation itself before we even solve it.

Furthermore, while there is reasonably well-established theory regarding the existence and uniqueness of solutions to initial value problems, theoretical discussions regarding BVPs are not as common and are slightly more involved (see Bailey, Shampine and Waltman for further discussion).

For the sake of this course we will move forward in attempting to solve a particular problem and, when possible, decide whether there are potentially multiple solutions. This means that our inability to solve a problem may or may not coincide with the existence of a solution.

B.2.1 Eigenfunctions

A common ordinary differential equation that arises in the study of partial differential equations is

$$y'' + ky = 0$$

with $y(0) = 0$ and $y(L) = 0$. This is a special case of what is known as a *Sturm-Liouville* problem. The question that comes along with this equation is: For what values of k does the BVP have non-trivial solutions?

The solution method is no different than before, we assume $y = e^{mx}$ to get a characteristic equation

$$m^2 + k = 0$$

meaning $m = \pm\sqrt{-k}$. This gives us 3 cases:

Case 1: $k < 0$

In this scenario we can rewrite as $m = \pm\sqrt{|k|}$ which gives the general solution

$$y(x) = c_1 e^{-\sqrt{|k|}x} + c_2 e^{\sqrt{|k|}x}$$

Applying the first boundary condition we get

$$0 = c_1 + c_2 \quad \Rightarrow \quad c_2 = -c_1$$

Using this and applying the second boundary condition we get

$$\begin{aligned} 0 &= c_1 e^{-\sqrt{|k|}L} - c_1 e^{\sqrt{|k|}L} \\ 0 &= c_1 e^{-\sqrt{|k|}L}(1 - e^{2\sqrt{|k|}L}) \end{aligned}$$

so either $c_1 = 0$ meaning $c_2 = 0$ and we have the trivial solution or $1 - e^{2\sqrt{|k|}L} = 0$ meaning $\sqrt{|k|}L = 0$ so that either $L = 0$ (which reduces to the other boundary condition) or $k = 0$.

But $k < 0$ so this is not valid either. Thus we are left with $c_1 = c_2 = 0$. Therefore the case $k < 0$ only generates the trivial solution $y(x) = 0$.

Case 2: $k = 0$

In this case we have the DE $y'' = 0$ which means our solution is a straight line

$$y = Ax + B$$

Given that $y(0) = y(L) = 0$ then the only possibility is the flat line $y = 0$. Thus $k = 0$ also gives the trivial solution.

Case 3: $k > 0$

Here we can rewrite as $m = \pm i\sqrt{k}$ meaning, as discussed in Chapter 2, our general solution is

$$y = c_1 \sin(\sqrt{k}x) + c_2 \cos(\sqrt{k}x)$$

Applying the first boundary condition, $y(0) = 0$ gives

$$0 = c_2$$

Using $y(L) = 0$ this leaves us with

$$0 = c_1 \sin(\sqrt{k}L)$$

which means that either $c_1 = 0$ or $\sin(\sqrt{k}L) = 0$. If $c_1 = 0$ then we are left with the trivial solution yet again.

Our hope for a nontrivial solution to the BVP $y'' + ky = 0, y(0) = y(L) = 0$ thus lies with the stament

$$\sin(\sqrt{k}L) = 0$$

Using our knowledge of the sin function we have that

$$\sqrt{k}L = n\pi, \quad n \in \mathbb{Z}$$

and thus

$$k = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{Z}$$

This means that we (finally) have a non-trivial solution. Specifically, the BVP $y'' + \left(\frac{n\pi}{L}\right)^2 y = 0, y(0) = y(L) = 0$ has solution

$$y = C \sin\left(\frac{n\pi}{L}x\right), \quad n \in \mathbb{Z}$$

The function above is sometimes called an eigenfunction of the BVP with associated eigenvalue $k = \left(\frac{n\pi}{L}\right)^2$.

Various scenarios are shown below:

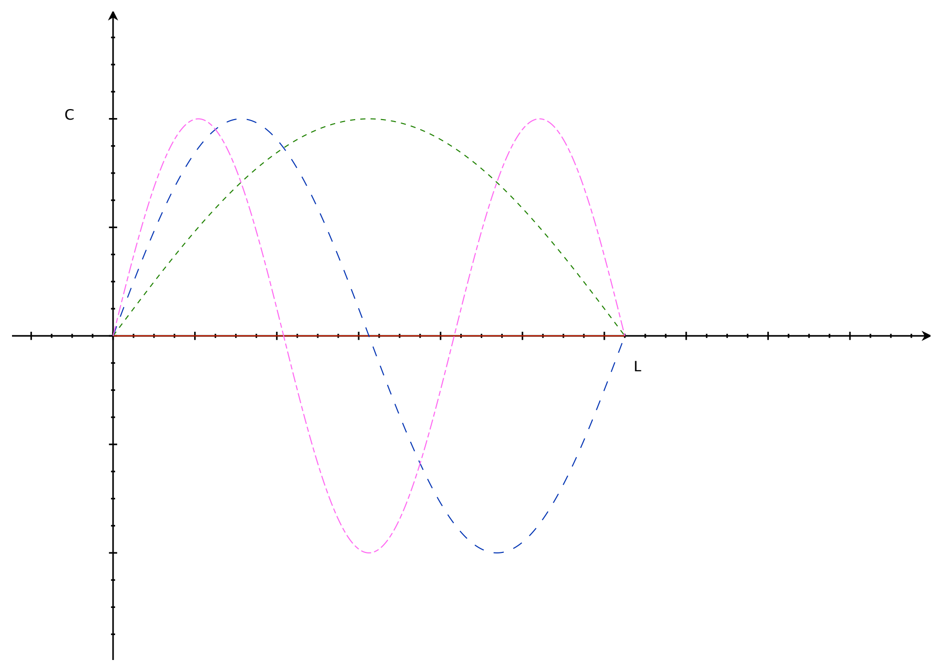


Figure B.3: Eigenfunctions of the BVP $y'' + ky = 0, y(0) = y(L) = 0$, corresponding to $n = 0, 1, 2, 3$

Unfortunately, for a given eigenvalue, we have an infinite set of solutions due to the constant C. While this is mathematically ok, in a real physical situation we would likely need to isolate a single unique solution.

It is beyond the scope of this course to discuss how one narrows down a specific value for C but suffice to say that extra conditions are required (as well as studying Fourier series). Part of the issue is that the equation discussed above ($y'' + ky = 0$) is usually a subproblem of a much larger investigation.

B.2.2 Solutions to physical problems

For many single variable problems the actual differential equation to solve will be rather simple compared to some of the ones that have been encountered in the main chapters. Do not be misled in thinking that boundary value problems are thus simpler than initial value problems; The truth is that most real world situations involve a combination of boundary and initial conditions on rather complex geometries.

The scenarios below arise mostly due to our inherent spatial restriction. Though relatively simple to solve they nonetheless allow for interesting analysis.

Fluid flow

The DE representing laminar flow as shown in figure B.1 was given by

$$\mu \frac{d^2 v}{dz^2} = p_g$$

where the variables are v (horizontal velocity) and z (vertical axis). Since the only appearance of the unknown function v involves the second derivative this particular equation can be solved by straightforward integration yielding

$$\mu \frac{dv}{dz} = p_g z + A$$
$$\mu v(z) = \frac{p_g z^2}{2} + Az + B$$

meaning we have a velocity function of

$$v(z) = \frac{p_g z^2}{2\mu} + az + b$$

where we have incorporated the μ constant into a and b.

Our boundary conditions (sometimes called no-slip conditions) were $v(0) = 0$ and $v(h) = v_{top}$. Using these in turn gives

$$0 = b$$

and

$$v_{top} = \frac{p_g h^2}{2\mu} + ah$$
$$\frac{v_{top}}{h} - \frac{p_g h}{2\mu} = a$$

meaning our final solution is

$$v(z) = \frac{p_g z^2}{2\mu} + \left(\frac{v_{top}}{h} - \frac{p_g h}{2\mu} \right) z$$

See the problem set for a discussion on various velocity profiles.

Temperature

The full steady-state temperature DE of a uniform "one dimesional" rod

$$0 = K_0 \frac{d^2 u}{dx^2} - \frac{Ph}{A}(u(x) - U_a)$$

has various sub-cases that are often considered separately. Here we discuss 2 such cases and leave the rest as exercises in the problem set.

If we assume that the lateral sides of the rod are perfectly insulated then no energy escapes and $h = 0$. This leaves us with a rather trivial DE

$$\frac{d^2u}{dx^2} = 0$$

Note we are allowing for the ends $x = 0$ and $x = L$ to be "controlled" as desired. For example if we prescribe a left and right endpoint temperatures of U_0 and U_L respectively then our problem is to solve

$$\frac{d^2u}{dx^2} = 0, u(0) = U_0, u(L) = U_L$$

The DE, not being dependent on the unknown $u(x)$ is easily integrated twice to give

$$u(x) = Ax + B$$

Applying our boundary conditions yields

$$u(x) = \frac{U_L - U_0}{L}x + U_0$$

which means that, for a laterally insulated rod with fixed temperature endpoints, the long-term temperature profile is a linear progression from U_0 to U_L throughout the rod.

On the other hand, if we also insulate the ends of the rod then we have the BVP

$$\frac{d^2u}{dx^2} = 0, u'(0) = 0, u'(L) = 0$$

Once again, since the DE is the same we have the result

$$u(x) = Ax + B$$

However in this case the boundary conditions require $A = 0$ and do not give any details about B.

This means that the solution $u(x) = B$ for any arbitrary constant B satisfies the BVP. This states that the long-term temperature profile of a fully insulated rod is constant. However it does not state the value of that constant. Surely this cannot be *any* constant we choose can it?

It turns out that the actual value of the constant depends on the initial temperature profile of the rod (i.e. on some initial condition $u(x,t) = f(x)$ at $t = 0$). Studying this is unfortunately beyond the scope of these notes (though you will surely see this in a future course).

Hanging wire

The general hanging wire DE was given by

$$\frac{d^2y}{dx^2} = \frac{\rho_\ell g}{T} + \frac{\rho_c g}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

This particular equation does not have an analytic solution unfortunately. We can consider limiting cases where

- the load is significantly more dense than the cable
- the cable is significantly more dense than the load (usually when there is no load whatsoever)

We will investigate the first scenario, that is $\rho_\ell >> \rho_c$ and leave the second as an exercise in the problem set.

If we assume $\rho_\ell >> \rho_c$ then rewriting the DE as

$$\frac{d^2y}{dx^2} = \frac{\rho_\ell g}{T}\left(1 + \frac{\rho_c}{\rho_\ell}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right)$$

gives $\dfrac{\rho_c}{\rho_\ell} << 1$ meaning the second term is negligible. Thus the DE becomes

$$\frac{d^2y}{dx^2} = \frac{\rho_\ell g}{T}$$

the solution of which is

$$y(x) = \frac{\rho_\ell g x^2}{2T} + Ax + B$$

At this point there are some options for boundary conditions. A common choice is to prescribe heights at 2 different values of x. For example if we let $y(0) = H_0$ and $y(L) = H_L$ then the solution becomes

$$y(x) = \frac{1}{2}\frac{\rho_\ell g x^2}{T} - \frac{1}{2}\frac{(\rho_\ell g L^2 + 2H_0T - 2H_\ell T)\,x}{LT} + H_0$$

Note that this is a parabola opening upward. Though this shape is expected, as you will see in the problem set, the solution is not always parabolic.

Epilogue

Differential equations is a large subject, and in Math 218 we have essentially examined the "tip of alarge iceberg". The subject of DEs splits into two branches:

I: Linear DEs
II: Non-linear DEs

"Solving a DE" means one of three things:

i) finding an *exact* (i.e. explicit) *solution* of the DE

ii) finding an *approximate numerical solution* of the DE using a computer, e.g. MATLAB,

iii) finding an *approximate analytic solution*, e.g. an expansion in terms of a small parameter, referred to as*perturbation methods*.

In all three approaches it is desirable to have a *graphical representation* of the solution.

In Math 218, we have almost exclusively worked with linear DEs, in particular *linear DEs with constant coefficients*, and have shown how to solve them explicitly. For other DEs, there are no general solution algorithms, but one can find solutions in special cases by making use of

- tables of solutions and solution methods

  e.g. D. Zwillinger, Handbook of DEs, 2nd ed., Section II A, pp. 185-363.

- computer software, e.g. MAPLE.

As regards the other methods, numerical solutions are considered in various courses and perturbation methods are introduced in **AMath 351**, the sequel to AMath 250 (the course on which these notes are closely based). A completely different approach, which complements solving a DE, is to study the properties of general classes of solutions, e.g. their long term behaviour. This approach, which dates back to Henri Poincaré at the turn of the 20th century, is called the *qualitative analysis of DEs*. We had a glimpse of this topic when we sketched orbits in state space in Chapter 5. To progress further one needs to study various theoretical issues, which begins in **AMath 351**. The emphasis in qualitative analysis is on *non-linear DEs*, and this subject is the main topic in **AMath 451**.

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Problem Sets

Review Problem Set

Note: Questions 1 and 2 are intended to give you practice with antiderivatives and curve-sketching, two aspects of Calculus that will arise frequently. Questions 3 and 4 are intended to give you practice with logs, exponentials and complex numbers. Questions 5-7 will also be relevant; they require more thought.

1. Express the following antiderivatives in elementary terms. Check your answers by differentiation.

(i) $\displaystyle\int te^{-t}\,dt$ (ii) $\displaystyle\int te^{-t^2}\,dt$ (iii) $\displaystyle\int \frac{1}{1+t}\,dt$

(iv) $\displaystyle\int \frac{t}{1-t}\,dt$ (v) $\displaystyle\int \frac{1-t}{t}\,dt$ (vi) $\displaystyle\int \frac{1}{t(1-t)}\,dt$

(vii) $\displaystyle\int \frac{1}{4+t^2}\,dt$ (viii) $\displaystyle\int \frac{1}{4-t^2}\,dt$ (ix) $\displaystyle\int t\sin(t^2)\,dt$

(x) $\displaystyle\int \sin^2 t\,dt$ (xi) $\displaystyle\int t\sin t\,dt$ (xii) $\displaystyle\int t^3e^{t^2}\,dt$

Hint for (xii): Make a change of variable $u = t^2$.

2. Give a qualitative sketch of the graphs of the following functions. The goal is to give a sketch of the graph, not drawn accurately to scale, but which shows the essential properties of the function. Think about symmetry, asymptotes and the behaviour as $x \to \pm\infty$, where appropriate.

(i) $f(x) = \dfrac{1}{1+x^2}$ (ii) $f(x) = \dfrac{1}{1-x^2}$ (iii) $f(x) = x + \dfrac{1}{x}$

(iv) $f(x) = x - \dfrac{1}{x}$ (v) $f(x) = e^{-x^2}$ (vi) $f(x) = e^{-|x|}$

(vii) $f(x) = x^3 + x$ (viii) $f(x) = x^3 - x$ (ix) $f(x) = \sin^2 x$

(x) $f(x) = e^{-x} + e^{2x}$ (xi) $f(x) = e^{-x} - e^{2x}$ (xii) $f(x) = \dfrac{e^x}{2+e^x}$

3. (i) Simplify the expression

$$e^{a\ln b - b\ln a},$$

where a and b are positive constants i.e. rewrite the expression so as to eliminate the logs and exponentials.

(ii) Find all solutions of the equation $e^x - e^{-x} = 2$.
Hint: Let $u = e^x$.

(iii) Suppose that

$$T(t) = T_0 e^{-kt}$$

and

$$T(t_1) = T_1, \qquad T(t_2) = T_2,$$

where k, t_1 and t_2 are positive. Verify that

$$t_2 = t_1 \frac{\ln(T_2/T_0)}{\ln(T_1/T_0)}.$$

4. Use Euler's formula

$$e^{i\theta} = \cos\theta + i \sin\theta$$

to derive the cosine and sine addition identities:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

Hint: Let $\theta = A + B$ and use $e^{i(A+B)} = e^{iA} e^{iB}$.

5. What does the graph of $f(x) = \sin^{2n} x$ look like for large n? Make a conjecture about the value of $\int_0^{\pi} \sin^{2n} x \, dx$ for large n.

6. We shall find that functions such as

$$f(t) = Ae^{-kt} \sin(\omega t + \beta)\,,$$

where $A, k > 0, \omega > 0$ and β are constants, describe *damped oscillations*.

a) (i) Sketch the graph of $f(t)$ for $A = 1, k = 1, \omega = 1$ and $\beta = 0$.
(ii) Describe how the graph changes as ω increases, and as k increases.

b) Show that $f(t)$ satisfies

$$f'' + 2kf' + (\omega^2 + k^2) f = 0\,.$$

7. Exponential growth of a population i.e. $\hat{N}(t) = N_0 \, e^{rt}$, $r > 0$, can only occur if the resources are essentially unlimited. If there are limited resources one encounters functions such as

$$N(t) = \frac{N_0 \, e^{rt}}{1 - \frac{N_0}{M} + \frac{N_0}{M} \, e^{rt}}\,,$$

where r, N_0, and M are positive constants.

(i) Evaluate $\lim\limits_{t \to +\infty} N(t)$.

(ii) Sketch the graph of $N(t)$ for $N_0 = \frac{1}{2}M$, $r = 1$. How does the shape of the graph change as r increases?

(iii) You will notice that $N(t)$ in (ii) is close to $\hat{N}(t) = N_0\, e^t$ over a subinterval of the t-axis. Find the restriction on t that will ensure that

$$\frac{\hat{N}(t) - N(t)}{\hat{N}(t)} < \frac{1}{100}\,.$$

(iv) In the general case show that if $0 < t < \frac{1}{r}\ln\left(\frac{\epsilon M}{N_0} + 1\right)$, then $0 < \frac{\hat{N}(t)-N(t)}{\hat{N}(t)} < \epsilon$.

Note: In the section of the course dealing with Laplace transforms we shall work with *improper integrals*. We shall use the following results.

8. Show that if $s > 0$ and a is constant, then

$$\int_0^\infty e^{-st}\sin at\, dt = \frac{a}{s^2+a^2}, \qquad \int_0^\infty e^{-st}\cos at\, dt = \frac{s}{s^2+a^2}.$$

Hint: Evaluate the complex antiderivative

$$\int e^{(-s+ia)t} dt,$$

and use Euler's formula.

9. (i) Derive the reduction formula

$$\int t^n e^{-st} dt = -\frac{1}{s}e^{-st}t^n + \frac{n}{s}\int t^{n-1}e^{-st}\, dt\,, \quad \text{for} \quad s > 0\,.$$

(ii) Let $I_n(s)$ denote the improper integral

$$I_n(s) = \int_0^\infty t^n e^{-st}\, dt\,, \quad s > 0\,.$$

Show that

$$I_n(s) = \frac{n}{s}\, I_{n-1}(s)\,, \quad n = 1, 2, \cdots$$

Hence show that

$$I_n(s) = \frac{n!}{s^{n+1}}\,.$$

Problem Set 1
First Order Differential Equations

1. Each equation below defines a one-parameter family of curves. The parameter k assumes all real values.

(i)	$y = e^x + k$	(ii)	$y = 2 + ke^{-2x}$	(iii)	$y = x + ke^{-x}$
(iv)	$y = x + \frac{k}{x}$	(v)	$y = \ln(e^x + k)$	(vi)	$k(x^2 + y^2) = 2y$

   a) Derive the DE that is satisfied by the family of curves. State whether the DE is separable, linear or neither.

   b) Use information deduced from the DE and from the given equation to give a qualitative sketch of each family.

2. Consider the following first order DEs:

(i)	$\frac{dy}{dx} = -2y + e^{-x}$	(ii)	$\frac{dy}{dx} = y \sin x$	(iii)	$\frac{dy}{dx} = x(1-y)$
(iv)	$\frac{dy}{dx} = y(1-y)$	(v)	$\frac{dy}{dx} = \frac{y}{1+x^2}$	(vi)	$x^2\frac{dy}{dx} + 3xy = 1$
(vii)	$\frac{dy}{dx} = -y - x$	(viii)	$\frac{dy}{dx} = -2xy + 2x^3$	ix)	$\frac{dy}{dx} = 1 - y^2$

   a) Find the general solution of each DE.

   Hint: For (viii), see the Review Set, #1 (xii).

   b) By using the form of the DE and the solution, give a qualitative sketch of the family of solutions of each DE, showing some typical solutions and all exceptional solutions. In your diagram, indicate as a dotted curve the set of all points at which the slope $\frac{dy}{dx}$ is zero.

3. Repeat question 2 for the DE

$$x\frac{dy}{dx} + (x+1)y = e^{-x}.$$

   Note: The sketch of the family of solution curves has an intricate structure, and it will take some effort to discover it.

4. Solve each initial value problem. Specify the interval in which your solution is valid. Sketch your solution.

(i)	$\frac{dy}{dx} = y^2 \cos x, \quad y(0) = \frac{1}{2}$	(ii)	$\frac{dy}{dx} = y^2 \cos x, \quad y(0) = 2$
(iii)	$\frac{dy}{dx} = \frac{2y}{x} + x, \quad y(1) = e$	(iv)	$\frac{dy}{dx} = e^{y-x}, \quad y(0) = \ln 2$

5. Suppose that $y = y(x)$ is a solution of the DE $\frac{dy}{dx} + 2xy = 2e^{-x^2}$.

   If $y(0) = 2$, find $y(1)$. Evaluate $\lim_{x\to+\infty} y(x)$, if the limit exists. Does $y(x)$ attain a maximum value for $x \geq 0$? Sketch the graph of $y(x)$.

6. Use the method of undetermined coefficients to find the general solution for each DE *where applicable*. Give the reason if not applicable, but do not solve.

   (i) $\dfrac{dy}{dx} + 2y = 3 - 2x$ (ii) $\dfrac{dy}{dx} - 2y = 2 + e^{-x}$

   (iii) $\dfrac{dy}{dx} + y = \sin 2x$ (iv) $\dfrac{dy}{dx} + 2xy = x^2$

   (v) $\dfrac{dy}{dx} + y = e^{-x}$ (vi) $\dfrac{dy}{dx} + 3y = y^2$

   (vii) $\dfrac{dy}{dx} + 2y = xe^{x}$ (viii) $\dfrac{dy}{dx} + y = \tan x$

7. Solve the DE

$$\frac{dy}{dt} + ky = A\sin\omega t,$$

   with initial condition $y(0) = y_0$, where k, A and ω are positive constants.

   a) Are all/any of the solutions periodic in t?

   b) Evaluate $\lim_{t\to+\infty} y(t)$, if the limit exists.

   c) Write the solution $y(t)$ as the sum of a *transient term* and a *steady state term.*

8. a) When a coil of steel is removed from an annealing furnace its temperature is 684° C. Four minutes later its temperature is 246°C. How long will it take to reach 100° C? Assume that *Newton's law of cooling* holds, which states that the time rate of change of temperature of a cooling body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium. Assume that room temperature is 27°.

   b) You will find it quite tedious to solve part a) because of all the numbers. The problem can be solved efficiently by formulating it more generally, as follows.

   Let T_A be the temperature of the surrounding medium (called the ambient temperature). Let T_0 be the temperature of the coil when it is removed from the furnace at time $t = 0$. The temperature is measured to be T_1 at time t_1. The problem is to find the time t_2 at which the temperature is T_2. So the given quantities are T_A, T_0, T_1, T_2 and t_1 and the unknown is t_2. The idea is to solve the DE for the temperature function $T(t)$ and show that

$$t_2 = t_1 \left[\frac{\ln\left(\frac{T_2 - T_A}{T_0 - T_A}\right)}{\ln\left(\frac{T_1 - T_A}{T_0 - T_A}\right)} \right].$$

9. The velocity $v(t)$ of a sky-diver falling towards the earth's surface satisfies the DE

$$m\frac{dv}{dt} = mg - \alpha v,$$

where m is the mass, g the acceleration due to gravity (assumed constant), and α is the drag coefficient (see the notes).

   a) Find $v(t)$ assuming an initial velocity $v(0) = v_0 > 0$.
   b) Show that as time passes, $v(t)$ approaches the terminal velocity $v_{\text{term}} = mg/\alpha$. Does $v(t)$ ever equal v_{term}?
   c) Find the distance y fallen as a function of time t.
   (d) Find the distance y fallen as a function of velocity v.

10. The velocity v of a projectile fired vertically up from the surface of a planet and travelling only under the influence of gravity, satisfies the DE

$$v\frac{dv}{dr} = -\frac{gR^2}{r^2},$$

where r is the distance of the projectile from the centre of the planet, R is the radius of the planet and g is the acceleration due to gravity on the surface of the planet (see the notes).

   a) Find v as a function of r, assuming the initial condition $v(R) = v_{\text{init}}$.
   b) Find the escape velocity for the planet.

11. A student carrying a flu virus returns to an isolated college campus of 1,000 students. The rate at which the virus spreads is proportional to the product of the number of infected students and the number not infected. Predict the number of infected students after 7 days, if it is found that after 3 days 40 students are infected. When is the infection spreading most rapidly? Illustrate your solution by graphing the number of infected students as a function of time t.

    Suggestion: Formulate and solve this problem more generally, as in #8b).

12. Suppose that a corpse with temperature 27°C is discovered at midnight, and that the ambient temperature is a constant 17°C. The body is moved quickly to a morgue where the ambient temperature is 7°. After one hour the body temperature is 20°C. Estimate the time of death.

    Suggestion: Formulate and solve this problem more generally, as in #8b).

13. A tank is used in certain hydrodynamic experiments. After one experiment, the tank contains 200 litres of a dye solution with a concentration of 1 g/litre. To prepare for the next experiment, the tank is to be rinsed with clear water flowing in at a rate of 2 litres/min, the well-stirred mixture flowing out at the same rate. How long will it take to reduce the concentration of dye to 1% of its original value?

14. The DE

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N - h,$$

where r, K and h are positive constants, describes a population of fish with natural growth rate coefficient r, carrying capacity K and a constant harvest rate h.

a) Show that if $h < \frac{1}{4}rK$ there are two equilibrium solutions $N(t) = N_1$ and $N(t) = N_2$ where N_1 and N_2 are constants with $0 < N_1 < N_2$. Find N_1 and N_2.

b) Give a qualitative sketch of the solution curves in the case $h < \frac{1}{4}rK$. Discuss the longterm behaviour of $N(t)$ in the two cases (i) $N(0) > N_1$ and (ii) $N(0) < N_1$.

15. A projectile of mass m is fired straight up from the earth's surface with initial velocity v_0. If v_0 is small compared to the escape velocity it is reasonable to assume that the acceleration due to gravity g is constant throughout the motion. After rising to a certain height, the projectile will momentarily stop before returning to the earth's surface. The ascent time t_a is the time taken to reach maximum height.

(a) Calculate t_a in two cases:

 (i) neglecting air resistance,

 (ii) assuming that the force due to air drag is proportional to the velocity.

(b) Which time is shorter?

16. An object of mass m is released from rest at a height of h metres above the earth's surface, and strikes the ground after falling for t_h seconds. Assume that the force due to air drag is proportional to the velocity. Is it possible to determine the drag coefficient knowing m, h and t_h?

17. Experiments show that the rate of decrease of atmospheric pressure p with height h is proportional to the product of the acceleration due to gravity (assumed constant) and the pressure.

a) Express the above result as a differential equation for the unknown function $p(h)$.

b) Let p_0 denote the pressure at sea-level, p_1 denote the pressure at a reference altitude of h_1. Show that the dependence of height h on pressure p can be written in the form

$$h = h_1 \frac{\ln[p/p_0]}{\ln[p_1/p_0]}.$$

c) Suppose that the pressure at sea level is 104 kPa and at an altitude of 3000 m is 70 kPa. Most people will lose consciousness if the pressure falls below 50 kPa. At what altitude does this occur? (kPa means kilopascals, and 1 Pa = 1 N/m^2).

18. Consider a population of size $N(t)$ which grows exponentially at a rate r, but is harvested at a constant rate H per day. Then $N(t)$ satisfies the DE

$$\frac{dN}{dt} = rN - H.$$

a) Suppose that $r = 0.01$ days^{-1} and $N(0) = 4000$. Show that if $H > 40$ per day, then the population becomes extinct in a finite time, but if $H < 40$, the population will increase without bound.

b) Referring to a), if $H > 40$ per day, find the time of extinction.

c) In a), $H = 40$ represents the critical harvest rate. Find the critical harvest rate for arbitrary $r > 0$ and arbitrary $N(0)$.

19. A mixing tank of capacity $V_{\max}$ litres initially contains V_0 litres of pure water. A salt solution of constant concentration c_{in} gm/litres flows in at a rate $2f$ litres/min and the contents of the tank flow out at f litres/min. For this physical situation it is reasonable to define a characteristic time by $t_c = \frac{V_0}{f}$ (the time to drain the initial volume, with zero inflow) and a characteristic mass by $m_c = V_{\max} c_{in}$ (the mass of salt in the tank if it were completely filled by the inflow). Then one can define a dimensionless (unitless) time τ and dimensionless mass $\mathcal{M}$ by

$$\tau = \frac{t}{t_c}, \quad \mathcal{M} = \frac{m}{m_c},$$

where m is the mass of salt in the tank at time t.

a) Show that $\mathcal{M}$ satisfies the DE

$$\frac{d\mathcal{M}}{d\tau} = -\frac{\mathcal{M}}{1+\tau} + 2b,$$

where $b = \frac{V_0}{V_{\max}}$.

b) Show that when the tank is filled it contains $(1 - b^2)V_{\max} c_{\text{in}}$ grams of salt.

20. Joe and Dave sit down to talk and enjoy a cup of coffee. When the coffee is served, Joe immediately adds a teaspoon of cream to his coffee. Dave waits 5 minutes before adding a teaspoon of cream (which has been kept at a constant temperature). The two now begin to drink their coffee. *Who has the hotter coffee?* Assume that the cream is cooler than the air and use Newton's law of cooling.

Problem Set 2
Second Order Linear Differential Equations

1. Find the general solution of each *homogeneous* linear DE:

   (i) $y'' + y' - 2y = 0$ (ii) $y'' + 4y' + 4y = 0$

   (iii) $y'' + 2y' + 5y = 0$ (iv) $y'' - 6y' + 10y = 0$.

2. Find the general solution of the following *inhomogeneous* linear DEs.

   (i) $y'' + y' - 6y = 6t$

   (ii) $y'' + y' - 6y = 5\cos t$

   (iii) $y'' + y' - 6y = 5e^{4t}$

3. a) Find the unique solution of the initial value problems

   (i) $y'' + 4y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$

   (ii) $y'' + 2y' + 5y = 0, \quad y(0) = 0 \;, \quad y'(0) = 1.$

   b) Give a qualitative sketch of the solutions in a).

4. Write the general solution of the linear DE

$$y'' + \omega^2 y = b,$$

   where ω and b are constants, *by inspection*.

5. A particle undergoes simple harmonic motion (SHM) described by

$$y'' + \omega^2\, y = 0,$$

   with initial conditions $y(0) = y_0$, $y'(0) = v_0$. Show that the amplitude of the SHM is

$$y_{\max} = \sqrt{y_0^2 + \tfrac{v_0^2}{\omega^2}}.$$

6. Find the general solution of the DE

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + b^2 y = 0,$$

   where k and b are constants.

   Note: There are a number of different cases, depending on the values of k and b.

7. Find the general solution of the DE

$$y'' + y' - 6y = \alpha e^{rt},$$

   where r and α are constants. Which values of r have to be treated as special cases?

8. Find the general solution of the DE

$$y'' + 2y' + 2y = \alpha \cos \omega t$$

where α and ω are constants.

9. a) Find the general solution of the DE

$$y'' + \omega_0^2\, y = \alpha \cos \omega t,$$

where ω_0, α and ω are constants. Which values(s) of ω have to be treated as special cases?

b) Find the unique solution of the DE in a) with $\omega^2 \neq \omega_0^2$, subject to the initial condition $y(0) = 0,\ y'(0) = 0,$

c) Show that the solution in b) can be written in the form

$$y = A(t) \sin \left[\tfrac{1}{2}(\omega_0 + \omega)t\right],$$

where

$$A(t) = \frac{2\alpha}{\omega_0^2 - \omega^2} \sin \left[\tfrac{1}{2}(\omega_0 - \omega)t\right].$$

Give a qualitative sketch of the graph of $y(t)$ in the case where $\omega_0 - \omega$ is small compared to ω_0. If you want to use specific values, use $\omega_0 = 11, \omega = 9$, but don't try to sketch the curve exactly.

d) Show that the unique solution of the DE in a) with $\omega^2 = \omega_0^2$, subject to the initial condition $y(0) = 0,\ y'(0) = 0$ is

$$y = \frac{\alpha}{2\omega_0} t \sin \omega_0 t.$$

Give a qualitative sketch of the graph.

10. The homogeneous linear DE

$$y'' + py' + qy = 0 \qquad (1)$$

has $y(t) = 0$ as an equilibrium solution. In studying a physical system one is interested under what conditions the general solution will approach the equilibrium solution as time $t \to \infty$. Show that the general solution of (1) approaches the equilibrium solution as $t \to +\infty$ if and only if $p > 0$ and $q > 0$.

11. a) An object of mass 2 kg, hanging from a spring, extends that spring 0.1 m from its equilibrium position. Calculate the spring constant k in Newtons per metre. Assuming that the mass moves with no damping and no driving force, calculate the period of the resulting Simple Harmonic Motion (SHM).

b) A resistive medium exerts a damping force of 200 Newtons when acting on the above object moving with a velocity of 10 metres per second. Calculate the damping constant in kilograms per second.

c) Consider a damped mass–spring system with spring as in a) and damping as in b). Will the system undergo damped oscillations?

Note: A force of 1 Newton will give a mass of 1 kilogram an acceleration of 1 metre/second^2.

12. A mechanical system undergoes damped oscillations governed by the DE

$$y'' + 2\lambda y' + w_0^2 y = 0.$$

Suppose that $\zeta = \frac{\lambda}{w_0} = 10^{-5}$. How many oscillations will take place in the time interval during which the amplitude decays by 1%? In this situation the motion of the system approximates SHM over a restricted time interval.

13. Consider the DE

$$y'' + 2\lambda y' + \omega^2 y = 0,$$

with initial conditions $y(0) = y_0,\ y'(0) = -v_0$, where $\lambda \geq \omega > 0$, so that oscillations do not occur.

(i) One expects that if v_0 is sufficiently large, then the mass will pass through the equilibrium position before coming to rest. Find the restriction on v_0 that will ensure that this happens, in the case of critical damping $\lambda = \omega$. Also find the time t_{zero} at which y is zero.

(ii) Find the time t_{crit} at which y attains its minimum value $y_{\min}$. Show that t_{crit} satisfies

$$t_{\text{crit}} = t_{\text{zero}} + \frac{1}{\omega},$$

and that

$$y_{\min} = -(\frac{v_0}{\omega} - y_0)e^{-\omega t_{\text{crit}}}.$$

(iii) Sketch the graphs of the displacement $y(t)$ and the velocity $y'(t)$, and label t_{zero}, t_{crit}, v_0 and $y_{\min}$.

14. Referring to #13(i) show that in the case $\lambda > \omega$, the restriction on v_0 is

$$v_0 > (\lambda + \sqrt{\lambda^2 - \omega^2})y_0.$$

15. We wish to find a particular solution of the DE

$$y'' + 2\lambda y' + \omega_0^2 y = f_0 \cos(\omega t). \tag{2}$$

Consider the complex DE

$$z'' + 2\lambda z' + \omega_0^2 z = f_0 e^{i\omega t} \tag{3}$$

The DE (2) is the real part of this DE. Consider a trial function

$$z = Re^{i(\omega t - \phi)}, \tag{4}$$

where R and ϕ are real constants.

(i) Show that (4) is a solution of (3) if and only if

$$R = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\lambda^2\omega^2}},$$
$$\cos\phi = \frac{1}{f_0}A(\omega_0^2 - \omega^2), \quad \sin\phi = \frac{2}{f_0}A\lambda\omega.$$

(ii) Hence conclude that

$$y = R\cos(\omega t - \phi),$$

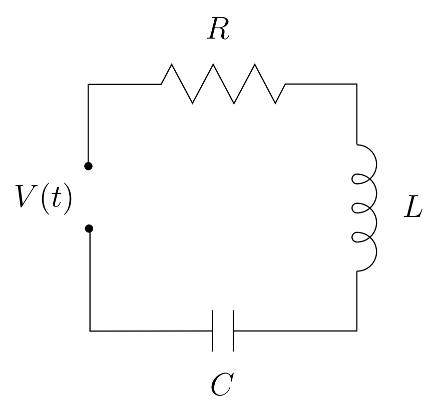
where R and ϕ are given in part (i), is a particular solution of (2).

16. The charge $Q(t)$ on the capacitor in the electrical circuit shown satisfies

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t),$$

$L = 0.5$ henrys is the coil's inductance
$R = 6$ ohms is the resistor's resistance
$C = 0.02$ farads is the capacitor's capacitance.

and $V(t)$ is the applied voltage.

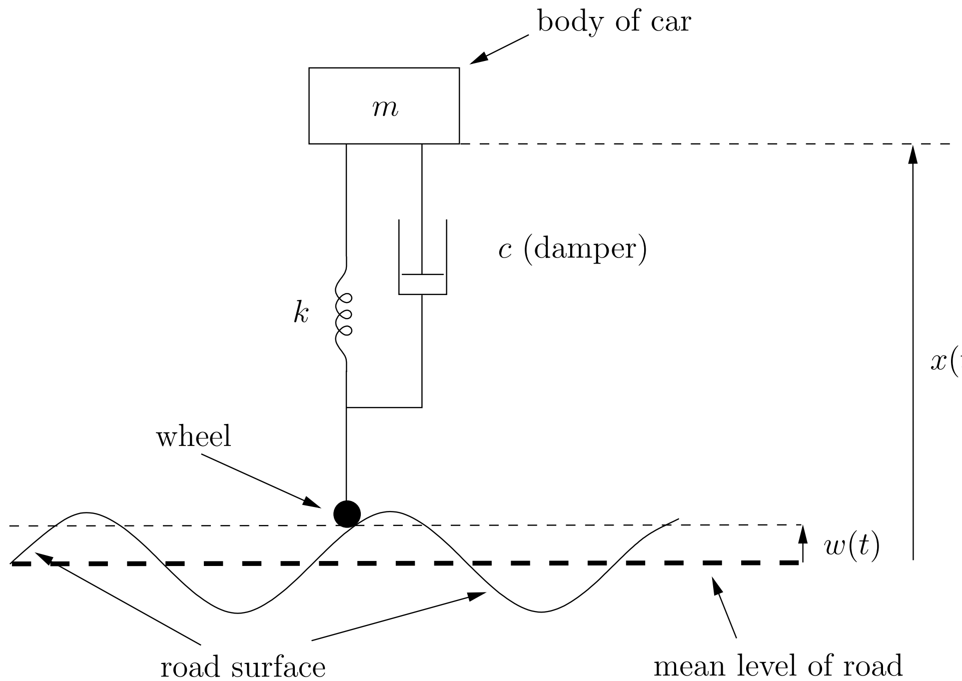


(i) Is the circuit oscillatory?

(ii) If $V(t) = 24\sin 10t$ volts and $Q(0) = 0 = Q'(0)$, find $Q(t)$.

(iii) Sketch the transient solution, the steady state solution and the full solution $Q(t)$.

17. A car suspension system may be modelled by a mass-spring-damper model as shown; the wheel is assumed to ride on a sinusoidally-undulating road, that is the wheel's position $w(t)$ is assumed to be given by $w(t) = w_0\cos(at)$ at time t.



a) Show, using Newton's Second Law, that a suitable differential equation for describing the motion of the mass m is

$$m\ddot{x} + c(\dot{x} - \dot{w}) + k(x - w - \ell) = 0, \qquad (*)$$

where ℓ is the equilibrium length of the spring, and where $\dot{\xi} \equiv d\xi/dt$.

b) Show that the problem may be solved by considering it to be the same as that discussed in the lectures, but with a modified applied force; that is, show that $(*)$ may be written as

$$m\ddot{z} + c\dot{z} + kz = F(t),$$

with suitable choices of z and F. Find $F(t)$ in the form $F(t) = F_0 \cos(at - \phi)$.

18. An electric charge q of mass m moves with velocity $\mathbf{v} = (v_1, v_2, v_3)$ in a region of space where there is a magnetic field $\mathbf{B} = (0, 0, B)$ of constant strength. According to the laws of physics, the motion of the charge is given by the following equations:

$$m\frac{dv_1}{dt} = \frac{qB}{c}v_2, \quad m\frac{dv_2}{dt} = -\frac{qB}{c}v_1, \quad m\frac{dv_3}{dt} = 0.$$

where c is the velocity of light in a vacuum. Assume an initial condition of the form $\mathbf{v}(0) = (u_1, u_2, u_3)$.

a) Eliminate v_2 from the first two equations and get a single second order DE for v_1.

b) Show that the quantity $\omega_c = \frac{qB}{mc}$ has the dimensions (units) of a frequency, thus justifying its name, which is the "*cyclotron frequency*" (or "*gyrofrequency*").

c) Solve the resulting DE for v_1, and then find v_2.

d) Identify the curve representing the path of the particle.

Problem Set 3

Laplace Transforms and Differential Equations

1. Calculate the Laplace transform $Y = \mathcal{L}[y]$ of each function y, using the definition. Give the domain of validity. Use complex variables for the trig. functions.

   (i) $y(t) = t^2$ (ii) $y(t) = te^{-2t}$ (iii) $y(t) = t\cos \pi t$ (iv) $y(t) = t\sin \pi t$

2. Use the fact that the Laplace transform operator $\mathcal{L}$ is *linear*, and the results from #1 to write down the Laplace transform of $y(t) = t(2t + 3e^{-2t} + \pi \sin \pi t)$.

3. Knowing $\mathcal{L}[e^{\alpha t}] = \frac{1}{s-\alpha}$ for $s > Re(\alpha)$, find the Laplace transforms of

$$\mathcal{L}[e^{at}\cos bt] \quad \text{and} \quad \mathcal{L}[e^{at}\sin bt].$$

4. Show that $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$, for $s > 0$, where n is a non-negative integer.
   Hint: See #9 on the Review Problem Set.

5. Use the Formula Sheet to find the inverse Laplace transforms of

$$\text{(i)} \quad Y(s) = \frac{1}{(s+a)(s+b)} \qquad \text{(ii)} \quad Y(s) = \frac{1}{(s^2+a^2)(s^2+b^2)}, \qquad \text{(iii)} \quad Y(s) = \frac{1}{(s+a)(s^2+b^2)},$$

   where a and b are real constants with $a \neq b$.

6. Find each inverse Laplace transform using the first shift theorem:

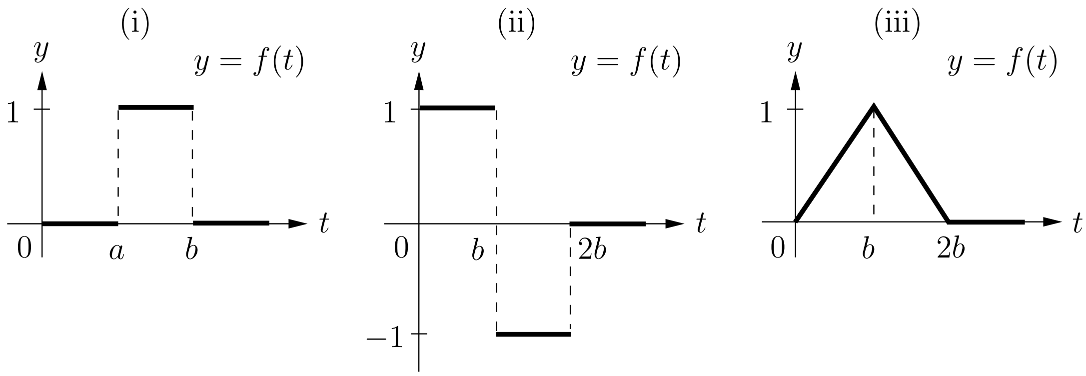
$$\text{If} \quad \mathcal{L}^{-1}[F(s)] = f(t) \quad \text{then} \quad \mathcal{L}^{-1}[F(s-c)] = e^{ct}f(t).$$

$$\text{(i)} \ \mathcal{L}^{-1}\left[\frac{5}{s(s^2+4s+5)}\right] \qquad \text{(ii)} \quad \mathcal{L}^{-1}\left[\frac{s-1}{2s^2+5s+2}\right] \qquad \text{(iii)} \quad \mathcal{L}^{-1}\left[\frac{1}{s^2-4s+8}\right]$$

7. Find the Laplace transform of each piecewise defined function $f(t)$ in two ways:

   (a) by directly using the definition,

   (b) by expressing $f(t)$ in terms of the Heaviside step function, $H(t)$.



8. Find the inverse Laplace transform using the second shift theorem, and sketch the graph of the resulting function:

$$\text{If} \quad \mathcal{L}^{-1}[F(s)] = f(t), \quad \text{then} \quad \mathcal{L}^{-1}[e^{-cs}F(s)] = H(t-c)f(t-c).$$

(i) $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s+2}\right]$ (ii) $\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+1}\right]$ (iii) $\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2-1}\right]$.

9. Derive the equation

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0)$$

for a suitably restricted function $f(t)$. State the hypothesis on $f(t)$.
Hint: Very little work is required.

10. Solve the following initial value problems using the Formula Sheet

(i) $y' + 2y = e^{-3t}$, $y(0) = 5$ (ii) $y' + 2y = 4\cos 2t$; $y(0) = 1$.

(iii) $y'' - y' - 2y = 0$; $y(0) = 1$, $y'(0) = 0$.

(iv) $y'' + y = 3\sin 2t$; $y(0) = 6$, $y'(0) = 1$.

11. A mixing tank with constant volume V_0 and flow rate k is initially filled with pure water. If the inflow concentration is a constant c_{in} for $0 \le t < T$, and is then zero afterwards, calculate the mass of chemical in the tank at time $2T$.

12. Consider a population $y(t)$ which undergoes exponential growth with growth factor r. Starting at time $t = 0$, the population is harvested at a constant rate h (number per unit time), for a period of time T.

(i) Show that this system is governed by the DE

$$y' - ry = -h + hH(t-T),$$

where H is the Heaviside step function.

(ii) If the initial population is y_0, find the population y at time t.

(iii) Sketch the family of solutions with y_0 as the parameter. What happens if $y_0 < \frac{h}{r} - \frac{h}{r}e^{-rT}$? What is the physical interpretation?

13. Consider the system described by the DE

$$y' + y = g(t),$$

where the input function $g(t)$ is the saw-tooth function in #7(iii), and the initial condition is $y(0) = 0$.

(i) Based on the graph of the input function, make an educated guess as to the graph of the response $y(t)$.

(ii) Show that the response at time t is

$$y(t) = \tfrac{1}{b}[f(t) - 2H(t-b)f(t-b) + H(t-2b)f(t-2b)],$$

where $f(t) = t - 1 + e^{-t}$.

(iii) Confirm your "educated guess" in (i) by appropriate analysis. In particular show that the maximum response occurs at time $t_{\max} = \ln(2e^b - 1)$, and is given by $y_{\max} = \frac{1}{b}(2b - t_{\max})$.

14. Consider the undamped oscillator DE with a driving force that is a rectangular pulse of duration b:

$$y'' + y = 1 - H(t-b),$$

where H is the Heaviside step function.

(i) Use the Laplace transform to show that the unique solution satisfying the initial conditions $y(0) = 0 = y'(0)$ is

$$y(t) = 1 - \cos t - H(t-b)[1 - \cos(t-b)],$$

(ii) Show that for $t > b$, the response $y(t)$ can be expressed as

$$y(t) = A\sin(t - \delta),$$

i.e. simple harmonic motion. Express the amplitude A and phase δ in terms of the pulse duration b.

(iii) Sketch the full response for $b = 2\pi, 3\pi$ and $2\pi + \epsilon$, where $0 < \epsilon \ll 1$.

(iv) For what values(s) of b is the amplitude of the long-term response a maximum? zero?

15. Let $*$ denote the convolution operation. Show that

i) $e^{\alpha t} * e^{\alpha t} = te^{\alpha t}$ for any $\alpha \in \mathbb{C}$.

ii) $e^{\alpha t} * e^{\beta t} = \dfrac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta}$ for any $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$.

16. Use complex functions to verify that

i) $2\cos at * \sin at = t\sin at$.

ii) $\cos at * \cos at - \sin at * \sin at = t\cos at$.

iii) $\cos at * \cos at + \sin at * \sin at = \frac{1}{a}\sin at$.

17. Prove that the convolution operation is commutative, i.e. $f * g = g * f$.

18. Use the convolution theorem to find the inverse Laplace transform of

$$\text{i) } G(s) = \frac{1}{(s+1)^2(s+2)} \qquad \text{ii) } G(s) = \frac{s}{(s^2+1)(s+2)} \qquad \text{iii) } G(s) = \frac{1}{(s^2+1)(s+2)}.$$

Hint: Use complex functions for ii) and iii).

19. Use the Convolution Theorem to prove the *time integration formula*:

$$\text{if} \quad \mathcal{L}[f(t)] = F(s), \quad \text{then} \quad \mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s).$$

20. i) Derive the *frequency differentiation formula*:

$$\text{if} \quad \mathcal{L}[f(t)] = F(s), \quad \text{then} \quad \mathcal{L}[-tf(t)] = F'(s), \quad \text{for} \quad s > a.$$

It is assumed that $f(t) = O(e^{ct})$ as $t \to +\infty$. You may assume that differentiating inside the integral is valid.

ii) Generalize i) to give a formula for the n^{th} derivative $F^{(n)}(s)$.

21. Prove that if f is periodic of period T, and piecewise continuous for $t \geq 0$, then

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)dt.$$

22. i) Express the solution of the initial value problem

$$y'' - \omega^2 y = u(t),$$
$$y(0) = 0, \quad y'(0) = 0$$

as a convolution. Give the answer in terms of hyperbolic functions.

ii) Find the asymptotic form of the response as $t \to +\infty$ to the input $u(t) = \delta(t)$.

23. The current $y(t)$ in an RLC circuit with applied voltage $e(t)$ satisfies the DE

$$y'' + \frac{R}{L}y' + \frac{1}{LC}y = \frac{1}{L}e'(t).$$

Find the transfer function $G(s)$, regarding $e(t)$ as the input and $y(t)$ as the response. Assume that $e(t) = 0$ for $t \leq 0$.

24. A linear time-invariant system is of the form

$$y'' + a_1 y' + a_0 y = f(u(t), u'(t), \ldots).$$

Given that the transfer function of the system is $G(s) = \frac{s+5}{s^2+4s+3}$, find the DE relating $y(t)$ and $f(t)$.

25. i) Consider the linear time invariant system described by the scalar DE

$$y' + ky = 0,$$

where k is a positive constant.

At time $t = 0$, the state is $y(0) = y_0$. At time $t_1 > 0$, an instantaneous impulse of magnitude p is applied to the system. Find the response $y(t)$ for $t > 0$, and sketch its graph.

ii) Suppose that an idealized impulse of magnitude p is applied periodically with period T, to the system in i). Find the periodic steady state response $y_{\text{per}}(t)$, and sketch its graph. In particular, give the maximum and minimum values of y for this response.

26. Consider the linear time-invariant system described by the DE

$$y'' + \omega^2 y = 0,$$

with initial condition $y(0) = 0$, $y'(0) = v_0$. At time $t = t_0$ an idealized impulse $p\delta(t-t_0)$ is applied which brings the system momentarily to rest.

i) Solve the DE $y'' + \omega^2 y = p\delta(t - t_0)$ subject to the ICs $y(0) = 0$, $y'(0) = v_0$.

ii) Find the value of p which will bring the system momentarily to rest at time $t = t_0$.

iii) Sketch the graphs of $y(t)$ and $y'(t)$, for $t > 0$, using the value of p obtained in the previous part.

27. Suppose that the transfer function of a linear time-invariant system is

$$G(s) = \frac{s}{(s^2+1)(s+2)}.$$

Find the response of the system in the time domain to

i) an idealized unit impulse at time $t_0 > 0$,

ii) a unit step input at time $t_0 > 0$.

28. i) Calculate $\mathcal{L}\left[\delta_\varepsilon(t-a)\right]$, where δ_ε is the pulse function, defined by

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & \text{if} \quad 0 \le t < \varepsilon \\ 0, & \text{otherwise,} \end{cases}$$

and $a \ge 0$.

ii) Show that $\lim\limits_{\varepsilon \to 0^+} \mathcal{L}\left[\delta_\varepsilon(t-a)\right] = e^{-as}$, thereby providing further justification for the definition

$$\mathcal{L}\left[\delta(t-a)\right] = e^{-as}.$$

29. Use the Laplace transform to show that the two initial value problems

i) $my'' + cy' + ky = p\delta(t), \qquad y(0) = 0, \quad y'(0) = 0,$

ii) $my'' + cy' + ky = 0, \qquad y(0) = 0, \quad y'(0) = \frac{p}{m},$

have the same solution for $t > 0$.

Comment: This result shows that an idealized impulse of magnitude p imparts an initial velocity of $v_0 = p/m$ to a mass-spring system that is initially at rest.

30. An undamped mass-spring system of natural frequency ω is initially at rest. At each time $t = 2\pi n/\omega$, $n = 0, 1, 2, \ldots$ the mass is struck with a hammer which imparts an impulse of magnitude p per unit mass in the positive direction. Determine the resulting motion. Comment on the long term behaviour. Sketch the graphs of $y(t)$ and $y'(t)$. How smooth is $y(t)$?

Problem Set 4
Linear Vector DEs

1. In each case verify that the given vector-valued function satisfies the vector DE $\mathbf{x}' = A\mathbf{x}$.

$$\text{i)}\ A = \begin{pmatrix} 2 & 6 \\ -2 & -5 \end{pmatrix}; \qquad \mathbf{x}(t) = e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\text{ii)}\ A = \begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix}; \qquad \mathbf{x}(t) = e^{-t}\begin{pmatrix} 5\cos t \\ -2\cos t - \sin t \end{pmatrix}$$

$$\text{iii)}\ A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}; \qquad \mathbf{x}(t) = e^{t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right].$$

2. a) Give a description of the *eigenvalue method* for solving a vector DE.

   b) Find the general solution of each vector DE using the eigenvalue method, referring to your explanation in a).

$$\text{i)}\ \mathbf{x}' = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}\mathbf{x} \qquad \text{ii)}\ \mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$$

$$\text{iii)}\ \mathbf{x} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}\mathbf{x} \qquad \text{iv)}\ \mathbf{x}' = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\mathbf{x}.$$

3. Find the unique solution for each DE in #2 that satisfies each given initial conditions (i.e. for each DE, two initial conditions are given, and each initial condition will give a unique solution):

$$\text{i)}\quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \mathbf{x}(\ln 2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{ii)}\quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \mathbf{x}\left(\tfrac{\pi}{4}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\text{iii)}\quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{iv)}\quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \mathbf{x}(\ln 2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

4. Give a qualitative sketch of the orbits for each DE in #2.

5. a) Give a description of the Laplace transform method for solving an inhomogeneous linear vector DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t). \qquad (*)$$

   b) Find the general solution of the DE (*) for the 2×2 matrices given in #2, and the input functions given below:

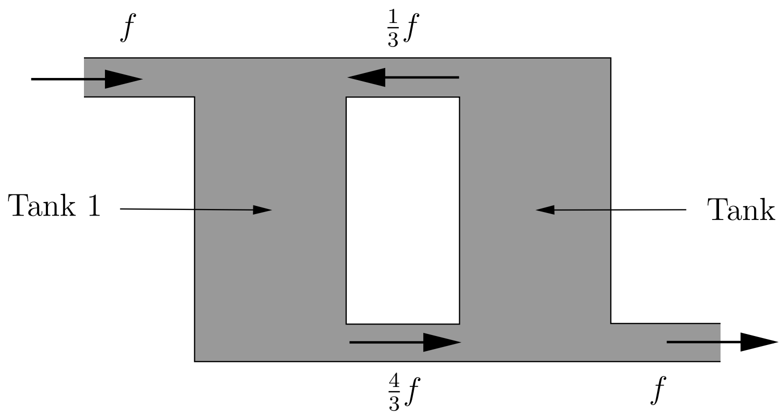
$$\text{(i)}\quad \mathbf{f}(t) = e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{(ii)}\quad \mathbf{f}(t) = \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix}$$

$$\text{(iii)}\quad \mathbf{f}(t) = e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{(iv)}\quad \mathbf{f}(t) = \cos t\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

6. Consider the coupled constant volume mixing tank system as shown with inflow concentration $c_{in}(t)$, with state vector

$$\mathbf{x} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix},$$

where m_1 and m_2 denote the mass of chemical in tanks 1 and 2 respectively. Let V be the volume of each tank.



a) Show that the vector DE governing the state of the system is

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}, \qquad A = \begin{pmatrix} -4b & b \\ 4b & -4b \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 3bV c_{in} \\ 0 \end{pmatrix},$$

where $b = \frac{f}{3V}$ (this simplifies the algebra).

b) Find the solution to the homogeneous DE $\mathbf{x}' = A\mathbf{x}$.

c) Find the solution for the following initial conditions, assuming $c_{in}(t) = 0$:

i) $m_1(0) = M$, $m_2(0) = 0$ ii) $m_1(0) = \frac{1}{3}M$, $m_2(0) = \frac{2}{3}M$ iii) $m_1(0) = 0$, $m_2(0) = M$.

In each case give a qualitative sketch of the mass functions $m_1(t)$ and $m_2(t)$ on the same axes. Use the graphs to give a physical interpretation of the behaviour of the system, discussing whether the mass of chemical in each tank is increasing or decreasing and whether the masses are ever equal.

d) Referring to c), in which case does the system flush most rapidly, i.e. in which case does the total mass in the system tend to zero most rapidly? First make an "educated guess", and then give a mathematical analysis.

e) Sketch typical orbits of the DE in $\mathbb{R}^2$, subject to the restriction $m_1 \geq 0$, $m_2 \geq 0$.

(i) Mark the orbits corresponding to the three solutions in part c) on your sketch.

(ii) Consider an initial state with $m_2(0) < m_1(0)$. Use the sketch to describe the future evolution of the system.

(iii) Do the same for an initial state with $m_2(0) > 4m_1(0)$.

f) Find the solution of the non-homogeneous DE assuming $c_{in}(t) = c$, a constant, and an arbitrary initial state $\mathbf{x}(0) = \mathbf{a}$. What is the asymptotic behaviour as $t \to +\infty$?

Appendix Problems

Series Solutions of Differential Equations

1. Find two linearly independent series solutions to the DE
$$y'' - xy' = 0.$$

2. Apply the power series method to the DE
$$y'' - xy' + y = 0.$$

3. Find two linearly independent series solutions to each of the following DEs:

   (a) $y'' + 2xy' + 2y = 0$

   (b) $(x-2)y'' + xy' - y = 0$

4. Find a power series solution to the IVP
$$(x^2+1)y'' + 2xy' = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

Boundary Value Problems

1. Find the solutions to the following boundary value problems. If there are no solutions state the reason why.

   (a) $y'' - y = 0, \quad y(0) = 1, y(\ln(2)) = 0$

   (b) $y'' + 4y = 0, \quad y(0) = 4, y(\pi) = 0$

   (c) $y'' + 4y = 0, \quad y(0) = 4, y(\frac{\pi}{4}) = 0$

   (d) $y'' + y = 1, \quad y'(0) = 1, y(\pi) = 0$

   (e) $y'' - 6y' + 9y = 0, \quad y(0) = 1, y(\ln(3)) = -1$

   (f) $y'' - y' = 0, \quad y(0) + y'(0) = 1, y'(5) = e^5$

2. Find the values of k the will give non-trivial solutions to the following BVPs. Furthermore, find the solutions for those values of k.

   (a) $y'' + ky = 0, \quad y'(0) = 0, y(L) = 0$

   (b) $y'' + ky = 0, \quad y'(0) = 0, y'(L) = 0$

   (c) $y'' + ky = 0, \quad y(0) = 0, y'(L) = 0$

3. The DE for laminar flow between two infinite flat plates separated by a distance h is given by
$$\mu \frac{d^2v}{dz^2} = p_g$$
where v is the horizontal velocity, z is vertical axis, μ is the fluid viscosity and p_g is the pressure gradient (see figure B.1).

(a) Determine the solution in the case where $v(0) = 0, v(h) = v_{top}$ and there is no pressure gradient. This is sometimes called *Couette flow*.

(b) Determine the solution in the case where $v(0) = 0, v(h) = 0$ and there is a negative pressure gradient (i.e. pressure is decreasing linearly along the channel). This is sometimes called *Poiseuille flow*.

(c) On a vz plane (i.e. z as the vertical axis, v as the horizontal axis) sketch the velocity profiles of (a) and (b). Assume $\mu = 1$.

(d) In each of (a) and (b) what is the value of the maximum velocity and where does it occur?

4. (a) Consider the laminar flow between infinite flat plates as described in question 3. In the case where there is both a negative pressure gradient and non-zero motion of the upper plate (i.e. $p_g < 0$ and $v(h) = v_{top} > 0$) what is the maximum velocity and where does it occur?

(b) Sketch a velocity profile on the vz plane for the situation described above.

(c) What happens if there is a positive pressure gradient? (i.e. $p_g > 0$ and $v_{top} > 0$). Is there still a maximum absolute velocity? In which direction? Where does it occur.

(d) Sketch a velocity profile on the vz plane for the situation described in (c)

(e) In the $p_g < 0, v_{top} > 0$ case how is the velocity profile affected by changes to the viscosity μ?

5. Assuming the flow through a tube of radius R is laminar (i.e. it has become steady at some point away from the ends) then it can be shown that the velocity v is only in the x direction and depends only on the radial component r.

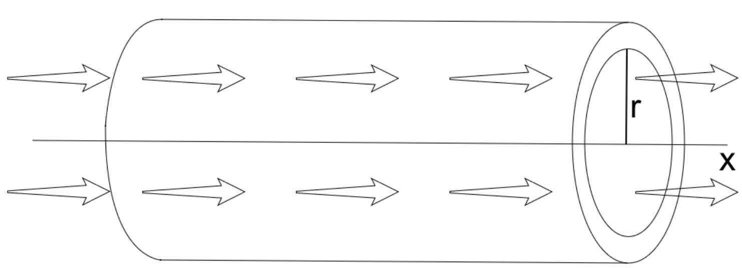


Figure B.4: Circular Poiseuille flow

The DE representing this situation is given by

$$r\frac{d^2v}{dr^2} + \frac{dv}{dr} = -\frac{rp_g}{\mu}$$

where p_g is the pressure gradient in the x-direction (assumed constant) and μ is the viscosity.

Solve this DE subject to the no-slip boundary condition $v(R) = 0$ and the restriction that v is bounded within the tube. Sketch the velocity profile on the vr plane (i.e. v as the horizontal axis and r as the vertical axis). What is the maximum velocity and where does it occur?

6. Consider the steady state temperature DE for a uniform rod

$$0 = K_0 \frac{d^2u}{dx^2} - \cancel{\frac{Ph}{A}(u(x) - U_a)}$$

Assume the rod is laterally insulated (i.e. $h = 0$) and we prescribe the left side ($x = 0$) to have a temperature of U_0 and the right side ($x = L$) to be open to the environment such that $u'(L) = -h_L(u(L) - U_a)$.

   (a) Solve this BVP.

   (b) Determine the behaviour as $h_L \to 0$ and $h_L \to \infty$.

   (c) Compare your answers to the cases of an insulated right endpoint and a prescribed temperature at the right endpoint.

7. Consider the steady state temperature DE for a uniform rod but this time without lateral insulation

$$0 = K_0 \frac{d^2u}{dx^2} - \frac{Ph}{A}(u(x) - U_a)$$

If the rod is cylindrical of radius r then, assuming $K_0 = h = 1$:

   (a) solve the BVP in the case where the endpoints are insulated (i.e. $u'(0) = u'(L) = 0$). Recall that P and A come from a cross section of the rod.

   (b) Further assume $r = 2$ and the ambient temperature is $U_a = 0$. Solve the BVP when the left and right endpoints are held at the constant temperature $U > 0$. Sketch this solution for x between 0 and L.

   (c) Resolve part (b) where we keep h as a parameter. Determine how the solution plots change as h varies.

8. A cable is suspending a load of uniform density ρ_ℓ of horizontal length L. We will assume that the cable mass is negligible and thus we have the simplified DE

$$\frac{d^2y}{dx^2} = \frac{\rho_\ell g}{T}$$

If we wish to suspend the ends at a height of H determine the smallest value of H that will guarantee that the cable is at least 10 length units above the ground.

9. The general uniform load, uniform cable equation is given by

$$\frac{d^2y}{dx^2} = \frac{\rho_\ell g}{T} + \frac{\rho_C g}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

If we assume there is no load other than the cable itself we arrive at

$$\frac{d^2y}{dx^2} = \gamma\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where $\gamma = \dfrac{\rho_C g}{T}$.

(a) Use a substitution $v = y'$ to solve the DE. Show that the solution can be written as $v = \sinh(\gamma x + d)$

(b) Solve for $y(x)$ and use the boundary conditions $y(-L) = H$ and $y(L) = H$ to solve for d.

(c) Show that full solution can be written as

$$y(x) = \frac{2}{\gamma}\sinh\left(\frac{\gamma(x+L)}{2}\right)\left(\frac{\gamma(x-L)}{2}\right) + H$$

Formula Sheets

Trigonometric Identities

1. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \quad \Rightarrow \quad \sin 2\theta = 2 \sin\theta \cos\theta$

2. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \quad \Rightarrow \quad \begin{cases} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{cases}$

3. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
4. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Integrals

5. $\int \sec^2 u \, du = \tan u + C$

6. $\int \tan u \, du = -\ln|\cos u| + C$

7. $\int \frac{1}{\sqrt{1-u^2}} \, du = \arcsin u + C$

8. $\int \frac{1}{1+u^2} \, du = \arctan u + C$

9. $\int \sec u \, du = \ln|\sec u + \tan u| + C$

10. $\int \frac{1}{\sqrt{u^2 \pm 1}} \, du = \ln\left|u + \sqrt{u^2 \pm 1}\right| + C$

11. $\int \sqrt{1-u^2} \, du = \frac{1}{2}\arcsin u + \frac{1}{2}u\sqrt{1-u^2} + C$

12. $\int \sqrt{u^2 \pm 1} \, du = \frac{u}{2}\sqrt{u^2 \pm 1} \pm \frac{1}{2}\ln\left|u + \sqrt{u^2 \pm 1}\right| + C$

Partial Fraction Expansions

$$\frac{p(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

$$\frac{p(x)}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b} = \frac{\bar{A}x + \bar{B}}{(x-a)^2} + \frac{C}{x-b}$$

$$\frac{p(x)}{(x^2+a^2)(x-b)} = \frac{Ax+B}{x^2+a^2} + \frac{C}{x-b}$$

note: $p(x)$ is a polynomial with degree less than that of the polynomial in the denominator (3, in each of these examples). If the degree of the numerator is greater than or equal to that in the denominator, long division must be done first.

Laplace Transforms

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\sin kt$	$\dfrac{k}{s^2+k^2}$
$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f'(t)$	$sF(s) - f(0)$
$e^{at}f(t)$	$F(s-a)$ (First Shift Theorem)
$f(t-a)H(t-a)$	$e^{-as}F(s)$ (Second Shift Theorem)
$f(t)H(t-a)$	$e^{-as}\mathcal{L}\{f(t+a)\}$ (Second Shift Theorem)
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$\delta(t-a)$	e^{-as}