Murmurations and ratios conjectures

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Overview

`arXiv:2408.12723` : *Murmurations and ratios conjectures*

- Ratios conjectures: Expression for $\sum_{f\in\mathcal{F}} L'(s,f)/L(s,f)$; major topic in random matrix theory for the past 20ish years.
- This talk: systematic conversion from ratios conjectures results to murmurations.
- Other direction: Are murmurations interesting to random matrix theorists?

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Quadratic Dirichlet characters — (Proto-)explicit formula

Theorem (Montgomery–Vaughan [MV07, Thm. 12.10])

Suppose that $X \notin \mathbb{Z}$, that $2 < T < X$, and that χ is a primitive nontrivial Dirichlet character modulo d. Then

$$\sum_{\substack{p \text{ prime}, k \in \mathbb{Z}_{>0} \\ p^k < X}} \chi(p^k) \log p = -\frac{1}{2\pi i} \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \frac{L'(s,\chi)}{L(s,\chi)} X^s \frac{ds}{s} + O\left(X^{1+\varepsilon} T^{-1+\varepsilon} d^{\varepsilon}\right).$$

Sum of Dirichlet coefficients

$\updownarrow$

Inverse Mellin transform of logarithmic derivative of L-function.

Quadratic Dirichlet characters — Ratios conjecture

Let $\mathcal{F}_1(D) := \{d \,:\, 1 < d < D,\ d \text{ a fundamental discriminant}\}$.
Let χ_d denote the Kronecker symbol $\left(\frac{d}{\cdot}\right)$.

Theorem (Conrey–Snaith [CS07, Thm. 2.7])

Assume the ratios conjecture [CS07, Conj. 2.6]. Suppose that $\frac{1}{\log D} \ll \mathrm{Re}(r) < \frac{1}{4}$ and that $\mathrm{Im}(r) \ll D^{1-\varepsilon}$. Then

$$\sum_{d\in\mathcal{F}_1(D)} \frac{L'(\frac{1}{2}+r,\chi_d)}{L(\frac{1}{2}+r,\chi_d)} = \sum_{d\in\mathcal{F}_1(D)} \left[\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \sum_p \frac{\log p}{(p+1)(p^{1+2r}-1)} - \left(\frac{\pi}{d}\right)^r \frac{\pi^2}{6} \frac{\Gamma\left(\frac{1}{4}-\frac{r}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{r}{2}\right)} \frac{\zeta(1-2r)}{\zeta(2-2r)}\right] + O(D^{\frac{1}{2}+\varepsilon}).$$

Plan: plug this expression for $\frac{L'(\frac{1}{2}+r,\chi_d)}{L(\frac{1}{2}+r,\chi_d)}$ into the explicit formula.

Quadratic Dirichlet characters — Murmurations

Define $\mathcal{F}_\chi := \{d \,:\, D_1 < d < D,\ d \text{ a fundamental discriminant}\}$.
Let $\frac{1}{\log X} < \varepsilon < \frac{1}{4}$ and $T \ll X^{1-\varepsilon}, D_1^{1-\varepsilon}$.

Theorem

Assuming the ratios conjecture [CS07, Conj. 2.6],

$$\frac{1}{\#\mathcal{F}_\chi}\sum_{d\in\mathcal{F}_\chi}\sum_{\substack{p^k<X\\ k\ odd}}\chi_d(p)\log p$$

$$= \frac{X^{\frac{1}{2}}}{2\pi i}\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}\frac{\pi^2}{6}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\frac{\zeta(2-2s)}{\zeta(3-2s)}\frac{1}{\#\mathcal{F}_\chi}\sum_{d\in\mathcal{F}_\chi}\left(\frac{\pi X}{d}\right)^{s-\frac{1}{2}}\frac{ds}{s}$$

$$+\sum_{\substack{p^k<X\\ k\ even}}\frac{\#\{d\in\mathcal{F}_\chi \,:\, p\mid d\}}{\#\mathcal{F}_\chi}\log p$$

$$+\,O\left(X^{\frac{1}{2}+\varepsilon}T^{\varepsilon}D^{\frac{1}{2}+\varepsilon}\#\mathcal{F}_\chi^{-1}+X^{1+\varepsilon}T^{-1+\varepsilon}D^{\varepsilon}\right).$$

Quadratic Dirichlet characters — Narrow range of p?

Evaluate at X and $X + X^\delta$ and take the difference. For any $\frac{1}{2} < \delta < 1$ and any $\frac{3}{2} - \delta < \theta < 2\delta$, there exist $\alpha, \beta \in \mathbb{R}$ such that

1. $X^\theta \ll D \ll X^\theta,\ X^\alpha \ll T \ll X^\alpha,\ X^\beta \ll \#\mathcal{F}_\chi \ll X^\beta$,
2. $\alpha < 1$ and $\alpha < \theta$,
3. $\beta < \theta$, and
4. Ratio of main term to error term is $\gg X^\eta$ for some $\eta > 0$ as $X \to \infty$.

TL;DR: Can isolate range of primes of length $X^{\frac{1}{2}+\varepsilon}$.

Quadratic Dirichlet characters — Approximate form

$$\frac{1}{\#\mathcal{F}_\chi}\sum_{d\in\mathcal{F}_\chi}\frac{1}{X^{\frac{1}{2}}}\sum_{\substack{p^k<X\\ k \text{ odd}}}\chi_d(p)\log p \tag{1}$$

$$\approx \frac{1}{2\pi i}\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}\frac{\pi^2}{6}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\frac{\zeta(2-2s)}{\zeta(3-2s)}\frac{1}{\#\mathcal{F}_\chi}\sum_{d\in\mathcal{F}_\chi}\left(\frac{\pi X}{d}\right)^{s-\frac{1}{2}}\frac{ds}{s}$$

or

$$\frac{1}{\#\mathcal{F}_\chi}\sum_{d\in\mathcal{F}_\chi}\frac{1}{\sqrt{Dy}}\sum_{p<Dy}\chi_d(p)\log p$$

$$\approx \frac{1}{2\pi i}\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}\frac{\pi^2}{6}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\frac{\zeta(2-2s)}{\zeta(3-2s)}(\pi y)^{s-\frac{1}{2}}\frac{ds}{s}.$$

Quadratic Dirichlet characters — Example illustration

Taking $\mathcal{F}_\chi = \{d : 95{,}000 < d < 105{,}000,\ d \text{ a fundamental discriminant}\}$:

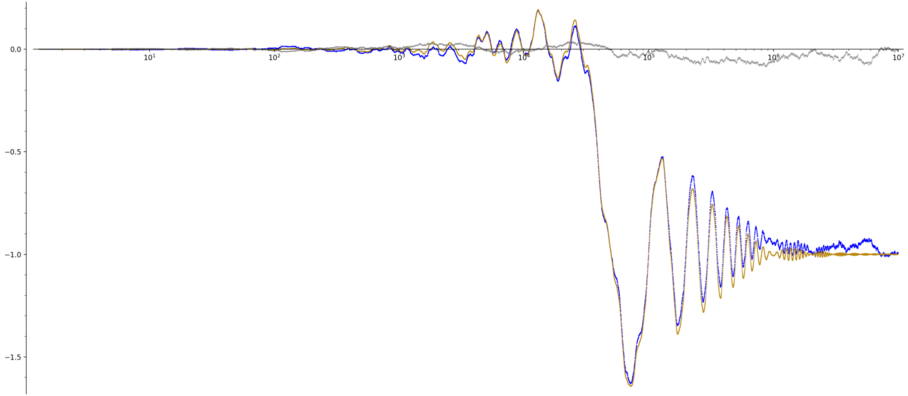

Figure: For $T = 900$ and $\varepsilon = 0.1$, the left and right hand sides of (1) in blue and gold respectively, as well as their difference in grey, as functions of X. The integral in (1) is approximated by Riemann sum evaluated at 180,000 equally-spaced points. In this example $\#\mathcal{F}_\chi = 3038$. Code: [Cow24b].

Quadratic Dirichlet characters — Sum of residues

Proposition (WIP)

Assume RH and that all zeros of ζ are simple. For $y \ll T^{1-\varepsilon}$,

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\pi^2}{6}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\frac{\zeta(2-2s)}{\zeta(3-2s)}(\pi y)^{s-\frac{1}{2}}\,\frac{ds}{s}$$

$$= \sum_{\substack{\gamma\in i\mathbb{R} \\ \zeta(\frac{1}{2}+i\gamma)=0}} \frac{\pi^2}{12}\frac{\Gamma(-\frac{1}{8}+\frac{i\gamma}{4})}{\Gamma(\frac{5}{8}-\frac{i\gamma}{4})}\frac{\zeta(-\frac{1}{2}+i\gamma)}{\zeta'(\frac{1}{2}+i\gamma)}\frac{(\pi y)^{\frac{3}{4}-\frac{i\gamma}{2}}}{\frac{5}{4}-\frac{i\gamma}{2}}$$

$$+\sum_{k=0}^{\infty} \frac{\pi^2}{12}\frac{\Gamma(-\frac{3}{4}-\frac{k}{2})}{\Gamma(\frac{5}{4}+\frac{k}{2})}\frac{\zeta(-3-2k)}{\zeta'(-2-2k)}\frac{(\pi y)^{2+k}}{\frac{5}{2}+k}$$

$$+\,O(T^{\frac{1}{2}-c+\varepsilon}).$$

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Ratios conjecture — $L(s,\chi_d)$ example 1/2

Following Conrey–Snaith "recipe", for $\dfrac{L(\frac12+\alpha,\chi_d)}{L(\frac12+\gamma,\chi_d)}$ with $d>0$ [CS07, §2.2]:

Step 1: Write

$$\frac{1}{L(\frac12+\gamma,\chi_d)}=\sum_n \frac{\mu(n)\chi_d(n)}{n^{\frac12+\gamma}}$$

and

$$L(\tfrac12+\alpha,\chi_d)=\sum_{n<t\sqrt{d/2\pi}}\frac{\chi_d(n)}{n^{\frac12+\alpha}}+\left(\frac{\pi}{d}\right)^{\alpha}\frac{\Gamma\left(\frac14-\frac{\alpha}{2}\right)}{\Gamma\left(\frac14+\frac{\alpha}{2}\right)}\sum_{n<\frac1t\sqrt{d/2\pi}}\frac{\chi_d(n)}{n^{\frac12-\alpha}}+\text{small}$$

(using the approximate functional equation; [IK04, Thm. 5.3]).

Ratios conjecture — $L(s, \chi_d)$ example 2/2

Step 2: Multiply and then average over d using

$$\sum_{d<D} \chi_d(n) \approx \begin{cases} \prod_{p|n} \frac{p}{p+1} & \text{if } n \text{ is square} \\ 0 & \text{otherwise.} \end{cases}$$

Step 3: Extend Dirichlet series summation ranges to infinity, recognize and manipulate Euler products.

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Quadratic Dirichlet characters II — Setup

Let g be an even Schwartz function such that

$$\hat{g}(\xi) \coloneqq \frac{1}{2\pi} \int_{\mathbb{R}} g(x) e^{-2\pi i \xi x}\, dx$$

is supported on $(-\sigma, \sigma)$. Define

$$\mathcal{F}_1(D) \coloneqq \{d \,:\, 0 < d < D,\ d \text{ a fundamental discriminant}\}$$
$$\mathcal{F}_2(D) \coloneqq \left\{8d \,:\, 0 < d < \tfrac{D}{8},\ d \text{ an odd positive squarefree integer}\right\}.$$

Lemma (Miller [Mil08, Lemma B.1])

$\#\mathcal{F}_1(D) = \frac{3}{\pi^2} D + O(D^{\frac{1}{2}})$ and $\#\mathcal{F}_2(D) = \frac{1}{2\pi^2} D + O(D^{\frac{1}{2}})$.

Quadratic Dirichlet characters II — Murmurations

Theorem (based on Miller [Mil08])

Assume GRH. With the notation of the previous slide and $j \in \{1,2\}$,

$$\frac{1}{\#\mathcal{F}_j(D)} \sum_{d \in \mathcal{F}_j(D)} \sum_p \sum_{k=1}^{\infty} \hat{g}\left(\frac{\log p^k}{\log D}\right) \frac{\chi_d(p)^k \log p}{\sqrt{p^k}}$$

$$= \int_0^{\infty} \hat{g}\left(1 + \frac{\log y}{\log D}\right) \int_{-\infty}^{\infty} \frac{\pi^2}{6} \frac{\Gamma(\frac{1}{4} - \pi i t)}{\Gamma(\frac{1}{4} + \pi i t)} \frac{\zeta(1 - 4\pi i t)}{\zeta(2 - 4\pi i t)}$$

$$\cdot \frac{1}{\#\mathcal{F}_j(D)} \sum_{d \in \mathcal{F}_j(D)} \left(\frac{\pi D y}{d}\right)^{2\pi i t} dt \, \frac{dy}{y}$$

$$- \log D \int_{-\infty}^{\infty} g(t \log D) \left(\frac{\zeta'}{\zeta}(1 + 4\pi i t) + \sum_p \frac{\log p}{(p+1)(p^{1+4\pi i t} - 1)}\right) dt$$

$$+ O\left(D^{-\frac{1}{2}+\varepsilon} + \begin{cases} D^{-\frac{1-\sigma}{2}+\varepsilon} & \text{if } j = 1 \\ D^{-\frac{2-3\sigma}{2}+\varepsilon} + D^{-\frac{3-3\sigma}{4}+\varepsilon} & \text{if } j = 2 \end{cases}\right).$$

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Quadratic Dirichlet characters III — Density under GRH

Theorem (Čech [Čec24, Thm. 1.4])

Assume GRH. Let f be a smooth, fast-decaying (as in [Čec24]) weight function with Mellin transform $\mathcal{M}f$, and let χ_n denote the Jacobi symbol $\left(\frac{\cdot}{n}\right)$. For $\varepsilon < \mathrm{Re}(r) < \frac{1}{4}$ and $N \to \infty$,

$$\sum_{\substack{n>0 \\ n \text{ odd, squarefree}}} \frac{1}{N} f\left(\frac{n}{N}\right) \frac{L'(\frac{1}{2}+r,\chi_n)}{L(\frac{1}{2}+r,\chi_n)} + O\left(|r|^{\varepsilon} N^{-2\mathrm{Re}(r)+\varepsilon}\right)$$

$$= \frac{2\mathcal{M}f(1)}{3\zeta(2)} \left(\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \sum_{p>2} \frac{\log p}{(p+1)p^{1+2r}-1} \right)$$

$$- \mathcal{M}f(1-r)\left(\frac{\pi}{N}\right)^{r} \left(\frac{\Gamma\left(\frac{\frac{1}{2}-r}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+r}{2}\right)} + \frac{\Gamma\left(\frac{\frac{3}{2}-r}{2}\right)}{\Gamma\left(\frac{\frac{3}{2}+r}{2}\right)} \right) \frac{\zeta(1-2r)}{4\zeta^{(2)}(2-2r)}.$$

Quadratic Dirichlet characters III — Murmurations

Theorem (based on Čech [Čec24])

Assume GRH. Let f be a smooth, fast-decaying weight function with Mellin transform $\mathcal{M}f$, and let χ_n denote the Jacobi symbol $\left(\frac{\cdot}{n}\right)$. For any N, y, T, c such that $2 < T < Ny$ and $\frac{1}{2} + \varepsilon < c < \frac{3}{4}$,

$$\sum_{\substack{n>0 \\ n \text{ odd, squarefree}}} \frac{1}{N} f\left(\frac{n}{N}\right) \frac{1}{\sqrt{Ny}} \sum_{p^k < Ny} \chi_n(p^k) \log p$$

$$= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \mathcal{M}f\left(\tfrac{3}{2} - s\right) \left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)} \right) \frac{\zeta(2-2s)}{4\zeta^{(2)}(3-2s)} (\pi y)^{s-\frac{1}{2}} \frac{ds}{s}$$

$$+ \frac{1}{\sqrt{Ny}} \sum_{p^k < \sqrt{Ny}} \frac{2\mathcal{M}f(1)}{3\zeta(2)} \log p + O\left(N^{\frac{1}{2}-c+\varepsilon} y^{c-\frac{1}{2}} T^{\varepsilon} + N^{\frac{1}{2}+\varepsilon} y^{\frac{1}{2}+\varepsilon} T^{-1+\varepsilon} \right).$$

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Elliptic curves — Setup 1/2

Let $E_{a,b} : y^2 = x^3 + ax + b$. Suppose $3 \nmid r$ and $2 \nmid t$. Define

$$\mathcal{F}(H) := \big\{ E_{a,b} \,:\, a = r \bmod 6,\ b = t \bmod 6,\ |a| \leq H^{\frac{1}{3}},\ |b| \leq H^{\frac{1}{2}}, \\ p^4 \mid a \implies p^6 \nmid b \big\}. \tag{2}$$

Write the L-function attached to E as

$$L(s, E) := \prod_{p \mid N_E} \left(1 - \frac{a_p(E)}{p^{s+\frac{1}{2}}}\right)^{-1} \prod_{p \nmid N_E} \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha}_p}{p^s}\right)^{-1}$$

with $|\alpha_p| = |\overline{\alpha}_p| = 1$ and $\alpha_p + \overline{\alpha}_p = \frac{a_p(E)}{\sqrt{p}}$.

The (proto-)explicit formula is, for $2 < T < X$ and $X \notin \mathbb{Z}$ [Fio14, Lemma 2.1],

$$\sum_{\substack{p \text{ prime},\, k \in \mathbb{Z}_{>0} \\ p^k < X,\, p \nmid N_E}} \left(\alpha_p^k + \overline{\alpha}_p^k\right) \log p = -\frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \frac{L'(s,E)}{L(s,E)} X^s \frac{ds}{s} \\ + O\big(X^{1+\varepsilon} T^{-1+\varepsilon} N_E^{\varepsilon}\big).$$

Elliptic curves — Setup 2/2

Let $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > 0$, and let $\mathrm{Tr}_k(p)$ denote the trace of the Hecke operator at p on the space of level 1 weight k holomorphic cusp forms.

Definition (David–Huynh–Parks [DHP15, (3.38)])

$$A(\alpha,\gamma) := \frac{\zeta(1+\alpha+\gamma)}{\zeta(1+2\gamma)} \prod_{p=2,3} \frac{1 - a_p(E_{r,t})p^{-1-\gamma} + p^{-1-2\gamma}}{1 - a_p(E_{r,t})p^{-1-\alpha} + p^{-1-2\alpha}}$$

$$\cdot \prod_{p>3} \Bigg[1 + \left(1 - \frac{p^9-1}{p^{10}-1}\right)\Bigg(\frac{1}{p^{1+2\gamma}} - \frac{1}{p^{1+\alpha+\gamma}} + \frac{p^{-2-\alpha-\gamma} - p^{-2-2\gamma}}{p^{2+2\alpha}-1}$$

$$+ \frac{p^{1+2\alpha+\gamma} - p^{1+\alpha+2\gamma} + p^{\gamma} - p^{\alpha}}{p^{\frac{3}{2}+\alpha+2\gamma}} \sum_{m=5}^{\infty} \frac{\mathrm{Tr}_{2m+2}(p)}{p^{2m(\alpha+1)+\frac{1}{2}}}\Bigg)\Bigg].$$

The function $A(\alpha,\gamma)$ extends to a nonzero holomorphic function on $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > -\frac{1}{4}$, and $A(r,r) = 1$ in this region.

Elliptic curves — Ratios conjecture

Theorem (David–Huynh–Parks [DHP15, Thm. 3.8])

Assume the ratios conjecture [DHP15, Conj. 3.7]. Suppose $\mathrm{Re}(r) \gg \frac{1}{\log H}$ *and* $\mathrm{Im}(r) \ll H^{1-\varepsilon}$. *Then*

$$\sum_{E\in\mathcal{F}(H)} \frac{L'(\frac{1}{2}+r,E)}{L(\frac{1}{2}+r,E)} = \sum_{E\in\mathcal{F}(H)} \Bigg[-\frac{\zeta'(1+2r)}{\zeta(1+2r)} + A_\alpha(r,r) - \omega_E \left(\frac{4\pi^2}{N_E}\right)^r \frac{\Gamma(1-r)}{\Gamma(1+r)} \zeta(1+2r) A(-r,r)\Bigg] + \mathcal{R}(H)$$

with $\mathcal{R}(H) \ll H^{\frac{1}{3}+\varepsilon}$.

Elliptic curves — Conductor distribution

David–Huynh–Parks ratios conjecture is for $\mathcal{F}(H)$: a family ordered by height. Murmurations care about conductor. We need a conversion.

Theorem

For any $\lambda_1 > \lambda_0 > \frac{4464}{\log H}$,

$$\frac{\#\left\{E \in \mathcal{F}(H) \,:\, \lambda_0 < \frac{N_E}{H} < \lambda_1\right\}}{\#\mathcal{F}(H)} = F_N(\lambda_1) - F_N(\lambda_0) + O((\log H)^{-1+\varepsilon}). \tag{3}$$

Theorem

For any $\lambda > \frac{4464}{\log H}$,

$$\frac{\#\left\{E \in \mathcal{F}(H) \,:\, \frac{N_E}{H} < \lambda\right\}}{\#\mathcal{F}(H)} \ll \lambda^{\frac{5}{6}}. \tag{4}$$

Elliptic curves — Conductor distribution plot

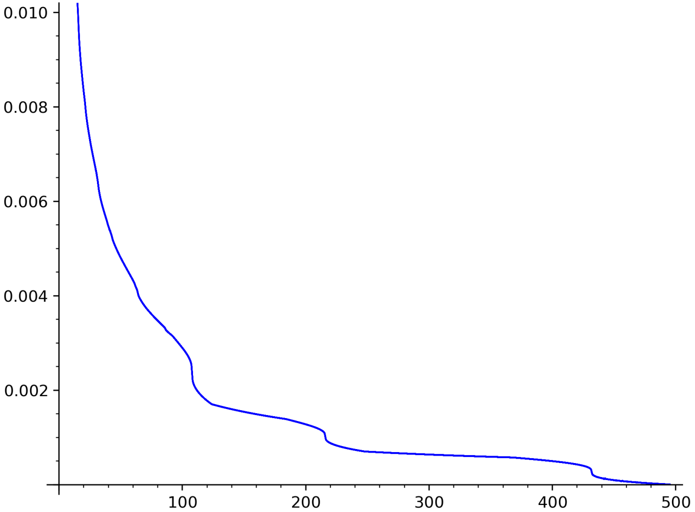

Figure: $F'_N(\lambda)$, computed numerically with $\Delta\lambda = 0.496$ [Cow24a].

Elliptic curves — Murmurations

Theorem

Let $\mathcal{F}(H)$ be the family (2) of elliptic curves ordered by height, take $\omega \in \{\pm 1\}$, and set $\mathcal{F}(H)^\omega := \{E \in \mathcal{F}(H) : \omega_E = \omega\}$. Assume that (3), (4), and the ratios conjecture [DHP15, Conj. 3.7] hold with $\mathcal{F}(H)$ replaced with $\mathcal{F}(H)^\omega$.

For any H, y, T, ε such that $0 < \varepsilon < \frac{5}{6}$ and $(Hy)^{\frac{1}{2}+\varepsilon} \ll T < Hy$,

$$\frac{1}{\#\mathcal{F}(H)^\omega} \sum_{E \in \mathcal{F}(H)^\omega} \frac{1}{\sqrt{Hy}} \sum_{\substack{p^k < Hy \\ p \nmid N_E}} \left(\alpha_p^k + \overline{\alpha}_p^k\right) \log p$$

$$= \frac{\omega}{2\pi i} \int_{\mathbb{R}} \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \frac{\Gamma(\frac{3}{2}-s)}{\Gamma(\frac{1}{2}+s)} \zeta(2s) A(\tfrac{1}{2}-s, s-\tfrac{1}{2}) \left(\frac{4\pi^2 y}{\lambda}\right)^{s-\frac{1}{2}} \frac{ds}{s}\, F_N'(\lambda)\, d\lambda$$

$$- \frac{1}{\sqrt{Hy}} \sum_{p^k < \sqrt{Hy}} \log p + O\left(H^\varepsilon y^\varepsilon T^\varepsilon \mathcal{R}(H) \#\mathcal{F}(H)^{-1} + (\log H)^{-\frac{5}{6}}\right).$$

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Theorem (based on Miller–Montague [MM11])

Assume GRH. For fixed $\pm \in \{+,-\}$, *fixed* $k \in 2\mathbb{Z}_{>0}$, *and* $N \to \infty$ *prime,*

$$\sum_{f \in H_k^{\pm}(N)} \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f, f\rangle} \sum_{p \neq N} \hat{g}\left(\frac{\log p}{\log N}\right) \frac{\lambda_f(p)\log p}{\sqrt{p}}$$

$$= \pm 2 \lim_{\delta \to 0^+} \int_0^\infty \hat{g}\left(1 + \frac{\log y}{\log N}\right) \int_{-\infty}^{\infty} \frac{\Gamma(\frac{k}{2} - 2\pi i t)}{\Gamma(\frac{k}{2} + 2\pi i t)}$$

$$\cdot \prod_p \left(1 + \frac{1}{(p-1)p^{4\pi i t + \delta}}\right) (4\pi^2 y)^{2\pi i t} dt \, \frac{dy}{y} + O\left(N^{\frac{\sigma}{2} - 1 + \varepsilon}\right).$$

Can take $\hat{g}(\xi)$ sharply peaked at $\xi = 1$ while keeping the error term small.

$$\sum_{f \in H_k^{\pm}(N)} \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f, f\rangle} \sim \frac{1}{2}, \qquad N^{-1-\varepsilon} \ll \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f, f\rangle} \ll N^{-1+\varepsilon}.$$

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Quadratic twists of an elliptic curve — Setup

Let N_E be an odd prime and let E be an elliptic curve of conductor N_E with even functional equation. Define

$$\mathcal{F}_E(D) \coloneqq \{0 < d < D \,:\, d \text{ a fundamental discriminant}, \chi_d(-N_E)\omega_E = 1\}\,.$$

Let A_E be the arithmetic factor given by the infinite product in [HMM11, (3.2)].

Quadratic twists of an elliptic curve — Murmurations

Theorem (based on Huynh–Miller–Morrison [HMM11])

Assume GRH. With the notation of the previous slide,

$$\frac{1}{\#\mathcal{F}_E(D)} \sum_{d\in\mathcal{F}_E(D)} \sum_{p\neq N_E} \sum_{\substack{k>0 \\ k \text{ odd}}} \hat{g}\left(\frac{\log p^k}{\log \frac{N_E D^2}{4\pi^2}}\right) \chi_d(p) \frac{(\alpha_p^k + \overline{\alpha}_p^k)\log p}{\sqrt{p^k}}$$

$$= \frac{g(0)}{2}\log\left(\frac{\sqrt{N_E}D}{2\pi}\right) + \int_{-\infty}^{\infty} \frac{\Gamma(1-\pi i t)}{\Gamma(1+\pi i t)} \frac{\zeta(1+2\pi i t) L_E(\mathrm{sym}^2, 1-2\pi i t)}{L_E(\mathrm{sym}^2, 1)}$$

$$\cdot A_E(-\pi i t, \pi i t) \frac{1}{\#\mathcal{F}_E(D)} \sum_{d\in\mathcal{F}_E(D)} \int_0^\infty \hat{g}\left(1 + \frac{\log y}{\log \frac{\sqrt{N_E}D}{2\pi}}\right) \left(\frac{Dy}{d}\right)^{2\pi i t} \frac{dy}{y}\, dt$$

$$+ O\left(D^{-\frac{1-\sigma}{2}}\right).$$

Can't take $\hat{g}(\xi)$ sharply peaked at $\xi = 1$ while keeping the error term small $\implies$ Need to assume ratios conjecture to get murmurations for now.

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Future directions? — Similar work

Many similar examples seem possible using existing results from the literature, e.g.

- [Mil09] Miller — *An orthogonal test of the L-functions ratios conjecture.*
- [GJM$^+$10] Goes, Jackson, Miller, Montague, Ninsuwan, Peckner, Pham — *A unitary test of the ratios conjecture.*
- [MP12] Miller, Peckner — *Low-lying zeros of number field L-functions.*
- [FM15] Fiorilli, Miller — *Surpassing the ratios conjecture in the 1-level density of Dirichlet L-functions.*

Future directions? — Ratios conjecture for ζ'/ζ

Theorem (Conrey–Snaith [CS07, Thm. 2.5])

Assume the ratios conjecture [CS07, Conj. 2.1]. Suppose that $\frac{1}{\log T} \ll \mathrm{Re}(\alpha), \mathrm{Re}(\alpha) < \frac{1}{4}$ *and that* $\mathrm{Im}(\alpha), \mathrm{Im}(\beta) \ll T^{1-\varepsilon}$. *Then*

$$\int_0^T \frac{\zeta'}{\zeta}\left(\frac{1}{2} + it + \alpha\right) \frac{\zeta'}{\zeta}\left(\frac{1}{2} - it + \beta\right) dt$$

$$= \int_0^T \zeta(1+\alpha+\beta)\zeta(1-\alpha-\beta)$$

$$\cdot \prod_p \left(1 - \frac{1}{p^{1+\alpha+\beta}}\right)\left(1 - \frac{2}{p} + \frac{1}{p^{1+\alpha+\beta}}\right)\left(1 - \frac{1}{p}\right)^{-2} \left(\frac{2\pi}{t}\right)^{\alpha+\beta} dt$$

$$+ T\left(\frac{\zeta'}{\zeta}\right)'(1+\alpha+\beta) - T\sum_p \left(\frac{\log p}{p^{1+\alpha+\beta} - 1}\right)^2 + O(T^{\frac{1}{2}+\varepsilon}).$$

Future directions? — ζ'/ζ murmurations

Idea (based on Conrey–Snaith [CS07])

Assume the *ratios conjecture* [CS07, Conj. 2.1]. Suppose that $\frac{1}{\log T} \ll c < \frac{1}{4}$ and that $A \ll T^{1-\varepsilon}$. Then

$$\int_{c-iA}^{c+iA} \int_{T-\Delta T}^{T} \frac{\zeta'}{\zeta}\left(\frac{1}{2}+it+\alpha\right) \frac{\zeta'}{\zeta}\left(\frac{1}{2}-it\right) dt\, X^{\alpha} \frac{d\alpha}{\alpha}$$

$$= \int_{c-iA}^{c+iA} \zeta(1+\alpha)\zeta(1-\alpha) \prod_p \left(1-\frac{1}{p^{1+\alpha}}\right)\left(1-\frac{2}{p}+\frac{1}{p^{1+\alpha}}\right)\left(1-\frac{1}{p}\right)^{-2} \cdot \int_{T-\Delta T}^{T} \left(\frac{2\pi X}{t}\right)^{\alpha} dt \frac{d\alpha}{\alpha}$$

$$+ \int_{c-iA}^{c+iA} \left[\Delta T \left(\frac{\zeta'}{\zeta}\right)'(1+\alpha) - \Delta T \sum_p \left(\frac{\log p}{p^{1+\alpha}-1}\right)^2 + O(T^{\frac{1}{2}+\varepsilon})\right] X^{\alpha} \frac{d\alpha}{\alpha}.$$

Future directions? — Correlations of $|\zeta|^2$

Theorem (Kovaleva [Kov24, Thm. 1.1])

Let $\alpha, \beta \geq 0$ *be such that* $\delta := \beta - \alpha \geq 0$.

$$\int_0^T |\zeta(1/2 + it + i\alpha)|^2 |\zeta(1/2 + it + i\beta)|^2 dt$$

$$= \int_1^T D(t; \alpha, \beta) + OD(t; \alpha, \beta)\, dt + E(T; \alpha, \beta),$$

$$D(t; \alpha, \beta) = 2\mathrm{Re} \sum_{s \in \{0, i\delta\}} \operatorname*{Res}_s \frac{\zeta^4(1+s)}{\zeta(2+2s)} \frac{(t/2\pi)^s}{s - i\delta}$$

$$OD(t; \alpha, \beta) = \frac{\partial^2}{\partial s_1 \partial s_2} \left(h(i\delta, s_1 + s_2) \left(\frac{t+\alpha}{2\pi}\right)^{s_1} \left(\frac{t+\beta}{2\pi}\right)^{s_2} \right) \Bigg|_{s_1 = s_2 = 0}$$

and $E(T; \alpha, \beta) \ll O((T+\alpha)^{2/3+\varepsilon} + (T+\alpha)^{1/2+\varepsilon}\delta^{1/3} + \delta^{1/2})$.

Future directions? — Skip the "ratios"?

Approximate functional equation [IK04, Thm. 5.3]:

$$L(s,f) = \sum_n \frac{\lambda_f(n)}{n^s} V_s\left(\frac{n}{t\sqrt{N}}\right) + \omega_f \sqrt{N} \frac{G_f(1-s)}{G_f(s)} \sum_n \frac{\overline{\lambda_f}(n)}{n^{1-s}} V_{1-s}\left(\frac{nt}{\sqrt{N}}\right) \left(\frac{1}{N}\right)^s + R.$$

Inverse Mellin transform:

$$\sum_{n<X} \lambda_f(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \sum_n \frac{\lambda_f(n)}{n^s} V_s\left(\frac{n}{t\sqrt{N}}\right) X^s + \omega_f \sqrt{N} \frac{G_f(1-s)}{G_f(s)} \sum_n \frac{\overline{\lambda_f}(n)}{n^{1-s}} V_{1-s}\left(\frac{nt}{\sqrt{N}}\right) \left(\frac{X}{N}\right)^s + RX^s \frac{ds}{s} + O(X^{1+\varepsilon} T^{-1+\varepsilon} N^{\varepsilon}).$$

Thanks!

Thanks for listening!

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