Murmurations
&
Elliptic curve conductor distributions

Alex Cowan

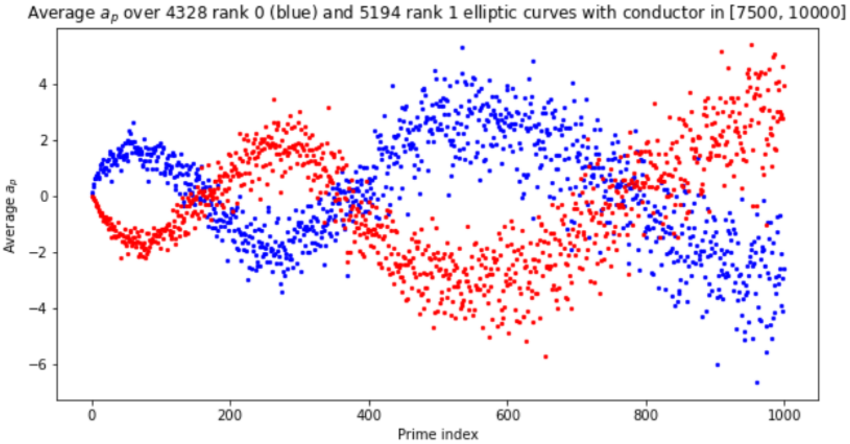
University of Waterloo

University of Toronto, December 4th 2024

Acknowledgments: Chantal David, Valeriya Kovaleva, Kimball Martin, Steve Miller, Alexey Pozdnyakov, Mike Rubinstein, Drew Sutherland, and Jerry Wang.

Murmurations? 1/2

Yang-Hui He, Kyu-Hwan Lee, Thomas Oliver, Alexey Pozdnyakov 2022:



Scale invariant: The curves are a function of $\frac{p}{N}$.

Murmurations? 2/2

Results so far:

- Zubrilina: **Holomorphic newspaces by level**
  Fixed weight and single prime, averaged over squarefree levels going to infinity. [Zub23]
- Lee, Oliver, Pozdnyakov: **Quadratic Dirichlet characters**
  Averaged over primes and discriminants, under GRH. [LOP23]
- Bober, Booker, Lee, Lowry-Duda: **Holomorphic newspaces by weight**
  Level 1, averaged over primes and over weights going to infinity, under GRH. [BBLLD23]
- Booker, Lee, Lowry-Duda, Seymour-Howell, Zubrilina: **Maass forms**
  Level 1, averaged over primes and over Laplace eigenvalues going to infinity, under GRH. [BLLD$^+$24]

Proofs use trace formulae. Murmuration functions unknown even heuristically in other cases.

Overview of this talk

This talk:

1. New approach to murmurations, via random matrix theory.
   - Many existing results can be adapted to prove murmurations,
   - sometimes under a *ratios conjecture* (standard in random matrix theory),
   - sometimes under GRH only.
   - Four examples given in this talk.
2. Distribution (as in histogram) of elliptic curve conductors in a big family ordered by height.

`arXiv:2408.12723` : *Murmurations and ratios conjectures*
`arXiv:2408.09745` : *Conductor distributions of elliptic curves*

Table of Contents

Quadratic Dirichlet characters — (Proto-)explicit formula

Theorem (Montgomery–Vaughan [MV07, Thm. 12.10])

Suppose that $X \notin \mathbb{Z}$, that $2 < T < X$, and that χ is a primitive nontrivial Dirichlet character modulo d. Then

$$\sum_{\substack{p \text{ prime},\, k \in \mathbb{Z}_{>0} \\ p^k < X}} \chi(p^k) \log p = -\frac{1}{2\pi i} \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \frac{L'(s,\chi)}{L(s,\chi)} X^s \frac{ds}{s} + O\left(X^{1+\varepsilon} T^{-1+\varepsilon} d^{\varepsilon}\right).$$

Sum of Dirichlet coefficients

$\updownarrow$

Inverse Mellin transform of logarithmic derivative of L-function.

Quadratic Dirichlet characters — Ratios conjecture

Let $\mathcal{F}_1(D) := \{d : 1 < d < D,\ d \text{ a fundamental discriminant}\}$.
Let χ_d denote the Kronecker symbol $\left(\frac{d}{\cdot}\right)$.

Theorem (Conrey–Snaith [CS07, Thm. 2.7])

Assume the ratios conjecture [CS07, Conj. 2.6]. Suppose that $\frac{1}{\log D} \ll \mathrm{Re}(r) < \frac{1}{4}$ *and that* $\mathrm{Im}(r) \ll D^{1-\varepsilon}$. *Then*

$$\sum_{d \in \mathcal{F}_1(D)} \frac{L'(\frac{1}{2}+r,\chi_d)}{L(\frac{1}{2}+r,\chi_d)} = \sum_{d \in \mathcal{F}_1(D)} \left[\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \sum_p \frac{\log p}{(p+1)(p^{1+2r}-1)} - \left(\frac{\pi}{d}\right)^r \frac{\pi^2}{6} \frac{\Gamma\left(\frac{1}{4}-\frac{r}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{r}{2}\right)} \frac{\zeta(1-2r)}{\zeta(2-2r)} \right] + O(D^{\frac{1}{2}+\varepsilon}).$$

Plan: plug this expression for $\frac{L'(\frac{1}{2}+r,\chi_d)}{L(\frac{1}{2}+r,\chi_d)}$ into the explicit formula.

Quadratic Dirichlet characters — Murmurations

Define $\mathcal{F}_\chi := \{d : D_1 < d < D,\ d \text{ a fundamental discriminant}\}$.
Let $\frac{1}{\log X} < \varepsilon < \frac{1}{4}$ and $T \ll X^{1-\varepsilon}, D_1^{1-\varepsilon}$.

Theorem

Assuming the ratios conjecture [CS07, Conj. 2.6],

$$\frac{1}{\#\mathcal{F}_\chi} \sum_{d \in \mathcal{F}_\chi} \frac{1}{X^{\frac{1}{2}}} \sum_{\substack{p^k < X \\ k \text{ odd}}} \chi_d(p) \log p$$

$$= \frac{1}{2\pi i} \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \frac{\pi^2}{6} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \frac{\zeta(2-2s)}{\zeta(3-2s)} \frac{1}{\#\mathcal{F}_\chi} \sum_{d \in \mathcal{F}_\chi} \left(\frac{\pi X}{d}\right)^{s-\frac{1}{2}} \frac{ds}{s}$$

$$+ \frac{1}{X^{\frac{1}{2}}} \sum_{\substack{p^k < X \\ k \text{ even}}} \frac{\#\{d \in \mathcal{F}_\chi : p \mid d\}}{\#\mathcal{F}_\chi} \log p$$

$$+ O\left(X^\varepsilon T^\varepsilon D^{\frac{1}{2}+\varepsilon} \#\mathcal{F}_\chi^{-1} + X^{\frac{1}{2}+\varepsilon} T^{-1+\varepsilon} D^\varepsilon\right).$$

Quadratic Dirichlet characters — Illustration

Taking $\mathcal{F}_\chi = \{d : 95{,}000 < d < 105{,}000,\ d \text{ a fundamental discriminant}\}$:

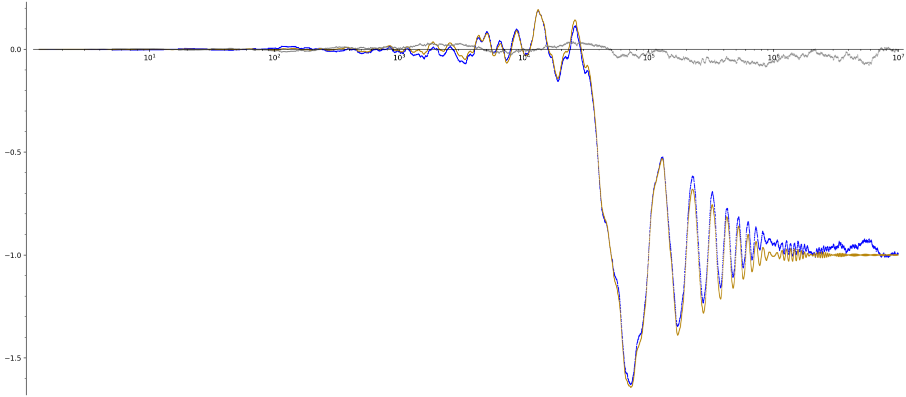


Figure: For $T = 900$ and $\varepsilon = 0.1$, the LHS and RHS main term of theorem in blue and gold respectively, as well as their difference in grey, as functions of X. In this example $\#\mathcal{F}_\chi = 3038$. Code: [Cow24b].

Table of Contents

Following Conrey–Snaith "recipe" for $\dfrac{L(\frac{1}{2}+\alpha,\chi_d)}{L(\frac{1}{2}+\gamma,\chi_d)}$ with $d>0$:

Step 1: Write

$$\frac{1}{L(\frac{1}{2}+\gamma,\chi_d)}=\sum_n\frac{\mu(n)\chi_d(n)}{n^{\frac{1}{2}+\gamma}}$$

and

$$L(\tfrac{1}{2}+\alpha,\chi_d)=\sum_{n<t\sqrt{d/2\pi}}\frac{\chi_d(n)}{n^{\frac{1}{2}+\alpha}}+\left(\frac{\pi}{d}\right)^{\alpha}\frac{\Gamma\left(\frac{1}{4}-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{\alpha}{2}\right)}\sum_{n<\frac{1}{t}\sqrt{d/2\pi}}\frac{\chi_d(n)}{n^{\frac{1}{2}-\alpha}}+\text{small}$$

(using the approximate functional equation; [IK04, Thm. 5.3]).

Ratios conjecture — $L(s, \chi_d)$ example 2/2

Step 1:

$$\frac{L(\frac{1}{2}+\alpha, \chi_d)}{L(\frac{1}{2}+\gamma, \chi_d)} = \frac{1}{L(\frac{1}{2}+\gamma, \chi_d)} L(\tfrac{1}{2}+\alpha, \chi_d)$$

Use approximate functional equation on second factor.

Step 2: Multiply and then average over d using

$$\sum_{d<D} \chi_d(n) \approx \begin{cases} \prod_{p|n} \frac{p}{p+1} & \text{if } n \text{ is square} \\ 0 & \text{otherwise.} \end{cases}$$

Step 3: Extend Dirichlet series summation ranges to infinity, recognize and manipulate Euler products.

Table of Contents

Quadratic Dirichlet characters II — Density under GRH

Theorem (Čech [Čec24, Thm. 1.4])

Assume GRH. Let f be a smooth, fast-decaying (as in [Čec24]) weight function with Mellin transform $\mathcal{M}f$, and let χ_n denote the Jacobi symbol $(\frac{\cdot}{n})$. For $\varepsilon < \mathrm{Re}(r) < \frac{1}{4}$ and $N \to \infty$,

$$\sum_{\substack{n>0 \\ n \text{ odd, squarefree}}} \frac{1}{N} f\left(\frac{n}{N}\right) \frac{L'(\frac{1}{2}+r,\chi_n)}{L(\frac{1}{2}+r,\chi_n)} + O\left(|r|^{\varepsilon} N^{-2\mathrm{Re}(r)+\varepsilon}\right)$$

$$= \frac{2\mathcal{M}f(1)}{3\zeta(2)} \left(\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \sum_{p>2} \frac{\log p}{(p+1)p^{1+2r}-1} \right)$$

$$- \mathcal{M}f(1-r) \left(\frac{\pi}{N}\right)^{r} \left(\frac{\Gamma\left(\frac{\frac{1}{2}-r}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+r}{2}\right)} + \frac{\Gamma\left(\frac{\frac{3}{2}-r}{2}\right)}{\Gamma\left(\frac{\frac{3}{2}+r}{2}\right)} \right) \frac{\zeta(1-2r)}{4\zeta^{(2)}(2-2r)}.$$

Quadratic Dirichlet characters II — Murmurations

Theorem (based on Čech [Čec24])

Assume GRH. Let f be a smooth, fast-decaying weight function with Mellin transform $\mathcal{M}f$, and let χ_n denote the Jacobi symbol $\left(\frac{\cdot}{n}\right)$. For any N, y, T, c such that $2 < T < Ny$ and $\frac{1}{2} + \varepsilon < c < \frac{3}{4}$,

$$\sum_{\substack{n>0 \\ n \text{ odd, squarefree}}} \frac{1}{N} f\left(\frac{n}{N}\right) \frac{1}{\sqrt{Ny}} \sum_{p^k < Ny} \chi_n(p^k) \log p$$

$$= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \mathcal{M}f\left(\tfrac{3}{2} - s\right) \left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)} \right) \frac{\zeta(2-2s)}{4\zeta^{(2)}(3-2s)} (\pi y)^{s-\frac{1}{2}} \frac{ds}{s}$$

$$+ \frac{1}{\sqrt{Ny}} \sum_{p^k < \sqrt{Ny}} \frac{2\mathcal{M}f(1)}{3\zeta(2)} \log p + O\left(N^{\frac{1}{2}-c+\varepsilon} y^{c-\frac{1}{2}} T^{\varepsilon} + N^{\frac{1}{2}+\varepsilon} y^{\frac{1}{2}+\varepsilon} T^{-1+\varepsilon} \right).$$

Table of Contents

Elliptic curves — Setup 1/2

Let $E_{a,b} : y^2 = x^3 + ax + b$. Suppose $3 \nmid r$ and $2 \nmid t$. Define

$$\mathcal{F}(H) := \{E_{a,b} \,:\, a = r \bmod 6,\ b = t \bmod 6,\ |a| \le H^{\frac{1}{3}},\ |b| \le H^{\frac{1}{2}},\ p^4 \mid a \implies p^6 \nmid b\}. \tag{1}$$

Write the L-function attached to E as

$$L(s, E) := \prod_{p \mid N_E} \left(1 - \frac{a_p(E)}{p^{s+\frac{1}{2}}}\right)^{-1} \prod_{p \nmid N_E} \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha}_p}{p^s}\right)^{-1}$$

with $|\alpha_p| = |\overline{\alpha}_p| = 1$ and $\alpha_p + \overline{\alpha}_p = \frac{a_p(E)}{\sqrt{p}}$.

Theorem ((Proto-)explicit formula, [Fio14, Lemma 2.1])

$$\sum_{\substack{p \text{ prime},\, k \in \mathbb{Z}_{>0} \\ p^k < X,\, p \nmid N_E}} \left(\alpha_p^k + \overline{\alpha}_p^k\right) \log p = -\frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \frac{L'(s, E)}{L(s, E)} X^s \frac{ds}{s} + O\left(X^{1+\varepsilon} T^{-1+\varepsilon} N_E^{\varepsilon}\right).$$

Elliptic curves — Setup 2/2

Let $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > 0$, and let $\mathrm{Tr}_k(p)$ denote the trace of the Hecke operator at p on the space of level 1 weight k holomorphic cusp forms.

Definition (David–Huynh–Parks [DHP15, (3.38)])

$$A(\alpha,\gamma) := \frac{\zeta(1+\alpha+\gamma)}{\zeta(1+2\gamma)} \prod_{p=2,3} \frac{1 - a_p(E_{r,t})p^{-1-\gamma} + p^{-1-2\gamma}}{1 - a_p(E_{r,t})p^{-1-\alpha} + p^{-1-2\alpha}}$$

$$\cdot \prod_{p>3} \left[1 + \left(1 - \frac{p^9 - 1}{p^{10} - 1}\right)\left(\frac{1}{p^{1+2\gamma}} - \frac{1}{p^{1+\alpha+\gamma}} + \frac{p^{-2-\alpha-\gamma} - p^{-2-2\gamma}}{p^{2+2\alpha} - 1}\right.\right.$$

$$\left.\left. + \frac{p^{1+2\alpha+\gamma} - p^{1+\alpha+2\gamma} + p^{\gamma} - p^{\alpha}}{p^{\frac{3}{2}+\alpha+2\gamma}} \sum_{m=5}^{\infty} \frac{\mathrm{Tr}_{2m+2}(p)}{p^{2m(\alpha+1)+\frac{1}{2}}}\right)\right].$$

The function $A(\alpha,\gamma)$ extends to a nonzero holomorphic function on $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > -\frac{1}{4}$, and $A(r,r) = 1$ in this region.

Elliptic curves — Ratios conjecture

Theorem (David–Huynh–Parks [DHP15, Thm. 3.8])

Assume the ratios conjecture [DHP15, Conj. 3.7]. Suppose $\mathrm{Re}(r) \gg \frac{1}{\log H}$ *and* $\mathrm{Im}(r) \ll H^{1-\varepsilon}$. *Then*

$$\sum_{E \in \mathcal{F}(H)} \frac{L'(\frac{1}{2}+r, E)}{L(\frac{1}{2}+r, E)} = \sum_{E \in \mathcal{F}(H)} \Bigg[-\frac{\zeta'(1+2r)}{\zeta(1+2r)} + A_\alpha(r,r) - \omega_E \left(\frac{4\pi^2}{N_E}\right)^r \frac{\Gamma(1-r)}{\Gamma(1+r)} \zeta(1+2r) A(-r,r) \Bigg] + \mathcal{R}(H)$$

with $\mathcal{R}(H) \ll H^{\frac{1}{3}+\varepsilon}$.

Elliptic curves — Conductor distribution

David–Huynh–Parks ratios conjecture is for $\mathcal{F}(H)$: a family ordered by height. Murmurations care about conductor. We need a conversion.

Theorem

For any $\lambda_1 > \lambda_0 > \frac{4464}{\log H}$,

$$\frac{\#\left\{E \in \mathcal{F}(H) \,:\, \lambda_0 < \frac{N_E}{H} < \lambda_1\right\}}{\#\mathcal{F}(H)} = F_N(\lambda_1) - F_N(\lambda_0) + O((\log H)^{-1+\varepsilon}). \quad (2)$$

Theorem

For any $\lambda > \frac{4464}{\log H}$,

$$\frac{\#\left\{E \in \mathcal{F}(H) \,:\, \frac{N_E}{H} < \lambda\right\}}{\#\mathcal{F}(H)} \ll \lambda^{\frac{5}{6}}. \quad (3)$$

Elliptic curves — Conductor distribution plot

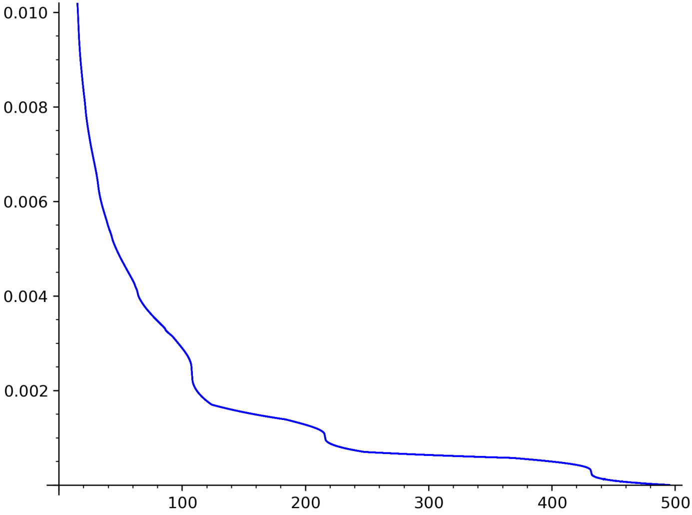


Figure: $F'_N(\lambda)$, computed numerically with $\Delta\lambda = 0.496$ [Cow24a].

Elliptic curves — Murmurations

Theorem

Let $\mathcal{F}(H)$ be the family (1) of elliptic curves ordered by height, take $\omega \in \{\pm 1\}$, and set $\mathcal{F}(H)^\omega := \{E \in \mathcal{F}(H) : \omega_E = \omega\}$. Assume that (2), (3), and the ratios conjecture [DHP15, Conj. 3.7] hold with $\mathcal{F}(H)$ replaced with $\mathcal{F}(H)^\omega$.

For any H, y, T, ε such that $0 < \varepsilon < \frac{5}{6}$ and $(Hy)^{\frac{1}{2}+\varepsilon} \ll T < Hy$,

$$\frac{1}{\#\mathcal{F}(H)^\omega} \sum_{E \in \mathcal{F}(H)^\omega} \frac{1}{\sqrt{Hy}} \sum_{\substack{p^k < Hy \\ p \nmid N_E}} \left(\alpha_p^k + \overline{\alpha}_p^k\right) \log p$$

$$= \frac{\omega}{2\pi i} \int_{\mathbb{R}} \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \frac{\Gamma(\frac{3}{2}-s)}{\Gamma(\frac{1}{2}+s)} \zeta(2s) A(\tfrac{1}{2}-s, s-\tfrac{1}{2}) \left(\frac{4\pi^2 y}{\lambda}\right)^{s-\frac{1}{2}} \frac{ds}{s} F_N'(\lambda)\, d\lambda$$

$$-1 + O\left(H^\varepsilon y^\varepsilon T^\varepsilon \mathcal{R}(H) \#\mathcal{F}(H)^{-1} + (\log H)^{-\frac{5}{6}}\right).$$

Table of Contents

Holomorphic newspaces — Setup

Let g be an even Schwartz function such that

$$\hat{g}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} g(x) e^{-2\pi i \xi x} \, dx$$

is supported on $(-\sigma, \sigma)$.

Let $H_k^+(N)$ and $H_k^-(N)$ be bases of normalized Hecke eigenforms of weight k, level N, and root number ± 1.

Let $\lambda_f(p)$ be such that $p^{\frac{k-1}{2}} \lambda_f(p)$ is the p^{th} Hecke eigenvalue of f.

Holomorphic newspaces — Murmurations

Theorem (based on Miller–Montague [MM11])

Assume GRH. For fixed $\pm \in \{+,-\}$, fixed $k \in 2\mathbb{Z}_{>0}$, and $N \to \infty$ prime,

$$\sum_{f \in H_k^{\pm}(N)} \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f, f\rangle} \sum_{p \neq N} \hat{g}\left(\frac{\log p}{\log N}\right) \frac{\lambda_f(p)\log p}{\sqrt{p}}$$

$$= \pm 2 \lim_{\delta \to 0^+} \int_0^{\infty} \hat{g}\left(1 + \frac{\log y}{\log N}\right) \int_{-\infty}^{\infty} \frac{\Gamma(\frac{k}{2} - 2\pi i t)}{\Gamma(\frac{k}{2} + 2\pi i t)}$$

$$\cdot \prod_p \left(1 + \frac{1}{(p-1)p^{4\pi i t + \delta}}\right) (4\pi^2 y)^{2\pi i t} dt \, \frac{dy}{y} + O\left(N^{\frac{\sigma}{2} - 1 + \varepsilon}\right).$$

Can take $\hat{g}(\xi)$ sharply peaked at $\xi = 1$ while keeping the error term small.

$$\sum_{f \in H_k^{\pm}(N)} \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f, f\rangle} \sim \frac{1}{2}, \qquad N^{-1-\varepsilon} \ll \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f, f\rangle} \ll N^{-1+\varepsilon}.$$

Table of Contents

Quadratic twists of an elliptic curve — Setup

Let N_E be an odd prime and let E be an elliptic curve of conductor N_E with even functional equation. Define

$$\mathcal{F}_E(D) \coloneqq \{0 < d < D \,:\, d \text{ a fundamental discriminant}, \chi_d(-N_E)\omega_E = 1\}.$$

Let A_E be the arithmetic factor given by the infinite product in [HMM11, (3.2)].

Quadratic twists of an elliptic curve — Murmurations

Theorem (based on Huynh–Miller–Morrison [HMM11])

Assume GRH. With the notation of the previous slide,

$$\frac{1}{\#\mathcal{F}_E(D)} \sum_{d \in \mathcal{F}_E(D)} \sum_{p \neq N_E} \sum_{\substack{k>0 \\ k \text{ odd}}} \hat{g}\left(\frac{\log p^k}{\log \frac{N_E D^2}{4\pi^2}}\right) \chi_d(p) \frac{(\alpha_p^k + \overline{\alpha}_p^k)\log p}{\sqrt{p^k}}$$

$$= \frac{g(0)}{2} \log\left(\frac{\sqrt{N_E} D}{2\pi}\right) + \int_{-\infty}^{\infty} \frac{\Gamma(1-\pi i t)}{\Gamma(1+\pi i t)} \frac{\zeta(1+2\pi i t) L_E(\mathrm{sym}^2, 1-2\pi i t)}{L_E(\mathrm{sym}^2, 1)}$$

$$\cdot A_E(-\pi i t, \pi i t) \frac{1}{\#\mathcal{F}_E(D)} \sum_{d \in \mathcal{F}_E(D)} \int_0^{\infty} \hat{g}\left(1 + \frac{\log y}{\log \frac{\sqrt{N_E} D}{2\pi}}\right) \left(\frac{Dy}{d}\right)^{2\pi i t} \frac{dy}{y}\, dt$$

$$+ O\left(D^{-\frac{1-\sigma}{2}}\right).$$

Can't take $\hat{g}(\xi)$ sharply peaked at $\xi = 1$ while keeping the error term small
$\implies$ Need to assume ratios conjecture to get murmurations for now.

Table of Contents

Similar work

Many similar examples seem possible using existing results from the literature, e.g.

- [Mil09] Miller — *An orthogonal test of the L-functions ratios conjecture.*
- [GJM$^+$10] Goes, Jackson, Miller, Montague, Ninsuwan, Peckner, Pham — *A unitary test of the ratios conjecture.*
- [MP12] Miller, Peckner — *Low-lying zeros of number field L-functions.*
- [FM15] Fiorilli, Miller — *Surpassing the ratios conjecture in the 1-level density of Dirichlet L-functions.*
- [CS07] Conrey–Snaith — Correlations of ζ'/ζ.
- [Kov24] Kovaleva — *Correlations of the squares of the riemann zeta on the critical line.*

Table of Contents

Elliptic curves — Conductor distribution plot

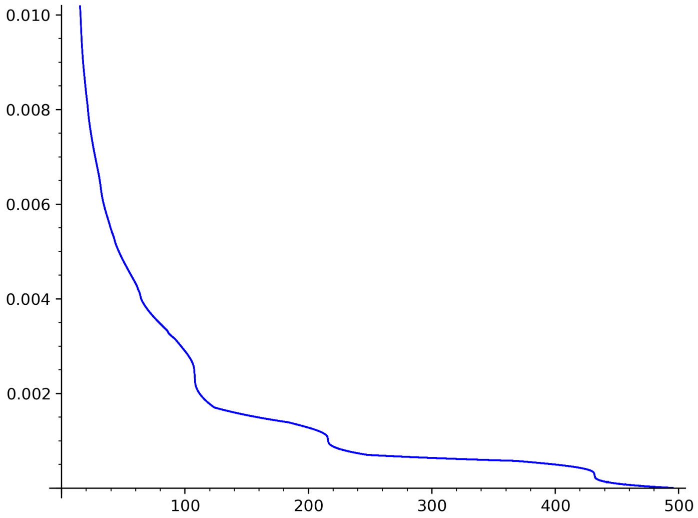


Figure: $F'_N(\lambda)$, computed numerically with $\Delta\lambda = 0.496$ [Cow24a].

Setup 1/2 — F_Δ

Let $E_{a,b} : y^2 = x^3 + ax + b$. Suppose $3 \nmid r$ and $2 \nmid t$. Define

$$\mathcal{F}(H) \coloneqq \left\{ E_{a,b} \,:\, a = r \bmod 6,\; b = t \bmod 6,\; |a| \leq H^{\frac{1}{3}},\; |b| \leq H^{\frac{1}{2}}, \right. \\ \left. p^4 \mid a \implies p^6 \nmid b \right\}.$$

Approximate discriminant distribution:

$$F_\Delta(\lambda) \coloneqq \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \begin{cases} 1 & \text{if } -16(4\alpha^3 + 27\beta^2) < \lambda \\ 0 & \text{otherwise} \end{cases} \, d\alpha \, d\beta$$

Setup 2/2 — $\rho(p, m)$

$\rho(p, m) :=$

$\gcd(m, p^\infty)$	$p \geq 5$	$p = 2$	$p = 3$
p^0	$1 - \frac{1}{p^2}$	$\frac{1}{2}$	1
p^1	$\frac{1}{p^2}\left(1 - \frac{1}{p}\right)$	$\frac{1}{4}$	
p^2	$\frac{1}{p^3}\left(1 - \frac{1}{p}\right)$	$\frac{1}{4}$	
p^3	$\frac{1}{p^4}\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p}\right)$		
p^4	$\frac{1}{p^5}\left(1 - \frac{1}{p}\right)\left(2 - \frac{1}{p}\right)$		
p^5	$\frac{1}{p^6}\left(1 - \frac{1}{p}\right)\left(2 - \frac{2}{p}\right)$		
p^6	$\frac{1}{p^7}\left(1 - \frac{1}{p}\right)\left(3 - \frac{2}{p}\right)$		
p^7	$\frac{1}{p^8}\left(1 - \frac{1}{p}\right)\left(3 - \frac{2}{p}\right)$		
p^8	$\frac{1}{p^9}\left(1 - \frac{1}{p}\right)\left(3 - \frac{2}{p}\right)$		
$p^n,\ n \geq 9$	$\frac{1}{p^{n+1}}\left(1 - \frac{1}{p}\right)\left(2 - \frac{2}{p}\right)$		

Theorem 1

Theorem (Conductor distribution)

For any $\lambda_1 > \lambda_0 > \frac{4464}{\log H}$,

$$\frac{\#\left\{E \in \mathcal{F}(H) \,:\, \lambda_0 < \frac{N_E}{H} < \lambda_1\right\}}{\#\mathcal{F}(H)} + O((\log H)^{-1+\varepsilon})$$

$$= \frac{\zeta^{(6)}(10)}{\zeta^{(6)}(2)} \sum_{m=1}^{\infty} \left(F_\Delta(m\lambda_1) - F_\Delta(m\lambda_0) + F_\Delta(-m\lambda_0) - F_\Delta(-m\lambda_1)\right)$$

$$\cdot\, \rho(2,m)\rho(3,m) \prod_{\substack{p \geq 5 \\ p \mid m}} \frac{\rho(p,m)}{1-p^{-2}}.$$

(Both sum and product are finite.)

Theorem 2

Theorem 1 says nothing when $N < \frac{4464}{\log H} H$.

Theorem (Bound on small conductors)

For any $\lambda > \frac{4464}{\log H}$,

$$\lambda^{\frac{5}{6}} \ll \frac{\#\{E \in \mathcal{F}(H) : N_E < \lambda H\}}{\#\mathcal{F}(H)} \ll \lambda^{\frac{5}{6}}.$$

(Take $\lambda \approx \frac{4464}{\log H}$ to bound how many small conductors there are in $\mathcal{F}(H)$)

Theorem 3

- Current best estimate for number of elliptic curves with conductor less than X is by Duke–Kowalski [DK00]: $\# \ll X^{1+\varepsilon}$.
- Widely-believed Brumer–McGuinness–Watkins [BM90, Wat08] heuristics predict $\# \sim cX^{\frac{5}{6}}$.
- Shankar–Shankar–Wang 2021 [SSW21]: $\# \sim c'X^{\frac{5}{6}}$ for certain large subfamilies.

Considering the counting problem in $\mathcal{F}(H)$ instead of "$\mathcal{F}(\infty)$" gives

Theorem (Counting elliptic curves by conductor)

$$X^{\frac{5}{6}} \ll \#\{E \in \mathcal{F}(H) \,:\, N_E < X\} \ll X^{\frac{5}{6}} \left(\frac{H}{X}\right)^{\frac{35}{54}} H^{\frac{7}{324}+\varepsilon} + H^{\frac{1}{2}}.$$

When $X \gg H^{\frac{217}{264}+\varepsilon} (\approx H^{0.822})$ this bound is tighter than the one of the same sort implied by Duke–Kowalski.

Plan: Height $\longrightarrow$ Discriminant $\longrightarrow$ Conductor

Main idea: Δ is "Archimedean", N is "non-Archimedean"

- Archimedean restrictions on Δ:
  e.g. $|a^3|, |b^2| < H \implies \Delta < 496H$.
- Non-Archimedean restrictions on N:
  e.g. when Δ is squarefree, must have $\Delta = N$.
- Intertwining Archimedean and non-Archimedean restrictions:
  e.g. if $\gcd(\Delta/N, 2^\infty) = 2$, then $N < 248H$.
  For $E_{a,b} \in \mathcal{F}(H)$, this happens iff $(a, b) \in \{(1, 3), (2, 1)\} \bmod 4$.

Key lemma

Archimedean: Let $H \in \mathbb{R}_{>0}$ and let $\lambda_0 < \lambda_1 \in \mathbb{R}$.
Non-Archimedean: Let r, t be as in the definition of $\mathcal{F}(H)$,
let $Q \in \mathbb{Z}_{>0}$, and let $S_Q \subseteq (\mathbb{Z}/Q\mathbb{Z})^2$.

Lemma (Intertwining Archimedean and non-Archimedean restrictions)

Suppose that $6 \mid Q$ *and that* $a = r \bmod 6$ *and* $b = t \bmod 6$ *for all* $(a,b) \in S_Q$. *Suppose also that, for all* $(a,b) \in S_Q$, *there does not exist an integer* d *such that* $\gcd(d,Q) > 1$ *and* $\gcd(d^4,Q) \mid \gcd(a,Q)$ *and* $\gcd(d^6,Q) \mid \gcd(b,Q)$. *Then*

$$\#\left\{E_{a,b} \in \mathcal{F}(H) \ : \ \lambda_0 < \frac{\Delta_E}{H} < \lambda_1, \ (a \bmod Q, \ b \bmod Q) \in S_Q\right\}$$

$$= \frac{4H^{\frac{5}{6}}}{\zeta^{(Q)}(10)}\Big(F_\Delta(\lambda_1) - F_\Delta(\lambda_0)\Big)\frac{\#S_Q}{Q^2} + O\left(H^{\frac{1}{2}}\frac{\#S_Q}{Q} + \#S_Q\right).$$

Outline of proof of theorem 1

Let $S_{Q,m}$ be the residue classes (a, b) which cause $\Delta/N = m$ to some specified (technical) non-Archimedean precision.

$$\#\left\{E \in \mathcal{F}(H) \,:\, \lambda_0 < \frac{N_E}{H} < \lambda_1\right\}$$

$$= \sum_{\substack{-\frac{496}{\lambda_0} < m < \frac{64}{\lambda_0} \\ m \neq 0}} \#\left\{E \in \mathcal{F}(H) \,:\, \lambda_0 < \frac{\Delta_E}{mH} < \lambda_1,\ \frac{\Delta_E}{N_E} = m\right\}$$

$$= \sum_{\substack{-\frac{496}{\lambda_0} < m < \frac{64}{\lambda_0} \\ m \neq 0}} \#\left\{E_{a,b} \in \mathcal{F}(H) \,:\, \lambda_0 < \frac{\Delta_E}{mH} < \lambda_1,\ (a, b) \in S_{Q,m}\right\}$$

$$- O\left(\#\left\{E \in \mathcal{F}(H) \,:\, \frac{496}{\lambda_0} < \frac{|\Delta_E|}{N_E},\ \lambda_0 < \frac{|\Delta_E|}{H}\right\}\right).$$

Use key lemma on main term.

Thanks!

Thanks for listening!

Jonathan Bober, Andrew R. Booker, Min Lee, and David Lowry-Duda.
Murmurations of modular forms in the weight aspect, 2023.
arXiv 2310.07746.

Andrew R. Booker, Min Lee, David Lowry-Duda, Andrei Seymour-Howell, and Nina Zubrilina.
Murmurations of maass forms, 2024.
arXiv 2409.00765.

Armand Brumer and Oisín McGuinness.
The behavior of the Mordell-Weil group of elliptic curves.
Bull. Amer. Math. Soc. (N.S.), 23(2):375–382, 1990.

Martin Čech.
The ratios conjecture for real Dirichlet characters and multiple Dirichlet series.
Trans. Amer. Math. Soc., 377(5):3487–3528, 2024.

Alex Cowan.
Conductor distribution code.

https://github.com/thealexcowan/conductordistribution, July 2024.

Alex Cowan.
Kronecker symbol murmurations code.
https://github.com/thealexcowan/murmurations, July 2024.

J. B. Conrey and N. C. Snaith.
Applications of the L-functions ratios conjectures.
Proc. Lond. Math. Soc. (3), 94(3):594–646, 2007.

Chantal David, Duc Khiem Huynh, and James Parks.
One-level density of families of elliptic curves and the Ratios Conjecture.
Res. Number Theory, 1:Paper No. 6, 37, 2015.

W. Duke and E. Kowalski.
A problem of Linnik for elliptic curves and mean-value estimates for automorphic representations.
Invent. Math., 139(1):1–39, 2000.
With an appendix by Dinakar Ramakrishnan.

Daniel Fiorilli.
Elliptic curves of unbounded rank and Chebyshev's bias.
Int. Math. Res. Not. IMRN, (18):4997–5024, 2014.

Daniel Fiorilli and Steven J. Miller.
Surpassing the ratios conjecture in the 1-level density of Dirichlet L-functions.
Algebra Number Theory, 9(1):13–52, 2015.

John Goes, Steven Jackson, Steven J. Miller, David Montague, Kesinee Ninsuwan, Ryan Peckner, and Thuy Pham.
A unitary test of the ratios conjecture.
J. Number Theory, 130(10):2238–2258, 2010.

Duc Khiem Huynh, Steven J. Miller, and Ralph Morrison.
An elliptic curve test of the L-functions ratios conjecture.
J. Number Theory, 131(6):1117–1147, 2011.

Henryk Iwaniec and Emmanuel Kowalski.
Analytic number theory, volume 53 of *American Mathematical Society Colloquium Publications*.

American Mathematical Society, Providence, RI, 2004.

Valeriya Kovaleva.
Correlations of the squares of the riemann zeta on the critical line, 2024.
arXiv 2401.13725.

Kyu-Hwan Lee, Thomas Oliver, and Alexey Pozdnyakov.
Murmurations of Dirichlet characters, 2023.
arXiv 2307.00256.

Steven J. Miller.
An orthogonal test of the L-functions ratios conjecture.
Proc. Lond. Math. Soc. (3), 99(2):484–520, 2009.

Steven J. Miller and David Montague.
An orthogonal test of the L-functions ratios conjecture, II.
Acta Arith., 146(1):53–90, 2011.

Steven J. Miller and Ryan Peckner.
Low-lying zeros of number field L-functions.
J. Number Theory, 132(12):2866–2891, 2012.

Hugh L. Montgomery and Robert C. Vaughan.
Multiplicative number theory. I. Classical theory, volume 97 of *Cambridge Studies in Advanced Mathematics*.
Cambridge University Press, Cambridge, 2007.

Ananth N. Shankar, Arul Shankar, and Xiaoheng Wang.
Large families of elliptic curves ordered by conductor.
Compos. Math., 157(7):1538–1583, 2021.

Mark Watkins.
Some heuristics about elliptic curves.
Experiment. Math., 17(1):105–125, 2008.

Nina Zubrilina.
Murmurations, 2023.
arXiv 2310.07681.