Non-existence of some Moore Cayley digraphs

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based on joint work with

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ORIGINAL PAPER

A Note on Moore Cayley Digraphs

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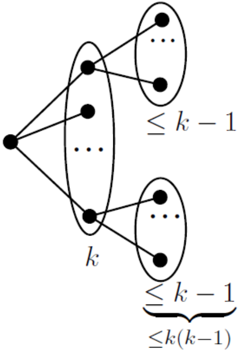
Degree/diameter problem

Let Γ be an undirected graph:

- regular of degree k;
- of diameter D;
- $|V(\Gamma)| \to$ max?

$$|V(\Gamma)| \leq 1 + k + k(k-1) + \ldots + k(k-1)^{D-1}$$



Moore graphs

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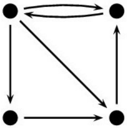
- of diameter D;
- $|V(\Gamma)| \leq 1 + k + k(k-1) + \ldots + k(k-1)^{D-1}$,

and if equality attains (Damerell, Bannai & Ito, 1973):

Diameter D	**Degree** k	**Graph**	**Vertex-transitive**
1	k	K_{k+1}	yes
D	2	C_{2D+1}	yes
2	3	Petersen	yes
2	7	Hoffman-Singleton	yes
2	57	?	no

Digraphs = Mixed graphs = Partially directed graphs

Digraphs may have arcs as well as (undirected) edges:



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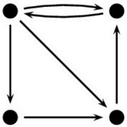
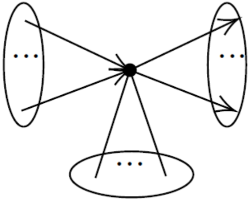


A digraph is regular with degrees (r, z) if every vertex:

- is incident to the same number r of undirected edges,
- has the same in-/out-degree z (counting only arcs not contained in digons).



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Moore digraphs of diameter 2 can be defined by the 'unique trail' property:

for every pair (x, y) of vertices of Δ,
there is a unique trail $x \longrightarrow \ldots \longrightarrow y$ of length at most 2.

Moore digraphs of diameter 2

Theorem (Bosák, 1979)

Let Δ be a Moore digraph of diameter 2 with degrees (r, z).
Then the number n of vertices of Δ equals

$$n = (r + z)^2 + z + 1$$

and one of the following cases occurs:

- $z = 1, r = 0$ (a directed 3-cycle);
- $z = 0, r = 2$ (an undirected 5-cycle);
- there exists an odd positive integer c such that

$$c \text{ divides } (4z - 3)(4z + 5) \text{ and } r = \frac{1}{4}(c^2 + 3).$$

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Admissible values of r: $1, 3, 7, 13, 21, \ldots,$
For given r: infinitely many admissible values of z.

Known Moore digraphs

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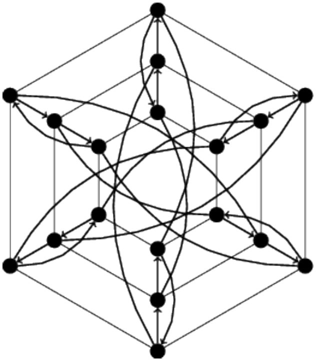

- $r > 1$: only three examples are known:
  - the Bosák graph on 18 vertices, $(r, z) = (3, 1)$;
  - two Jørgensen graphs on 108 vertices, $(r, z) = (3, 7)$.

All three examples with $r > 1$ are **Cayley** digraphs.

$S_3 \times C_3$ or $(C_3 \times C_3) \rtimes C_2$.

Cayley digraphs

Given a finite group G and a subset $S \subseteq G \setminus \{1\}$ s.t.

$$S = S_1 \cup S_2,\ S_1 = S_1^{-1},\ \text{and}\ S_2 \cap S_2^{-1} = \emptyset,$$

the **Cayley** (di-)graph $Cay(G, S)$ has:

- the vertex set G;
- the arcs $g \longrightarrow gs$ for every $g \in G$, $s \in S$;
- the undirected degree $r = |S_1|$;
- the directed degree $z = |S_2|$.

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for every pair (x, y) of vertices of Δ,
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it follows that if $\Delta = Cay(G, S)$ is Moore, then:

- for $g \in S$, $\nexists$ a pair $(s_1, s_2) \in S \times S$ such that $g = s_1 s_2$;
- for $g \notin S$, $\exists!$ a pair $(s_1, s_2) \in S \times S$ such that $g = s_1 s_2$.

Moore Cayley digraphs on at most 486 vertices, 1

Feasible parameters:

n	r	z	Exist	Transitive	Cayley
18	3	1	!		Yes
40	3	3	No[1]		
54	3	4	No[1]		
84	7	2	No[1]		
88	3	6	?	?	No[2]
108	3	7	≥ 2		Yes
150	7	5	?	?	No[2]
154	3	9	?	?	No[2]
180	3	10	?	?	No[2]

[1]: López, Miret, Fernández: *Non-existence of some mixed Moore graphs of diameter 2 using SAT* (2016).

[2]: Erskine: *Mixed Moore Cayley graphs* (2017).

Moore Cayley digraphs on at most 486 vertices, 2

n	r	z	Exist	Transitive	Cayley
204	7	7	?	?	No[2]
238	3	12	?	?	No[2]
270	3	13	?	?	No[2]
294	13	4	?	?	No[2]
300	7	10	?	?	No[2]
340	3	15	?	?	No[2]
368	13	6	?	?	No[2]
374	7	12	?	?	No[2]
378	3	16	?	?	No[2]
460	3	18	?	?	No[2]
486	21	1	?	No[3]	

[3]: Jørgensen: talk in Pilsen (2018).

The adjacency algebra of a Moore digraph Δ

The **adjacency matrix** $A = A(\Delta) \in \mathbb{R}^{V \times V}$:

$$(A)_{x,y} := \begin{cases} 1 & \text{if } x \to y, \\ 0 & \text{otherwise.} \end{cases}$$

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Recall that for every pair (x, y) of vertices of Δ, there is a unique trail $x \longrightarrow \ldots \longrightarrow y$ of length ≤ 2. Then:

$$\mathsf{A}^2 = (r-1)\mathsf{I} + \mathsf{J} - \mathsf{A}, \quad \text{and} \quad \mathsf{JA} = \mathsf{AJ} = k\mathsf{J},$$

so A is diagonalizable with 3 eigenspaces with eigenvalues $k = r + z$, and $\lambda_1, \lambda_2 \in \mathbb{Z}$, which can be computed from r and z.

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The **projection matrix** E_{λ_i} onto the (right) λ_i-eigenspace:

$$E_{\lambda_i} \in \langle A, I, J \rangle.$$

Duval (1988); Jørgensen (2003); Godsil, Hobart, Martin (2007)

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  - Higman: a degree 57 Moore graph is not transitive;
  - Benson: automorphisms of finite GQs.
  - De Winter, Kamischke, Wang: partial difference sets over Abelian groups.

Application to Moore **Cayley** digraphs

- Let Δ be a Moore **Cayley** digraph over G with degrees (r, z).

  Recall: $\exists$ an odd positive c which divides $(4z-3)(4z+5)$, and $r = \frac{1}{4}(c^2+3)$.
- Then $G \leq \mathrm{Aut}(\Delta)$ is a regular subgroup. Hence, for $g \neq 1$:

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$$\left(-\frac{1}{c}\mathbf{a}(g) + \frac{c^2-2c+4z+5}{4c}\right) \in \mathbb{Z} \tag{1}$$

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- Note that $\mathbf{a}(g)$ counts the number of arcs in the g-orbits that are (directed) cycles.
- For some $|G|$ and $|g|$, Condition (1) implies that $\mathbf{a}(g)$ is “too large” so that it contradicts:

  for every pair (x, y) of vertices of Δ,
  there is a unique trail $x \longrightarrow \ldots \longrightarrow y$ of length at most 2.

A few observations

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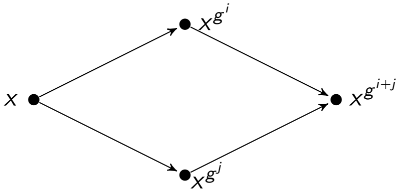

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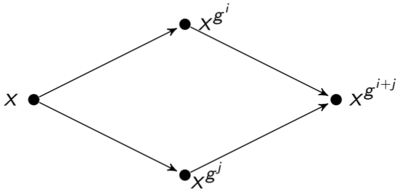

which violates the 'unique trail' property of Δ. Using this, we can show:

$$\sum_{i=1}^{p-1} \mathbf{a}(g^i) \leq 2n.$$

Example: $n = 88$, $(r, z) = (3, 6)$

Let $n = |G| = 88$ and $g \in G$ of a prime order p.

- By Cauchy's Lemma, we may assume $p = 11$.
- The Higman-Benson observation implies that

$$\frac{-\mathbf{a}(g) + 8}{3} \in \mathbb{Z},$$

thus $\mathbf{a}(g) \geq 11$.

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- Therefore, $\mathbf{a}(g) \geq 44$.
- Now we get a contradiction with

$$\sum_{i=1}^{p-1} \mathbf{a}(g^i) \leq 2n.$$

Example: $n = 88$, $(r, z) = (3, 6)$

Let $n = |G| = 88$ and $g \in G$ of a prime order p.

- By Cauchy's Lemma, we may assume $p = 11$.
- The Higman-Benson observation implies that

$$\frac{-\mathbf{a}(g) + 8}{3} \in \mathbb{Z},$$

  thus $\mathbf{a}(g) \geq 11$.
- Using the Sylow theorem, we can show that there exists an element of order 2 in $C_G(g)$, thus $\mathbf{a}(g) \geq 2 \cdot 11$.
- Therefore, $\mathbf{a}(g) \geq 44$.
- Now we get a contradiction with

$$\sum_{i=1}^{p-1} \mathbf{a}(g^i) \leq 2n.$$

- There are no Moore Cayley digraphs on 88 vertices.

Results

n	r	z	Cayley (by Erskine)	Cayley (by Higman)
40	3	3	No	No
54	3	4	No	
84	7	2	No	No
88	3	6	No	No
150	7	5	No	
154	3	9	No	No
180	3	10	No	

Results

n	r	z	Cayley (by Erskine)	Cayley (by Higman)
204	7	7	No	No
238	3	12	No	No
270	3	13	No	
294	13	4	No	
300	7	10	No	
340	3	15	No	No
368	13	6	No	No
374	7	12	No	No
378	3	16	No	
460	3	18	No	No
486	21	1		

It rules out 29 out of 58 feasible parameter sets for $v \leq 2000$.

Although it does not cover all results by Erskine, the proof is computer-free.

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