Algebraic bounds for sum-rank-metric codes

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Algebraic Graph Theory Seminar

Sum-rank metric and the size of a sum-rank-metric code

0 1 1 0

Hamming distance:

$$(0 \oplus 1) + (1 \oplus 1) + (1 \oplus 0) + (0 \oplus 0) = 2$$

1 1 0 0

$$\begin{pmatrix}1&0&0\\0&1&1\\1&1&1\end{pmatrix}, \begin{pmatrix}1&0&0\\1&0&1\end{pmatrix}, \begin{pmatrix}0&1\end{pmatrix}, 1$$

Sum-rank distance over $\mathbb{F}_2$:

$$\mathrm{rk}\left(\begin{pmatrix}1&0&0\\1&1&1\\0&1&0\end{pmatrix}-\begin{pmatrix}0&0&0\\1&0&0\\1&0&1\end{pmatrix}\right)+\mathrm{rk}\left(\begin{pmatrix}1&0&0\\1&0&1\end{pmatrix}-\begin{pmatrix}0&0&0\\1&1&1\end{pmatrix}\right)+$$

$$+\ \mathrm{rk}\left(\begin{pmatrix}0&1\end{pmatrix}-\begin{pmatrix}0&1\end{pmatrix}\right)+\mathrm{rk}(1-0)=$$

$$=\ 2\ +\ 2\ +\ 0\ +\ 1\ =\ 6$$

$$\begin{pmatrix}0&0&0\\1&0&0\\1&0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\1&1&1\end{pmatrix}, \begin{pmatrix}0&1\end{pmatrix}, 0$$

In general, the **sum-rank-metric space** is

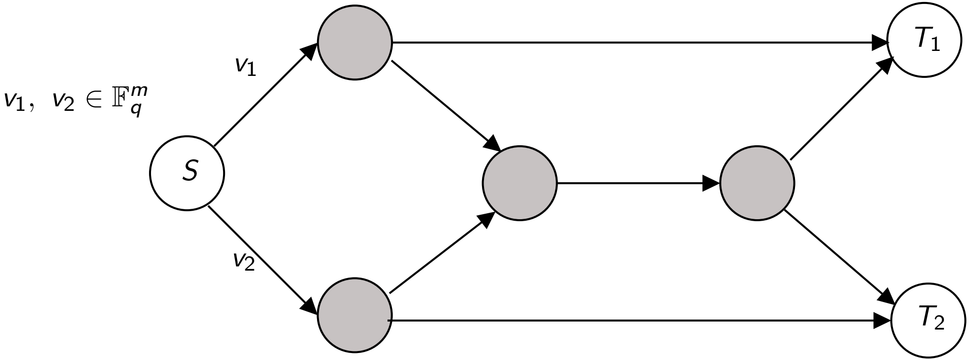
$$\mathbb{F}_q^{n_1\times m_1}\times\cdots\times\mathbb{F}_q^{n_t\times m_t}$$

with **sum-rank distance** between $\mathrm{A} := (A_1,\ldots,A_t)$ and $\mathrm{B} := (B_1,\ldots,B_t)$:

$$\mathrm{srkd}(\mathrm{A},\mathrm{B}) = \sum_{i=1}^{t}\mathrm{rk}(A_i - B_i).$$

Denoted by $\boxed{\mathbb{F}_q^{\mathbf{n}\times\mathbf{m}}}$, where $\mathbf{n} = [n_1,\ldots,n_t]$ and $\mathbf{m} = [m_1,\ldots,m_t]$.

- Network coding: transmitting messages $v_1, \dots, v_n \in \mathbb{F}_q^m$ over network $\mathcal{N}$ from source S to terminals $T_1, \dots, T_M$.
- The terminals demand **all** the messages.



Motivation from network coding

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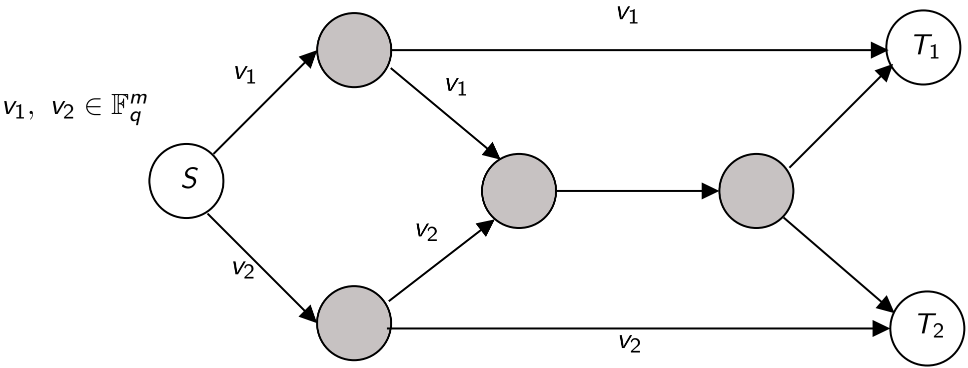

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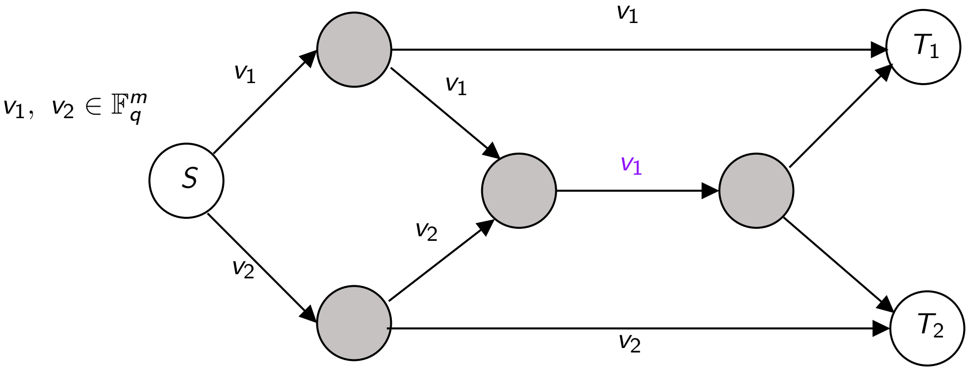

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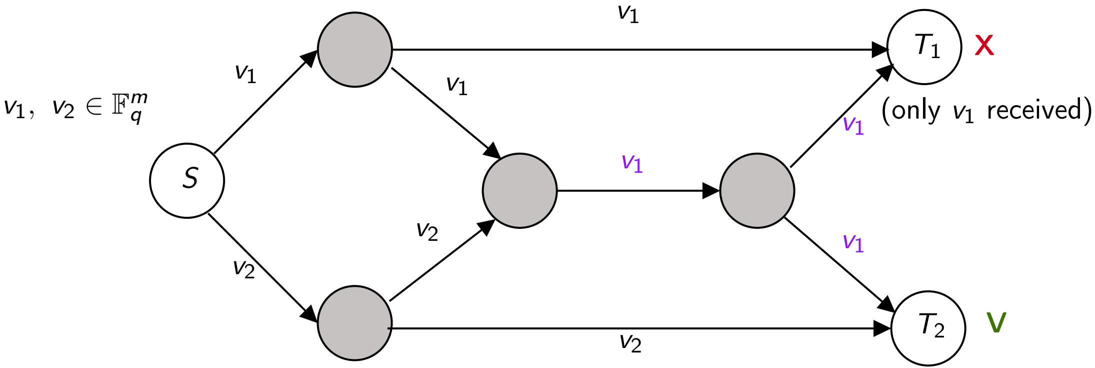

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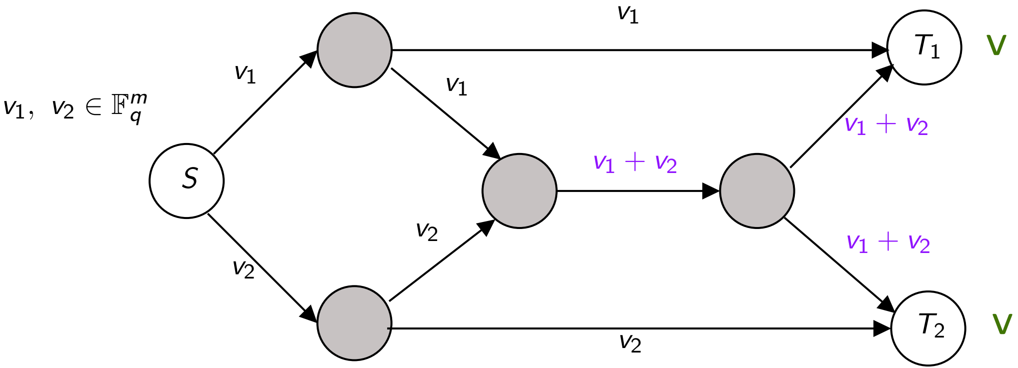

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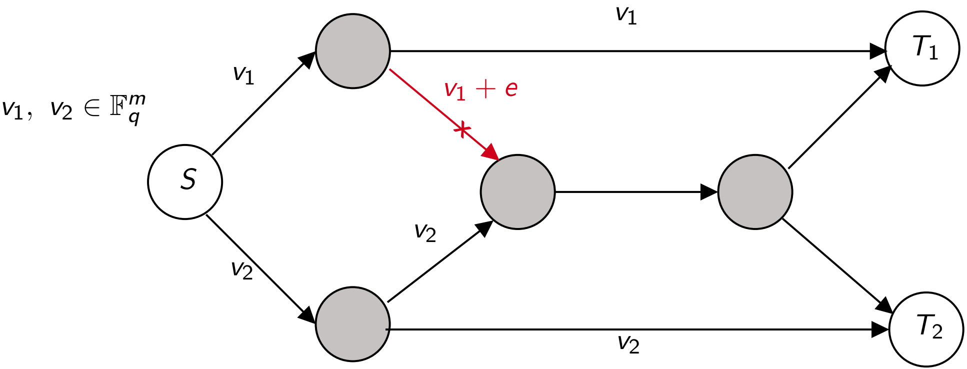

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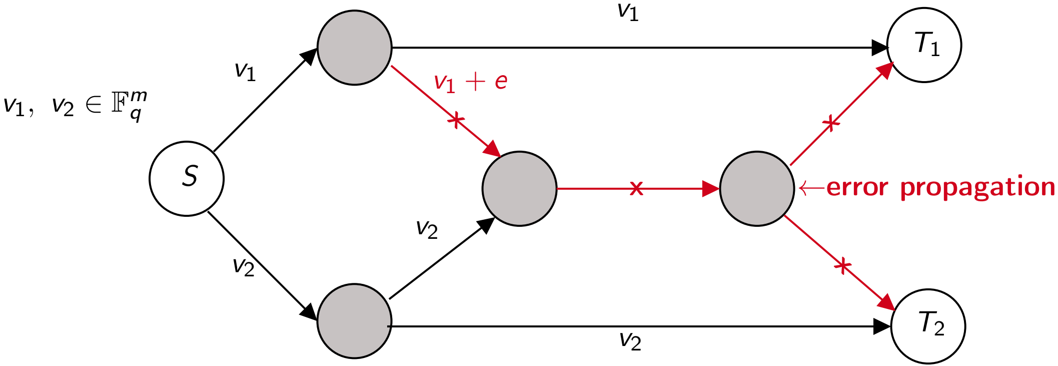


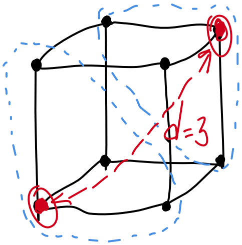
- When using rank-metric codes can help resolve the error propagation.

A **sum-rank-metric code $\mathcal{C}$ with minimum distance** d is a subset of $\mathbb{F}_q^{\mathbf{n}\times\mathbf{m}}$ such that:

$$\min_{A,B\in\mathcal{C}} \mathrm{srkd}(A,B) = d.$$

NB! The code is non-linear in general.

Question: What is the maximal size of a sum-rank-metric code with minimum distance d?



- Induced Singleton, induced Hamming, induced Plotkin, induced Elias, Singleton, Sphere-Packing, Projective Sphere-Packing, Total Distance **(Byrne, Gluesing-Luerssen, Ravagnani, 2021)**: come from adaptation of classical coding arguments.
- The Ratio-type bound **(Abiad, K, Ravagnani, 2024)**: a spectral bound from the connection to the k-independence number of the respective graph.
- The Delsarte's LP bound **(Abiad, Gavrilyuk, K, Ponomarenko, 2025)**: the new bound from constructing an association scheme and applying Delsarte's approach to it.

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Outline

1. Sum-rank metric and the size of sum-rank-metric code
2. Sum-rank-metric graph and Ratio-type bound
3. Delsarte's LP approach for sum-rank-metric codes
4. Conclusion and future research

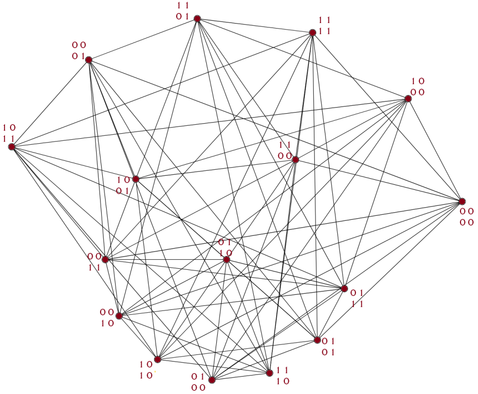
Sum-rank-metric graph and Ratio-type bound

joint work with Aida Abiad and Alberto Ravagnani (2024)

Sum-rank-metric graph $\Gamma := \Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$, $\mathbf{n} = [n_1, \ldots, n_t]$, $\mathbf{m} = [m_1, \ldots, m_t]$, with $m_i \geq n_i$ and $m_1 \geq \cdots \geq m_t$:

- *vertices* of Γ = t-tuples of matrices from $\mathbb{F}_q^{\mathbf{n} \times \mathbf{m}}$;
- A $:= (A_1, \ldots, A_t)$ and B $:= (B_1, \ldots, B_t)$ form an *edge* iff the sum-rank distance is 1:

$$\mathrm{srkd}(\mathrm{A},\mathrm{B}) = \sum_{i=1}^{t} \mathrm{rk}(A_i - B_i) = 1.$$



Sum-rank-metric graph

$\Gamma := \Gamma(2, 2, \mathbb{F}_2)$:

$V(\Gamma)$ = matrices 2×2 over $\mathbb{F}_2$.

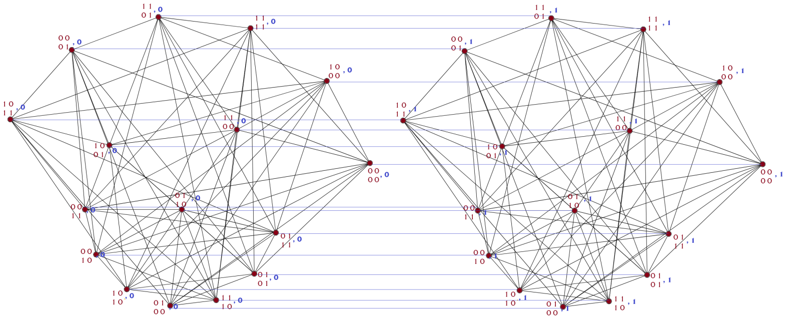
$A \sim B$ if $\mathrm{rk}(A - B) = 1$.

If $t = 1$ it is also a **bilinear forms graph**.

(Byrne, Gluesing-Luerssen, Ravagnani, 2022)
Geodesic distance between A and B in Γ = sum-rank distance $\mathrm{srkd}(A, B)$.

Sum-rank-metric graph $\Gamma := \Gamma([2,1],[2,1],\mathbb{F}_2)$:

- vertices: (A_1, A_2), A_1 is size 2×2 over $\mathbb{F}_2$, $A_2 \in \{0,1\}$;
- edges: $(A_1, A_2) \sim (B_1, B_2)$ if $rk(A_1 - B_1) + rk(A_2 - B_2) = 1$.



The graph is a *Cartesian product* of the first graph $\Gamma(2,2,\mathbb{F}_2)$ and $\Gamma(1,1,\mathbb{F}_2) = K_2$, a graph of two adjacent vertices.

For a graph G, its k-**independence number** α_k is the size of the largest set of vertices S such that distance between any $u, v \in S$ is more than k:

$$\alpha_k = \min_{u,v \in S} \mathrm{dist}_G(u, v) > k.$$

It is easy to see that α_{d-1} of $\Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q) =$
$=$ the maximal size of a code in $\mathbb{F}_q^{\mathbf{n} \times \mathbf{m}}$ with minimum distance d.

Question: What is an upper bound on α_{d-1} of the sum-rank-metric graph?

Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix A of a graph G.

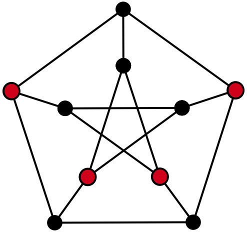
Ratio bound (Hoffman, 1974?): For a regular graph G, we have

$$\alpha_1 \leq n\frac{-\lambda_n}{\lambda_1 - \lambda_n}.$$

For example, the eigenvalues of Petersen graph are

$$\mathbf{3}, 1, 1, 1, 1, 1, -2, -2, -2, -\mathbf{2}.$$

Then the Ratio bound is $\alpha_1 \leq 10 \cdot \frac{2}{3+2} = 4$, and it is tight.



Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix A of a graph G.

The following result generalizes the Hoffman's bound:

Ratio-type bound (Abiad, Coutinho, Fiol, 2019): For a regular graph G and $p \in \mathbb{R}_{d-1}[x]$ let $W(p)$ be the largest element of the diagonal of $p(A)$. Then

$$\alpha_{d-1} \leq n \frac{W(p) - \min_{i \in [2,n]} p(\lambda_i)}{p(\lambda_1) - \min_{i \in [2,n]} p(\lambda_i)}.$$

How to define p so the right-hand expression is minimized?

The right-hand expression can be minimized e.g. by using the LP method **(Fiol, 2020)**, if we have the eigenvalues of the sum-rank-metric graph:

$$\begin{array}{lll} \text{minimize} & \sum_{i=0}^{r} m(\theta_i)x_i & \\ \text{subject to} & f[\theta_0,\dots,\theta_s]=0, & s=d,\dots,r \\ & x_i \geq 0, & i=1,\dots,r \\ & x_0 = 1 & \end{array}$$

where $f[\theta_0,\dots,\theta_m]$ is defined recursively: $f[\theta_i]=x_i$ for $i\in\{0,\dots,r\}$, and

$$f[\theta_i,\dots,\theta_j]=\frac{f[\theta_{i+1},\dots,\theta_j]-f[\theta_i,\dots,\theta_{j-1}]}{\theta_j-\theta_i}, \quad j>i.$$

In general, the method works for $(d-1)$-walk-regular graphs. All sum-rank-metric graphs are walk-regular **(Abiad, K, Ravagnani, 2024)**.

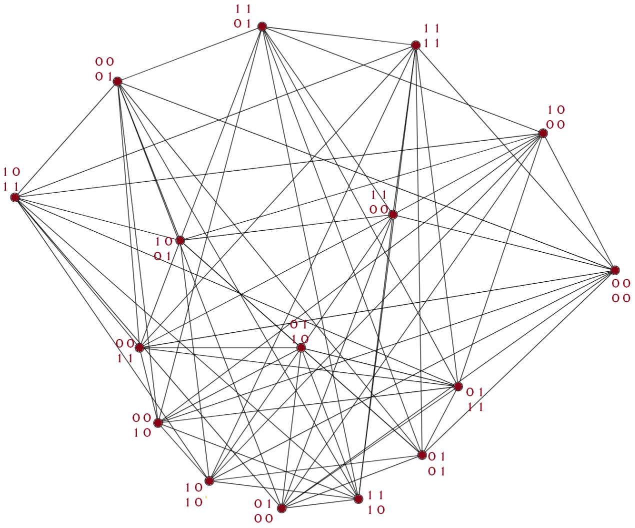
Connection to bilinear forms graphs

Let $\mathbf{n} = [n_1, \ldots, n_t]$, $\mathbf{m} = [m_1, \ldots, m_t]$.

(Abiad, K, Ravagnani, 2024) The sum-rank-metric graph $\Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ is the Cartesian product of graphs $\Gamma(n_i, m_i, \mathbb{F}_q)$ for $i = 1, \ldots, t$.

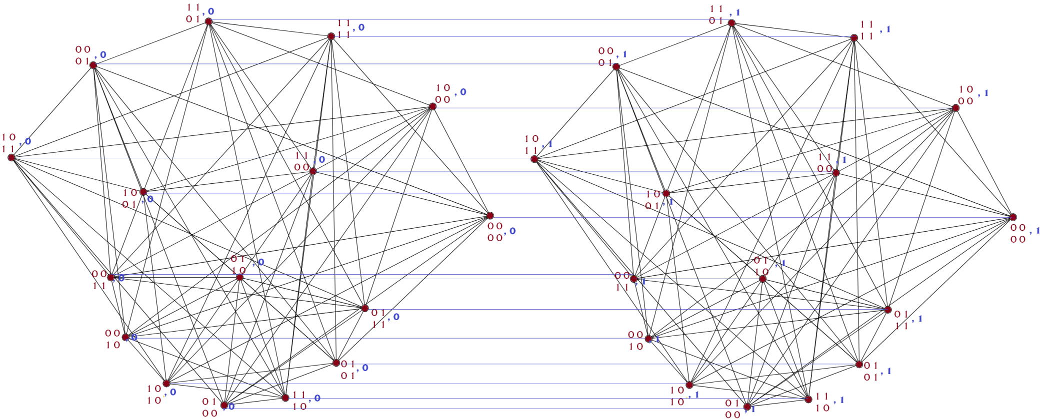
The graph $\Gamma(n_i, m_i, \mathbb{F}_q)$ is a *bilinear forms graph*

Bilinear forms graph $\Gamma(2, 2, \mathbb{F}_2)$, vertices are 2×2 matrices over $\mathbb{F}_2$:



Sum-rank-metric graph $\Gamma([2,1],[2,1],\mathbb{F}_2)$:

- vertices: (A_1, A_2), A_1 is 2×2 matrix over $\mathbb{F}_2$, $A_2 \in \{0,1\}$ (1×1 matrix);
- edges: $(A_1, A_2) \sim (B_1, B_2)$ if $rk(A_1 - B_1) + rk(A_2 - B_2) = 1$.



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The graph $\Gamma(n_i, m_i, \mathbb{F}_q)$ is a *bilinear forms graph*, with eigenvalues given by

$$\theta_j = \frac{(q^{n_i - j} - 1)(q^{m_i} - q^j) - q^j + 1}{q - 1}, \quad j = 0, \dots, n_i.$$

The eigenvalues of the Cartesian product are all possible sums of eigenvalues of the product's factors.

Let $\mathbf{n} = [n_1, \ldots, n_t]$, $\mathbf{m} = [m_1, \ldots, m_t]$.

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The eigenvalues of the Cartesian product are all possible sums of eigenvalues of the product's factors:

The graph $\Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ has the eigenvalues ($i_j = 0, \ldots, n_j$ for each $j \in [t]$)

$$\lambda_{(i_1,\ldots,i_t)} = \sum_{j=1}^{t} \frac{(q^{n_j - i_j} - 1)(q^{m_j} - q^{i_j}) - q^{i_j} + 1}{q - 1}.$$

$\Rightarrow$ From the full list of eigenvalues we can calculate the Ratio-type bound either from an explicit formula ($d = 3, 4$) or using the LP ($d \geq 5$).

Delsarte's LP bound

joint work with Aida Abiad, Alexander L. Gavrilyuk, and Ilia Ponomarenko (2025)

The Delsarte's LP bound is an efficient tool that has been used to estimate the maximal size of the code in multiple metrics:

- Hamming codes **(Delsarte, 1973)**;
- rank-metric codes **(Delsarte, 1978)**;
- bilinear alternating forms **(Delsarte, Goethals, 1975)**;
- Lee codes **(Astola, 1982)**;
- permutation codes **(Dukes, Ihringer, Lindzey, 2020)**;
- ...
- sum-rank-metric codes?

$\mathcal{A} = (X, \mathcal{R})$ is a **symmetric association scheme** on set X with relations $\mathcal{R} = \{R_0, \ldots, R_n\}$ that form a partition of $X \times X$ such that:

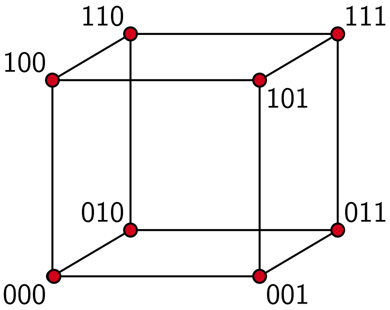
1. R_0 consists of all (x, x) for $x \in X$.
2. $(x, y) \in R_i$ means $(y, x) \in R_i$ for any R_i, x, y.
3. If $(x, y) \in R_k$, then the number of z such that $(x, z) \in R_i$ and $(y, z) \in R_j$ is a constant $p^k_{i,j}$ that does not depend on the choice of x, y.

If G is a distance-regular graph, then $(V(G), \mathcal{R})$ is a symmetric association scheme if we define relations by:

$$(x, y) \in R_i \Leftrightarrow d_G(x, y) = i.$$

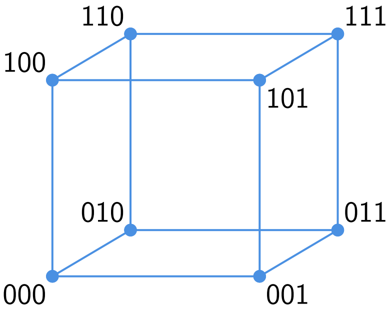
A well-known example of distance-regular graphs are *Hamming graphs*.

Vertices at distance 0:



	000	001	010	011	100	101	110	111
000	R_0							
001		R_0						
010			R_0					
011				R_0				
100					R_0			
101						R_0		
110							R_0	
111								R_0

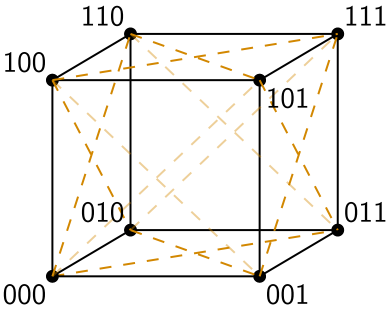
Vertices at distance 1:



	000	001	010	011	100	101	110	111
000	R_0	R_1	R_1		R_1			
001	R_1	R_0		R_1		R_1		
010	R_1		R_0	R_1			R_1	
011		R_1	R_1	R_0				R_1
100	R_1				R_0	R_1	R_1	
101		R_1			R_1	R_0		R_1
110			R_1		R_1		R_0	R_1
111				R_1		R_1	R_1	R_0

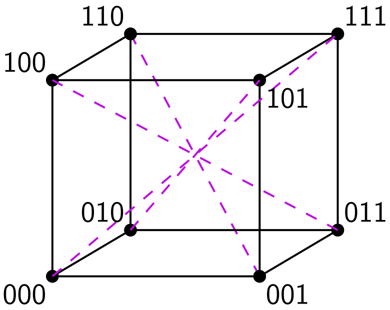
Example: Hamming scheme

Vertices at distance 2:



	000	001	010	011	100	101	110	111
000	R_0	R_1	R_1	R_2	R_1	R_2	R_2	
001	R_1	R_0	R_2	R_1	R_2	R_1		R_2
010	R_1	R_2	R_0	R_1	R_2		R_1	R_2
011	R_2	R_1	R_1	R_0		R_2	R_2	R_1
100	R_1	R_2	R_2		R_0	R_1	R_1	R_2
101	R_2	R_1		R_2	R_1	R_0	R_2	R_1
110	R_2		R_1	R_2	R_1	R_2	R_0	R_1
111		R_2	R_2	R_1	R_2	R_1	R_1	R_0

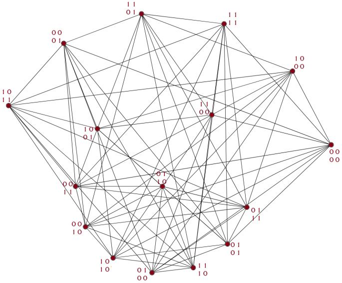
Vertices at distance 3:



	000	001	010	011	100	101	110	111
000	R_0	R_1	R_1	R_2	R_1	R_2	R_2	R_3
001	R_1	R_0	R_2	R_1	R_2	R_1	R_3	R_2
010	R_1	R_2	R_0	R_1	R_2	R_3	R_1	R_2
011	R_2	R_1	R_1	R_0	R_3	R_2	R_2	R_1
100	R_1	R_2	R_2	R_3	R_0	R_1	R_1	R_2
101	R_2	R_1	R_3	R_2	R_1	R_0	R_2	R_1
110	R_2	R_3	R_1	R_2	R_1	R_2	R_0	R_1
111	R_3	R_2	R_2	R_1	R_2	R_1	R_1	R_0

It is well-known that *bilinear forms graphs are distance-regular*.

A symmetric association scheme defined on a bilinear forms graph is called a **bilinear forms scheme**.



The Q-eigenmatrix in a bilinear forms graph

By considering the Bose-Mesner algebra, one can derive the Q-**eigenmatrix (the second eigenmatrix)** of a bilinear forms scheme.
Given eigenvalues $\theta_0, \ldots, \theta_n$ and intersection numbers a_i, b_i, c_i:

$$Q_{ij} = p_j(\theta_i) = \frac{1}{c_j}\left((\theta_i - a_{j-1})p_{j-1}(\theta_i) - b_{j-2}p_{j-2}(\theta_i)\right), \quad p_0(\theta_i) = 1,\ p_1(\theta_i) = \theta_i.$$

The values θ_i, a_i, b_i, c_i are all expressed in the parameters of the graph:

$$\theta_i = \frac{(q^{n-i}-1)(q^m-q^i)-q^i+1}{q-1}, \qquad c_i = p^i_{1,i-1} = \frac{q^{i-1}(q^i-1)}{q-1},$$

$$b_i = p^i_{1,i+1} = \frac{q^{2i}(q^{m-i}-1)(q^{n-i}-1)}{q-1}, \qquad a_i = p^i_{1,i} = b_0 - b_i - c_i.$$

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Let $\mathcal{A} = (X; R_0, \ldots, R_n)$ be an association scheme, and let $\Delta \subseteq X$. We define the **distribution vector a** of Δ with entries $a_i = \dfrac{|(\Delta \times \Delta) \cap R_i|}{|\Delta|}$, $i = 0, \ldots, n$.

(Delsarte, 1973) For the Q-eigenmatrix of $\mathcal{A}$, we have $\mathbf{a}Q \geq \mathbf{0}$ for any Δ.

This gives rise to the **Delsarte's LP bound** on the size of the code with minimum distance d:

$$\begin{array}{lll} \text{maximize} & \sum_{i=0}^{n} a_i (= |\Delta|) & \\ \text{subject to} & \mathbf{a}Q \geq \mathbf{0}, & \\ & \mathbf{a} \geq \mathbf{0}, & \\ & a_0 = 1, & \\ & a_i = 0, & 0 < i < d. \end{array}$$

Let $\mathcal{A} = (X; R_0, \ldots, R_n)$ be an association scheme, and let $\Delta \subseteq X$. We define the **distribution vector a** of Δ with entries $a_i = \dfrac{|(\Delta \times \Delta) \cap R_i|}{|\Delta|}$, $i = 0, \ldots, n$.

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\end{array}
$$

Delsarte's LP bound, 1973:

$$\begin{array}{lll} \text{maximize} & \sum_{i=0}^{n} a_i (= |\Delta|) & \\ \text{subject to} & \mathbf{a}Q \geq \mathbf{0}, & \\ & \mathbf{a} \geq \mathbf{0}, & \\ & a_0 = 1, & \\ & a_i = 0, & 0 < i < d. \end{array}$$

When an association scheme is defined, one can use *Delsarte's LP* to upper bound the size of the code with given minimum distance.

$\Rightarrow$ We can use Delsarte's LP bound if the graph is distance-regular (*e.g.* bilinear forms schemes).

Is sum-rank-metric graph distance-regular?

Is sum-rank-metric graph distance-regular?

(Abiad, K, Ravagnani, 2024) A sum-rank-graph on $t \geq 2$ blocks is distance-regular if and only if all of the blocks are of size $1 \times m$ for some positive integer m.

Hence *sum-rank-graph is not distance-regular in general*, unlike Hamming or rank-metric graphs.
↑ The main challenge in applying Delsarte's approach in sum-rank!

But can we still apply Delsarte's LP bound? Distance-regularity is sufficient, but not necessary.

Given two association schemes

$$\mathcal{A}_1 = (X, \{S_0, \dots, S_{D_1}\}) \text{ and } \mathcal{A}_2 = (Y, \{T_0, \dots, T_{D_2}\}),$$

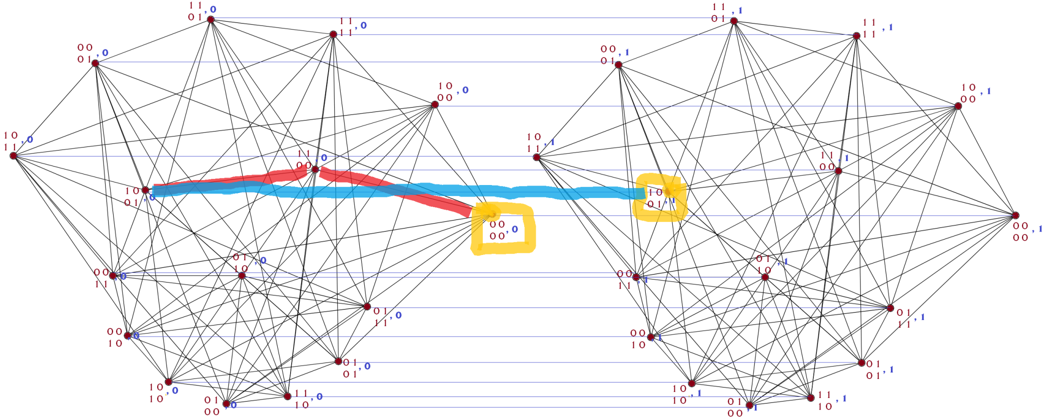
the **direct product** $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the association scheme $(X \times Y, \mathcal{R})$ such that:

- $\mathcal{R} = \{R_{0,0}, R_{0,1}, \dots, R_{0,D_2},\ R_{1,0}, \dots, R_{1,D_2},\ \dots,\ R_{D_1,0}, \dots, R_{D_1,D_2}\}$;
- If $(x_1, x_2) \in S_i$ and $(y_1, y_2) \in T_j$, then $((x_1, y_1), (x_2, y_2)) \in R_{i,j}$.

The Q-eigenmatrix of $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the Kronecker product of the Q-eigenmatrices of $\mathcal{A}_1$ and $\mathcal{A}_2$.

In the **direct product of bilinear forms schemes** for graphs Γ_1 and Γ_2, $((A_1, A_2), (B_1, B_2)) \in R_{i,j}$ means A_1 and A_2 are at (sum-rank) distance i, while B_1 and B_2 are at distance j.

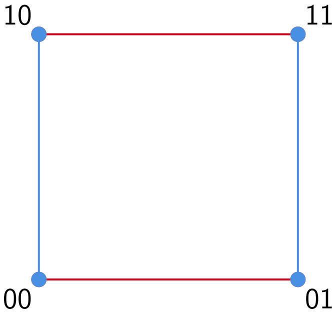
For example, in $\Gamma(2, 2, \mathbb{F}_2) \times \Gamma(1, 1, \mathbb{F}_2)$, the tuples $\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right)$ and $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right)$ are in the relation $R_{2,1}$.



Example: The Hamming cube

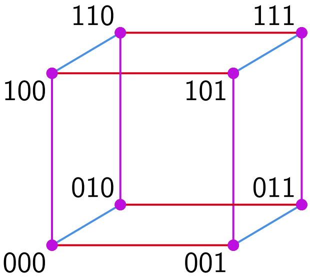
$$\begin{pmatrix} & 0 & 1 \\ 0 & R_0 & R_1 \\ 1 & R_1 & R_0 \end{pmatrix}$$

Example: The Hamming cube



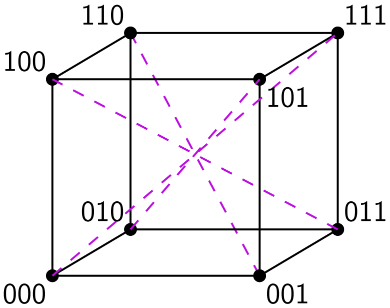
$$\begin{pmatrix} & 00 & 01 & 10 & 11 \\ 00 & R_{0,0} & R_{0,1} & R_{1,0} & R_{1,1} \\ 01 & R_{0,1} & R_{0,0} & R_{1,1} & R_{1,0} \\ 10 & R_{1,0} & R_{1,1} & R_{0,0} & R_{0,1} \\ 11 & R_{1,1} & R_{1,0} & R_{0,1} & R_{0,0} \end{pmatrix}$$

Example: The Hamming cube

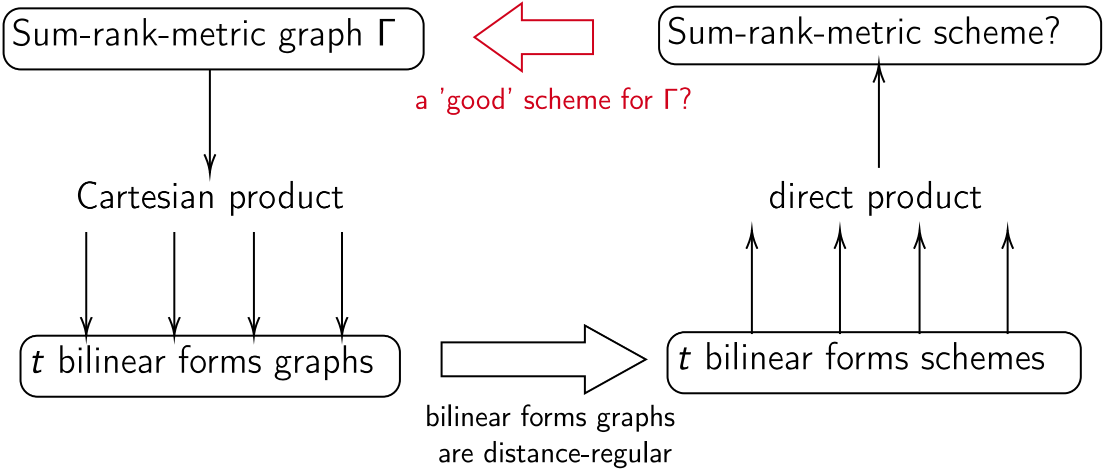


	000	001	010	011	100	101	110	111
000	$R_{0,0,0}$	$R_{0,0,1}$	$R_{0,1,0}$	$R_{0,1,1}$	$R_{1,0,0}$	$R_{1,0,1}$	$R_{1,1,0}$	$R_{1,1,1}$
001	$R_{0,0,1}$	$R_{0,0,0}$	$R_{0,1,1}$	$R_{0,1,0}$	$R_{1,0,1}$	$R_{1,0,0}$	$R_{1,1,1}$	$R_{1,1,0}$
010	$R_{0,1,0}$	$R_{0,1,1}$	$R_{0,0,0}$	$R_{0,0,1}$	$R_{1,1,0}$	$R_{1,1,1}$	$R_{1,0,0}$	$R_{1,0,1}$
011	$R_{0,1,1}$	$R_{0,1,0}$	$R_{0,0,1}$	$R_{0,0,0}$	$R_{1,1,1}$	$R_{1,1,0}$	$R_{1,0,1}$	$R_{1,0,0}$
100	$R_{1,0,0}$	$R_{1,0,1}$	$R_{1,1,0}$	$R_{1,1,1}$	$R_{0,0,0}$	$R_{0,0,1}$	$R_{0,1,0}$	$R_{0,1,1}$
101	$R_{1,0,1}$	$R_{1,0,0}$	$R_{1,1,1}$	$R_{1,1,0}$	$R_{0,0,1}$	$R_{0,0,0}$	$R_{0,1,1}$	$R_{0,1,0}$
110	$R_{1,1,0}$	$R_{1,1,1}$	$R_{1,0,0}$	$R_{1,0,1}$	$R_{0,1,0}$	$R_{0,1,1}$	$R_{0,0,0}$	$R_{0,0,1}$
111	$R_{1,1,1}$	$R_{1,1,0}$	$R_{1,0,1}$	$R_{1,0,0}$	$R_{0,1,1}$	$R_{0,1,0}$	$R_{0,0,1}$	$R_{0,0,0}$

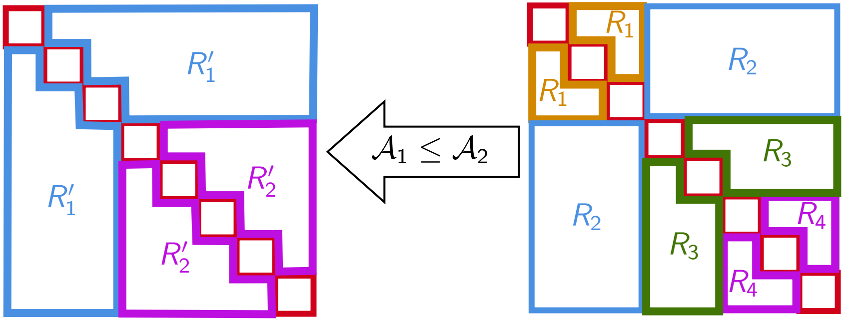
The Hamming scheme based on distances in the graph:



	000	001	010	011	100	101	110	111
000	R_0	R_1	R_1	R_2	R_1	R_2	R_2	R_3
001	R_1	R_0	R_2	R_1	R_2	R_1	R_3	R_2
010	R_1	R_2	R_0	R_1	R_2	R_3	R_1	R_2
011	R_2	R_1	R_1	R_0	R_3	R_2	R_2	R_1
100	R_1	R_2	R_2	R_3	R_0	R_1	R_1	R_2
101	R_2	R_1	R_3	R_2	R_1	R_0	R_2	R_1
110	R_2	R_3	R_1	R_2	R_1	R_2	R_0	R_1
111	R_3	R_2	R_2	R_1	R_2	R_1	R_1	R_0


Sum-rank-metric graph Γ
Sum-rank-metric scheme?
a 'good' scheme for Γ?
Cartesian product
direct product
t bilinear forms graphs
t bilinear forms schemes
bilinear forms graphs
are distance-regular

For two association schemes $\mathcal{A}_1$ and $\mathcal{A}_2$ on the same point set X, we say $\mathcal{A}_1$ is a **fusion** of $\mathcal{A}_2$ and write $\mathcal{A}_1 \leq \mathcal{A}_2$ if every relation of $\mathcal{A}_1$ is a union of some relations of $\mathcal{A}_2$.



Let G be a graph with the edge set $E(G)$.

A **Weisfeiler-Leman closure** $\mathrm{WL}(G)$ is an association scheme on vertices of G such that:

- there are relations, w.l.o.g. $R_1, R_2, \ldots, R_\ell$, such that

$$R_1 \cup R_2 \cup \cdots \cup R_\ell = E(G);$$

- it is the smallest such association scheme (in terms of fusion $\leq$).

$\mathrm{WL}(G)$ is a 'good' association scheme to run Delsarte LP on (smaller size of Q, the distances between vertices are taken into account).

How does the direct product of bilinear forms schemes compare to $\mathrm{WL}(G)$?

(Abiad, Gavrilyuk, K, Ponomarenko, 2025) If the graph G is a sum-rank-metric graph which is a Cartesian product of bilinear forms graphs $G_1, \ldots, G_t$, then $\mathrm{WL}(G)$ is a fusion of the direct product of bilinear forms schemes corresponding to $G_1, \ldots, G_t$.

$\Rightarrow$ We can define an association scheme for a sum-rank-metric graph G, possibly larger than $\mathrm{WL}(G)$, and apply Delsarte's LP bound to it.

Bound comparison: computational results

bold = best performing bound; underlined = RT-bound outperforms coding bounds.
For $|V| \leq 1024$ and $t \leq 7$ Delsarte's LP is never strictly outperformed.

t	q	**n**	**m**	d	$\lvert V\rvert$	Ratio-type	Delsarte LP	iS_d	iH_d	iE_d	S_d	SP_d	PSP_d
2	2	[2, 2]	[2, 2]	3	256	11	**10**	16	19	34	16	13	13
3	2	[2, 2, 1]	[2, 2, 1]	3	512	25	**20**	64	64	151	32	25	25
3	2	[2, 2, 1]	[2, 2, 1]	4	512	10	**6**	16	64	27	8	25	18
3	2	[2, 2, 1]	[2, 2, 2]	3	1024	38	**34**	64	64	151	64	46	46
3	2	[2, 2, 1]	[2, 2, 2]	4	1024	15	**8**	16	64	27	16	46	36
4	2	[2, 1, 1, 1]	[2, 2, 2, 1]	3	512	28	**24**	64	64	151	32	30	30
4	2	[2, 1, 1, 1]	[2, 2, 2, 1]	4	512	11	**6**	16	64	27	8	30	32
4	2	[2, 1, 1, 1]	[2, 2, 2, 2]	3	1024	44	**42**	64	64	151	64	53	53
4	2	[2, 1, 1, 1]	[2, 2, 2, 2]	4	1024	18	**10**	16	64	27	16	53	64
4	2	[2, 2, 1, 1]	[2, 2, 1, 1]	3	1024	46	**40**	256	215	529	64	48	48
4	2	[2, 2, 1, 1]	[2, 2, 1, 1]	4	1024	19	**12**	64	215	119	16	48	36
5	2	[2, 1, 1, 1, 1]	[2, 1, 1, 1, 1]	5	256	5	**2**	16	26	19	4	4	3
5	2	[2, 1, 1, 1, 1]	[3, 1, 1, 1, 1]	5	1024	8	**2**	64	336	240	4	6	3
5	2	[2, 1, 1, 1, 1]	[2, 2, 2, 1, 1]	3	1024	56	**49**	256	215	529	64	56	56
5	2	[2, 1, 1, 1, 1]	[2, 2, 2, 1, 1]	4	1024	22	**13**	64	215	119	16	56	64
6	2	[2, 1, 1, 1, 1, 1]	[2, 1, 1, 1, 1, 1]	4	512	16	**12**	256	512	407	16	34	32
6	2	[2, 1, 1, 1, 1, 1]	[2, 1, 1, 1, 1, 1]	5	512	8	**4**	64	77	99	8	6	5
6	2	[2, 1, 1, 1, 1, 1]	[2, 2, 1, 1, 1, 1]	5	1024	11	**6**	64	77	99	8	9	8
6	2	[2, 1, 1, 1, 1, 1]	[2, 2, 1, 1, 1, 1]	6	1024	7	**2**	16	77	14	4	9	3

(Schrijver, 1979) The Delsarte's LP does not perform worse than Lovász θ_k bound.

Conclusion and future research

? What are the codes that optimize the new bound? Recently there was progress in addressing the optimality of Delsarte's LP for Hamming and rank metrics, but not sum-rank.

? Can the Delsarte's LP approach be applied to other metrics? (In case the respective graph is not distance-regular.)

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Thank you for your attention!

The talk is based on:

Abiad, A., Khramova, A.P., Ravagnani A. Eigenvalue bounds for sum-rank-metric codes. *IEEE Transactions on Information Theory* (2024)
https://doi.org/10.1109/TIT.2023.3339808

Abiad, A., Gavrilyuk, A.L., Khramova, A.P., Ponomarenko I. The linear programming bound for sum-rank-metric codes. *IEEE Transactions on Information Theory* (2025)
https://doi.org/10.1109/TIT.2024.3488902

- 8 papers in graph, coding, group, and scheduling theory
- 28+1 conferences, seminars, workshops
- In the market for a postdoc from Nov 2025 ;)

Appendix: $\mathrm{WL}(G)$ equality

If G is a Cartesian product $G_1 \square \cdots \square G_t$, and $\mathcal{A}$ is the direct product of bilinear forms schemes, then $\mathrm{WL}(G) \leq \mathcal{A}$.
When do we have $\mathrm{WL}(G) = \mathcal{A}$? A sufficient condition:

(Abiad, Garvilyuk, K, Ponomarenko, 2025) Let G_1 and G_2 be graphs with precisely s_1 and s_2 pairwise distinct eigenvalues θ_{1j} and θ_{2k}, respectively, $j \in [s_1]$, $k \in [s_2]$. Then

$$\mathrm{WL}\left(G_1 \square G_2\right) = \mathrm{WL}(G_1) \otimes \mathrm{WL}(G_2)$$

if the set $S := \{\theta_{1j} + \theta_{2k} \mid j \in [s_1], k \in [s_2]\}$ is of cardinality $s_1 s_2$.

Checked computationally, this condition does not often hold for sum-rank-metric graphs. On the other hand, after checking Bannai-Muzychuk criterion for small graphs we found no counterexamples to $\mathrm{WL}(G) = \mathcal{A}$ when all blocks of the sum-rank-metric graph are of different sizes.