Minimum Spectral Radius

Vishal Gupta

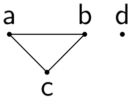
(joint work with Sebastian Cioabă, Dheer Noal Desai, and Celso Marques)
Department of Mathematical Sciences
University of Delaware

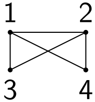
AGT Seminar, University of Waterloo

Introduction

A graph G is a pair of sets (V, E), where V is a non-empty set of elements called *vertices*, and E is a set of unordered pairs of distinct vertices called *edges*.

Example





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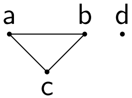
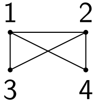




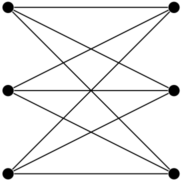
Degree of a vertex $v \in V$ $\deg(v) = |\{u \in V \mid \{u, v\} \in E\}|$.

Left : $deg(a) = deg(b) = deg(c) = 2, deg(d) = 0.$
Right : $deg(1) = deg(2) = 3, deg(3) = deg(4) = 2.$

Introduction

We can capture a graph $G = (V, E)$ on n vertices by a $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $\{i, j\} \in E$, else $a_{ij} = 0$. This matrix is called *adjacency matrix of* G.

Example



$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \{3, 0, 0, 0, 0, -3\}$$

The largest eigenvalue $\rho(G)$ of $A(G)$ is called the *spectral radius* or *index* of G.

Introduction

In 1986, Brualdi and Solheid posed the following problem.

Let $\mathcal{U}_n$ be the set of all $\{0,1\}$ matrices and let $\mathcal{P} \subset \mathbf{U}_n$. Determine

$$\rho_{min} = \min\{\rho(A) \ : \ A \in \mathcal{P}\}, \text{ and}$$

$$\rho_{max} = \max\{\rho(A) \ : \ A \in \mathcal{P}\}.$$

$\rho(A)$ denotes the largest eigenvalue of A.

Introduction

Set of all simple connected graphs on n vertices with fixed-

- #edges e
- #edges e and minimum degree δ
- diameter D
- chromatic number χ
- maximum degree Δ
- independence number α
- dissociation number τ
- matching number
- number of cut vertices
- forbidden subgraph

Part I: fixing order n and size e

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Bounds of eigenvalues of graphs*

Yuan Hong

Department of Mathematics, East China Normal University, Shanghai 200062, China

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Abstract

The eigenvalues of a graph are the eigenvalues of its adjacency matrix. This paper presents an algebraically defined invariant system of a graph. We get some bounds of the eigenvalues of graphs and propose a few open problems.

Problem Statement

$\Delta(G)$: maximum degree in G.

$\delta(G)$: minimum degree in G.

$\mathcal{G}_{n,e}$: the set of all simple connected graphs on n vertices with e edges.

Yuan Hong (1993)

Problem 3: If $G \in \mathcal{G}_{n,e}$ has the minimum spectral radius among all graphs in $\mathcal{G}_{n,e}$, then is it true that $\Delta(G) - \delta(G) \leq 1$?

We call such graphs *spectral minimizers.*

First observation

Courant-Fischer Theorem

If G is a graph on n vertices, then

$$\rho(G) = \max_{u \in \mathbb{R}^n, u \neq 0} \frac{u^T A u}{u^T u}$$

If G has n vertices and e edges, then

$$\rho(G) \geq \frac{\overrightarrow{\mathbf{1}}^T A \overrightarrow{\mathbf{1}}}{\overrightarrow{\mathbf{1}}^T \overrightarrow{\mathbf{1}}} = \frac{2e}{n} = \text{the average degree of } G.$$

Equality happens if and only if $n|2e$ and G is a $\frac{2e}{n}$-regular graph.

First observation

The case $2e/n \in \mathbb{N}$

If $2e/n = k \in \mathbb{N}$, then a k-regular graph G is a spectral minimizer in $\mathcal{G}_{n,e}$; $\Delta(G) - \delta(G) = 0$.

Our work deals with the cases when $n \not| 2e$. We measure the irregularity of a graph by

$$Ir(G) = \Delta(G) - \delta(G).$$

When $Ir(G) = 1$, $\Delta(G) = \lceil \frac{2e}{n} \rceil$ and $\delta(G) = \lfloor \frac{2e}{n} \rfloor$.

Known instances : $e = n - 1$ (trees)

Collatz and Sinogowitz (1957), Lovász and Pelikan (1973)

If G is a tree of order n, then

$$2\cos(\pi/(n+1)) = \rho(P_n) \leq \rho(G) \leq \rho(K_{1,n-1}) = \sqrt{n-1}.$$

The lower bound occurs only when G is the path P_n and the upper bound occurs only when G is the star $K_{1,n-1}$.

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$\implies$ Among all simple connected graphs P_n has the smallest spectral radius.

Figure: Path P_5 (left), Star $K_{1,4}$ (right).

$e = n$ and $e = n + 1$ (bicyclic graphs)

For $e = n$, by our previous observation for cases when $n | 2e$, the cycle graph C_n is the spectral minimizer in $\mathcal{G}_{n,n}$.

Simić (1989)

Among bicyclic graphs, $B(k, n+1-2k, k)$ and $P(k, n+1-2k, k)$, where $k = \lceil \frac{n}{3} \rceil$, are the spectral minimizers.

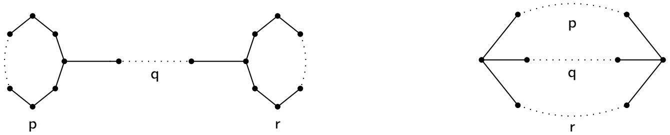


Figure: $B(p,q,r)$ (left) and $P(p,q,r)$ (right).

k-cyclic graphs

When $e = n + 2$ (tricyclic graphs) or when $e = n + k$ (k-cyclic graphs), for k some constant - still open.

Figure: Range of e.

Our results

Dense Graphs

- $e \geq \frac{n(n-2)}{2}$
- $e = \binom{n-1}{2}$
- $\binom{n-1}{2} < e \leq \frac{n(n-2)}{2}$
- $e = \frac{n(n-3)}{2} - 1$

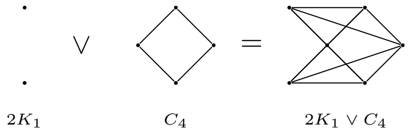
Sporadic Cases

- $e = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)$
- $e = \frac{n^2}{4} - 1$
- $e = \frac{n^2}{3} - 1$

Join of graphs

Join of two graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Their join $G = G_1 \vee G_2 = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{\{x, y\} \ : \ x \in V_1, y \in V_2\}$.



The case $e \geq \frac{n(n-2)}{2}$

The Cocktail Party graph CP_{2m} is the complement of a perfect matching of K_{2m}.

Proposition (Cioaba-Gupta-Marques, 2024)

For $n \in \mathbb{N}$, let $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$ and $e = \binom{n}{2} - p$. Then

$$\rho_{min}(n,e) = \frac{n - 3 + \sqrt{(n+1)^2 - 8p}}{2}.$$

For $p < \frac{n}{2}$, the spectral minimizer is $K_{n-2p} \vee CP_{2p}$ and for $p = \frac{n}{2}$, the spectral minimizer is the Cocktail Party graph CP_n.

The case $e \geq \frac{n(n-2)}{2}$

Proof.

For $e = \binom{n}{2} - 2p$, we observe any graph G has at least $n - 2p$ vertices of degree $n - 1$.
$G = K_{n-2p} \vee H$, where $|V(H)| = 2p$ and $|E(H)| = 2p(p-1)$.

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Proof.

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$G = K_{n-2p} \vee H$, where $|V(H)| = 2p$ and $|E(H)| = 2p(p-1)$.

$$Q = \begin{bmatrix} n-2p-1 & 2p \\ n-2p & 2(p-1) \end{bmatrix}.$$

$$P_Q(x) = x^2 - x(n-3) + 2(p-n+2),$$

$$\rho(Q) = \frac{n-3+\sqrt{(n+1)^2-8p}}{2}.$$

By eigenvalue interlacing it follows that $\rho(G) \geq \frac{n-3+\sqrt{(n+1)^2-8p}}{2}$ and equality happens if and only if H is a regular graph, that is, the Cocktail Party graph on $2p$ vertices. □

The case $\frac{n(n-3)}{2} \leq e \leq \frac{n(n-2)}{2}$

For $e = \binom{n}{2} - (n-1) = \binom{n-1}{2}$, Jack Koolen asked whether the join of the complement of a 2-regular graph on $n-2$ vertices and two isolated vertices is a minimizer graph.

The case $\frac{n(n-3)}{2} \leq e \leq \frac{n(n-2)}{2}$

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Theorem (Cioaba-Gupta-Marques, 2024)

For $e = \binom{n-1}{2}$, a spectral minimizer is of the form $G^3_{n-2} \vee (2K_1)$.

G^r_n denotes a $n-r$ regular graph on n vertices.

When $\frac{n(n-3)}{2} \leq e \leq \frac{n(n-2)}{2}$

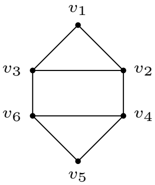
Theorem (Cioaba-Gupta-Marques, 2024)

n	e	*spectral minimizer*
$n \geq 5$ *odd*	$e = \binom{n}{2} - \frac{n+1}{2}$	$G^2_{n-3} \vee (P_2 \cup K_1)$
$n \geq 6$ *even*	$e = \binom{n}{2} - \frac{n+2}{2}$	$G^2_{n-4} \vee P_4$
$n \geq 5$ *is odd*	$e = \binom{n}{2} - (\frac{n+3}{2} + p)$, $1 \leq p \leq \frac{n-3}{2}$	$G^2_{n-2(p+1)+1} \vee G^3_{2(p+1)-1}$
$n \geq 6$ *is even*	$e = \binom{n}{2} - (\frac{n+2}{2} + p)$, $1 \leq p \leq \frac{n-4}{2}$	$G^2_{n-2(p+1)} \vee G^3_{2(p+1)}$

The case $e = \binom{n-1}{2} - 2$

Theorem (Cioaba-Gupta-Marques, 2024)

For $n \geq 6$ and $e = \binom{n-1}{2} - 2$, $G_1 \vee G^3_{n-6}$ is a spectral minimizer, where G_1 is any of the graphs in Figure below.



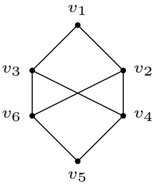


Figure: Minimizers on 6 vertices and 8 edges.

Sporadic cases

When $e = \frac{dn}{2} - 1$, for $d \in \{2, \frac{n}{2}, \frac{2n}{3}, n-3, n-2, n-1\}$, a spectral minimizer graph is a d-regular graph minus an edge. We believe the same is true for any value of d. We note that it is not necessarily true that when $e = \frac{dn}{2} + 1$ for some $2 \leq d \leq n-2$, a minimizer is obtained from a d-regular graph by adding an edge.

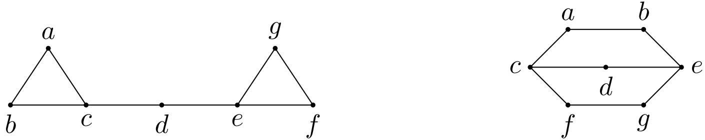


Figure: Bicyclic spectral minimizers on 7 vertices.

Open problems

Q1 Is it true that for $e = \frac{dn}{2} - 1$, a spectral minimizer is always a d-regular graph minus an edge?

Q2 Spectral minimizers for tricyclic, or k-cyclic graphs?

Q3 Is it true for the remaining cases of e that a spectral minimizer G satisfies $\Delta(G) - \delta(G) = 1$?

Kristina Kostić, Zorica Dražic, Aleksandar Savić, Zoran Stanić (2023)

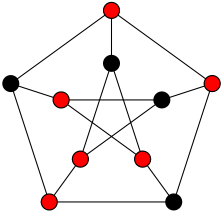
Verified for some special e values and $n \leq 100$ that $\Delta(\hat{G}) - \delta(\hat{G}) \leq 1$.

Part II: fixing the order n and the dissociation number τ

Definition

A set of vertices in a graph G that induces a subgraph of maximum degree at most 1 is called a *dissociation set*. The maximum order of a dissociation set is called the *dissociation number* $\tau(G)$ of G.

Example: $\tau(K_n) = 2$, $\tau(P_n) = \lceil \frac{2n}{3} \rceil$, or $\tau(\text{Petersen graph}) = 6$.



($\tau(G) \geq \max\{$independence number, twice the size of maximum induced matching of $G\}$).

Probabilistic lower bound on τ

Proposition (Desai-Gupta)

Let $G = (V, E)$ be a connected graph. Its dissociation number

$$\tau(G) \geq 2 \left\lceil \sum_{e=\{u,v\} \in E} \frac{1}{(d_u + d_v)\Delta(G) - 1} \right\rceil .$$

Probabilistic lower bound on τ

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Proof: Pick a total ordering $<$ of E uniformly at random. Define

$$I = \{e \in E \ : \ e < e' \text{ for all } e' \in D_2(e)\},$$

$D_2(e) =$ set of all edges at distance at most 2 from e.
Let X_e be the indicator random variable for $e \in I$ and
$X = \sum_{e \in E} X_e = |I|$.

Probabilistic lower bound on τ

$$\mathbf{E}[X] \geq \sum_{e=\{u,v\}\in E} \frac{1}{(d_u + d_v)\Delta(G) - 1}.$$

Therefore, there exists a total ordering for which

$$|I| \geq \left\lceil \sum_{e=\{u,v\}\in E} \frac{1}{(d_u + d_v)\Delta(G) - 1} \right\rceil.$$

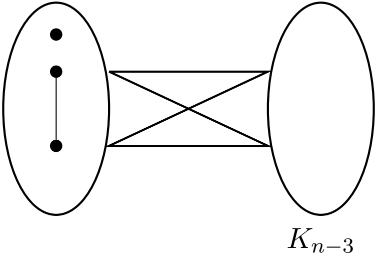
Note that the subgraph induced by the vertices incident to edges in I has maximum degree 1, therefore $\tau(G) \geq 2|I|$.

Example: $\tau(P_n) \geq 2\lceil \frac{n}{7} \rceil, \tau(K_n) \geq 2,\ \tau(C_3) \geq 2, \tau(C_4) \geq 2, \tau(C_8) \geq 4.$

Spectral Maximizer in $\mathcal{G}_{n,\tau}$

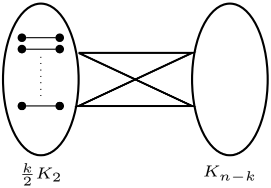
$\mathcal{G}_{n,\tau}$: the set of all simple connected graphs on n vertices with dissociation number τ.

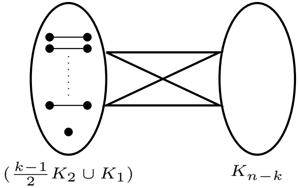
- For $\tau = 2$, the spectral maximizer in $\mathcal{G}_{n,2}$ is K_n.
- For $\tau = 3$, the spectral maximizer is $K_{n-3} \vee (K_2 \cup K_1)$.



Spectral Maximizer in $\mathcal{G}_{n,\tau}$

For $\tau = k$, the spectral maximizer is $K_{n-k} \vee (\frac{k}{2}K_2)$ if k is even and $K_{n-k} \vee (\frac{k-1}{2}K_2 \cup K_1)$ if k is odd.





Spectral Minimizer in $\mathcal{G}_{n,\tau}$

Theorem (Huang-Liu-Zhang, 2024)

Let G be a spectral minimizer in $\mathcal{G}_{n,\tau}$.

1. *If $\tau = 2$, then G is a Cocktail Party graph CP_n when n is even, and odd Cocktail Party graph $L_n = CP_{n-1} \vee K_1$ when n is odd.*
2. *If $\tau = \lfloor \frac{2n}{3} \rfloor, n \neq 0 (\mod 3)$, then G is the cycle C_n.*
3. *If $\tau = \lceil \frac{2n}{3} \rceil$, then G is the path P_n.*
4. *If $\tau = n - 1$, then $G \cong S(r, \lfloor (n-1)/2 \rfloor)$, where $r = 0$ if n is odd and $r = 1$ if n is even.*
5. *If $\tau = n - 2, n \geq 10$, then $G \cong H(n)$.*

Spectral Minimizer in $\mathcal{G}_{n,\tau}$

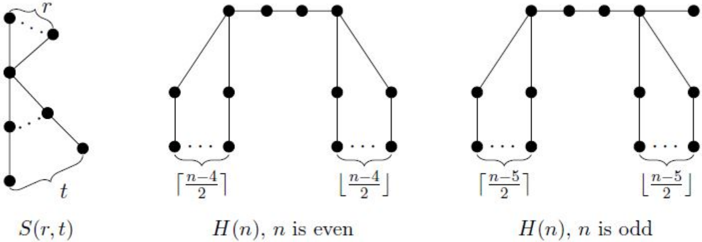


Figure 1: The graphs $S(r,t)$ and $H(n)$

$S(r,t)$: attaching t edges to the center of the star graph of order $r+1$.
$H(n)$: attaching edges to the end vertices of P_4.

Spectral Minimizer in $\mathcal{G}_{n,\tau}$

Theorem (Huang-Liu-Zhang, 2024)

If $\tau > \lceil \frac{2n}{3} \rceil$, then a spectral minimizer is a tree.

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Theorem (Desai-Gupta)

If G is an edge maximizer for CP_k (or $L_k = CP_{k-1} \vee K_1$, if k is odd), then $\tau(\overline{G}) = k - 1$.

Definition

Turán number $ex(n, F)$ of a graph F is the maximum number of edges in a graph on n vertices that does not contain F as a subgraph.

$\text{EX}(n, F)$ denotes the set of all edge maximizers that does not contain F as a subgraph.

Turán number for Odd Cocktail Party graphs

Let $k \geq 2$ be an integer and let $G_{n,2k}$ be the set of graphs we get after adding maximum matching in each part of the complete multipartite graph $K_{n_1,\ldots,n_k}$, where $\sum_{i=1}^{k} n_i = n$ and for all distinct $i, j \in [k]$, $|n_i - n_j| \leq 2$ (with $|n_i - n_j| = 2$ only when n_i, n_j are both even).

Theorem (Desai-Gupta)

For any order n, an edge maximizer for L_{2k+1} is a graph is the set $G_{n,2k}$.

Edge minimizer in $\mathcal{G}_{n,\tau}$ for even τ

For $4 \leq \tau \leq \lfloor \frac{2n}{3} \rfloor$ and $\tau = 2k$, let $M_{n,2k}$ be a set of graphs obtained by adding $k-1$ edges between k non-adjacent pairs- taking one pair from each of the k parts of the complement of graphs in $G_{n,2k}$ in which $|n_i - n_j| \leq 1$.

Theorem (Desai-Gupta)

Any graph in $M_{n,2k}$ is an edge minimizer in $\mathcal{G}_{n,2k}$ of size $e = \binom{n}{2} - ex(n, L_{2k+1}) + k - 1.$

Spectral minimizer in $\mathcal{G}_{n,4}$

Theorem (Desai-Gupta)

The graphs we get by adding an edge between the two parts of the complement of graphs in $G_{n,4}$ are the only edge minimizers in $\mathcal{G}_{n,4}$.

Theorem (Desai-Gupta)

$M_{n,4}$ is the unique spectral minimizer in $\mathcal{G}_{n,4}$.

$n \pmod 4$	*spectral minimizer*
0	$CP_{\frac{n}{2}} - CP_{\frac{n}{2}}$
1	$CP_{\frac{n-1}{2}} - L_{\frac{n+1}{2}}$
2	$L_{\frac{n}{2}} - L_{\frac{n}{2}}$
3	$CP_{\frac{n+1}{2}} - L_{\frac{n-1}{2}}$

Edge minimizer in $\mathcal{G}_{n,\tau}$ for odd τ

Let $K_m(r_1, r_2, \ldots, r_m)$ denote the complete m-partite graph with parts of sizes r_i for $1 \leq i \leq m$.

Theorem (Erdős-Simonovits, 1971)

Let $r_1, 1, 2$ or 3, $r_1 \leq r_2, \leq \ldots, r_{d+1}$ be given integers. If n is large enough, then each extremal graph G_n in $(n, K_{d+1}(r_1, \ldots, r_{d+1}))$ is a graph product:

$$G_n = \bigvee_{i=1}^{d} N_i$$

where

1. $n_i = v(N_i) = \frac{n}{d} + o(n)$;
2. N_1 *is an extremal graph for* $K_2(r_1, r_2)$;
3. $N_2, \ldots, N_d$ *are extremal graphs for* $K_2(1, r_2)$.

Edge minimizers in $\mathcal{G}_{n,\tau}$ for odd τ

Theorem (Desai-Gupta)

Let $r_2 \geq r_1^2 - r_1 + 2$, $r_1 \leq r_2, \leq \ldots, r_{d+1}$ be given integers. If n is large enough, then each edge extremal graph G_n in EX$(n, K_{d+1}(r_1, \ldots, r_{d+1}))$ is a graph product:

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Questions

Q1 Is it true in general that a spectral minimizer in $\mathcal{G}_{n,\tau}$ is also an edge minimizer in $\mathcal{G}_{n,\tau}$?

- For $\tau = 2$, the cocktail party graph CP_n or the odd cocktail party graph L_n is an edge minimizer in $\mathcal{G}_{n,2}$ depending on whether n is even or odd, respectively.
- For $\tau > \lceil \frac{2n}{3} \rceil$, an edge minimizer in $\mathcal{G}_{n,\tau}$ is a tree.

Q2 What are the spectral minimizers in $\mathcal{G}_{n,\tau}$ for the remaining cases of τ?

Thank you for your attention!