Diagonal coefficients of Kirchhoff polynomials of $2k$-regular graphs and the proof of the c_2 completion conjecture

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Feynman period

Define the first Symanzik or dual Kirchhoff polynomial to be

$$\Psi_G = \sum_{\substack{T \\ \text{sp.tr.}}} \prod_{e \notin T} a_e$$

and the period to be

$$P_G = \int_{a_e \geq 0} \frac{\prod_{e=1}^{|E|-1} da_e}{\Psi_G^2|_{a_{|E|}=1}}$$

It converges if $G = K - v$, K 4-regular, internally 6-edge-connected.

Why should I care?

physics . . . number theory . . . combinatorics?

contribution to β-function of theory of Feynman diagram G

P_σ — multiple zeta values

— special values of L-function of modular form

— ???

ψ_G game of spanning trees and spanning forests

Period symmetries

- duality (fourier transform)

G^* planar dual of G $\qquad P_{G^*} = P_G$

- completion

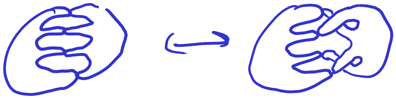
$P_{K-v} = P_{K-w}$

K complete of G

G decomplete of K

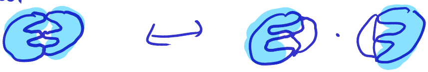
- twist

completed



- product

completed

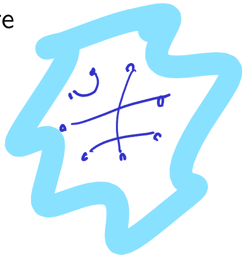


Martin recurrence

Take K $2k$-regular. A key recurrence, for $v \in V(K)$

$$F(K) = \sum_{\substack{\tau \text{ matching} \\ \text{of nbhd of } v}} F(K_\tau) \tag{1}$$

where



Martin invariance

The Martin recurrence came from Erik Panzer's Martin invariant M:

- M satisfies (1)
- $M($  $) = 0$
- $M($  $) = 1$
- (parenthetically) $M($  $) = \frac{1}{k!}$

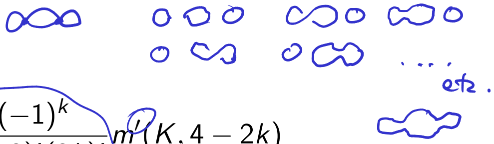
This satisfies the period symmetries (recursive proofs).

From Martin polynomial

The Martin invariant can be obtained from the Martin polynomial. Define

$$m(K, x) = \sum_P (x-2)^{|P|-1}$$

summing over result of resolving every vertex where $|P|$ is number of components.



Then for K $2k$-regular

$$M(K) = \frac{4(-1)^k}{(k-2)!(2k)!} m'(K, \underline{4-2k})$$

Diagonal coefficients

Key is this diagonal coefficient:

$$[a_1^{(k-1)r} \cdots a_{k(n-2)}^{(k-1)r}]\Psi_{K-v}^{kr}$$

It satisfies (1) so only differs from M by a normalization.

$$(r!)^{k(n-2)}[a_1^{(k-1)r} \cdots a_{k(n-2)}^{(k-1)r}]\Psi_{K-v}^{kr} = (kr)!M(K^{[r]})$$

where $K^{[r]}$ means r-duplicate each edge.

The c_2 invariant

For a prime p, K 4-regular, $G = K - v$ define

$$c_2^p(G) = \frac{[\Psi_G]_p}{p^2} \mod p$$

where $[\cdot]_p$ is the point count.
over $\mathbb{F}_p$

Why should you care?

↳ early computation of P_G all MZV
if always true should be a good reason
motivic ⇝ $[\Psi_G]_p$ poly in p so c_2 quad. coeff
FALSE but c_2 remains a measure of how false hence how exotic P_G.

c_2 and diagonal coefficients

partitns of $2p-2$ copies of edges of G into $p-1$ spanning trees

$$-3p^2 c_2^p(G) = [a_1^{p-1} \cdots a_{2(n-2)}^{p-1}]\Psi_G^{2(p-1)} \mod p^3$$

For $p = 3$ use a different more complicated polynomial that Simone Hu and I have worked with a lot.

$\Psi^{12,32} \cdot \Psi_2^{1,3}$

So

$$c_2^p(K - v) = \frac{M(K^{[p-1]})}{3p} \mod p$$

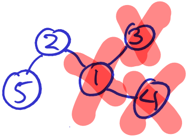
(use the more complicated argument for $p = 3$)

And so?

c_2 was conjectured to have all period symmetries, but completion was resisting proof. This finally proves it (and twist)

The graph permanent invariant also can be expressed as a diagonal coefficient and so treated similarly.

Recall classic Prüfer encoding

Eg: 

$a_1 = 3$ $b_1 = 1$

$a_2 = 4$ $b_2 = 1$

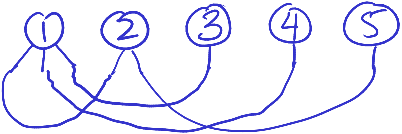
$a_3 = 1$ $b_3 = 2$

$(1, 1, 2)$

start $i = 1$

repeat
- pick smallest leaf a_i, let b_i be nbhr
- remove a_i, increment i

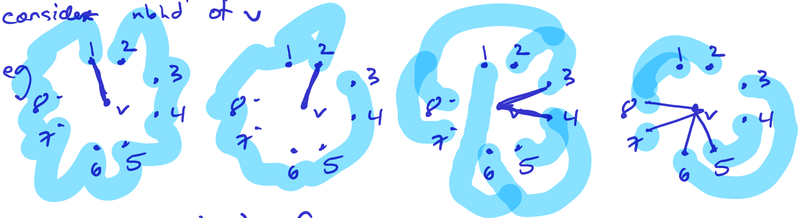
return $b = (b_1, \ldots, b_{n-2})$



turns out every $b \in \{1, \ldots, n\}^{n-2}$ exactly once

Trees partition neighbourhood into systems of distinct representatives

Eg: G 8-regular v vertex with no loops
Say have a partition of edges of G into 4 spanning trees
consider nbhd of v



so get partitions of nbhd of v

$P_1 = \{1,2,3,4,5,6,7,8\} = P_2$ $\quad P_3 = \{1,4,5,6\}, \{2,3,7,8\}$

$f(1) = 1 \quad f(3) = 3 \quad f(5) = 4 \quad f(7) = 4$

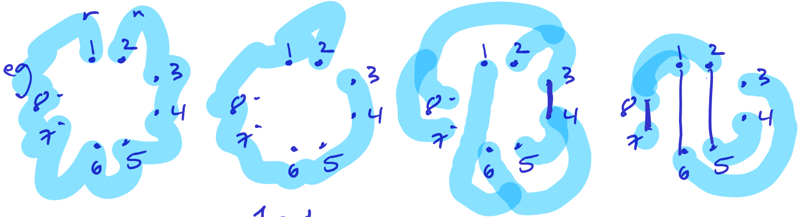
$f(2) = 2 \quad f(4) = 3 \quad f(6) = 4 \quad f(8) = 4$

$P_4 = \{1,2,8\}, \{3,6\}, \{4,5\}, \{7\}$

Use Prüfer on the matchings

Eg (continued):

the match to contribute to recurrence (1) turing the partition into trees



1 part 1 part

do nothing

r = # with 1 part = 2

$y_3 = ()$ $y_4 = (1,2)$

$X_3 = \{3,4\}$ $X_4 = \{5,6,7,8\}$

This works in general and proves the Martin recurrence for that diagonal coefficient. define f for $i > r$ $f(a) = i$ $\forall a \in X_i$

for $i \le r$ $f(b) = i$ for b ith entry $y_{r+1} y_{r+2} \cdots$