On the diameter and zero forcing number of some graph classes in the Johnson, Grassmann and Hamming association scheme

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Joint work with Aida Abiad and Robin Simoens

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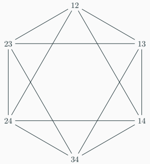
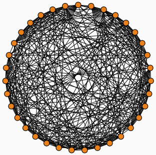
Background and motivation

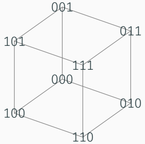
	Johnson	Grassmann	Hamming
Notation	$J(n,k)$	$J_q(n,k)$	$H(n,q)$
Vertices	$\binom{[n]}{k}$	k-dim. subspaces of $\mathbb{F}_q^n$	$\{0,1,\ldots,q-1\}^n$
Edges	$\lvert u \cap v\rvert = k-1$	$\dim u \cap v = k-1$	$q-1$ entries same



$J(4,2)$



$J_2(4,2)$



$H(3,2)$

Generalized Johnson and Grassmann graphs

Let $S \subseteq \{0, 1, \ldots, k-1\}$

	Johnson	Grassmann	Hamming
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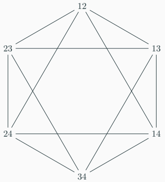
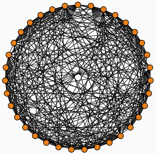
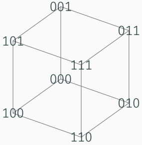


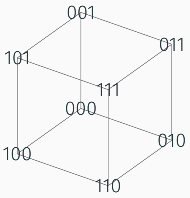




Why (generalized) Johnson, Grassmann and Hamming graphs?

Structure: distance-regular $\rightarrow$ very symmetric

Applications: designs, codes, association schemes, ...



Why (generalized) Johnson, Grassmann and Hamming graphs?

(Chen, Lih 1987) Hamiltonicity generalized Johnson graphs

(Van Dam, Haemers, Koolen, Spence 2006) Johnson and Grassmann graphs not determined by their spectrum

(Meagher, Bailey 2012) Metric dimension of Grassmann graphs

(Alspach 2013) Johnson graphs Hamiltonian connected

(Balogh, Cherkashin, Kiselev 2019) Coloring of generalized Kneser graphs

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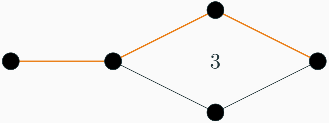
Diameter, zero forcing?

Graph diameter

Largest distance between two vertices

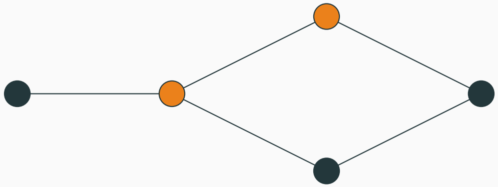


Polynomial-time computable, but

- our graphs are large;
- finding a closed expression is hard

Zero forcing on graphs

Graph $G = (V, E)$ with set $B \subseteq V$ of orange vertices



Force: unique uncolored neighbor of a orange vertex is colored orange

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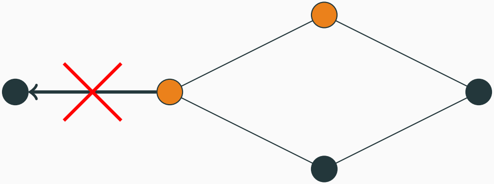

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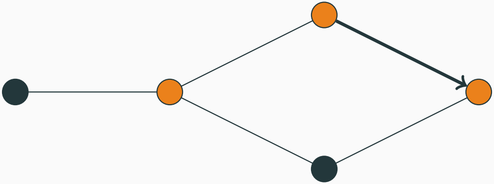

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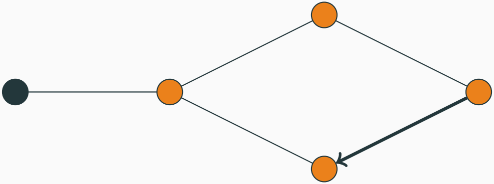

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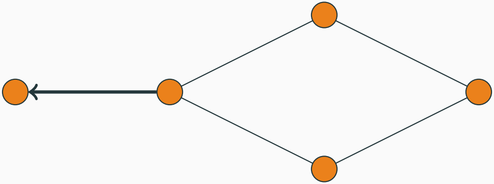

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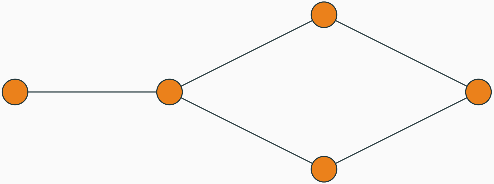


Zero forcing number $Z(G)$: minimum $|B|$ such that all of V is forced

(Yang 2013) In general, this is NP-hard

History and applications

(Haynes, Hedetniemi, Hedetniemi, Henning 2002) Power domination (placing Phasor Measurement Units in electrical networks)

(Burgarth, Giovannetti 2007) Zero forcing for quantum system control

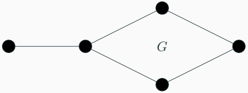
(AIM workshop 2008) Zero forcing as an upper bound for minimum rank

$\updownarrow$?

(Alon 2008) Zero forcing on Cayley graphs, relation minimum rank

Relation of Z to maximum nullity

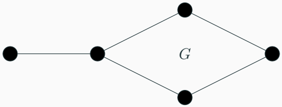
$\mathcal{S}^{\mathbb{F}}(G)$: symmetric matrices M over $\mathbb{F}$ with $m_{ij} = 0$ whenever vertices $i \neq j$ are not adjacent.



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 3 & 0 & 0 & 0 \\ 3 & 6 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0.7 & 1.1 \\ 0 & 1 & 0 & 1.1 & -1.4 \end{pmatrix}$$

Maximum nullity $M^{\mathbb{F}}(G)$: largest multiplicity of eigenvalue zero for any $M \in \mathcal{S}^{\mathbb{F}}(G)$

Relation of Z to maximum nullity



$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & a_{25} \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & a_{52} & 0 & a_{54} & a_{55} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

$$\Rightarrow \qquad \begin{aligned} a_{11}x_1 + a_{21}x_2 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{25}x_5 &= 0 \\ a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= 0 \\ a_{43}x_3 + a_{44}x_4 + a_{45}x_5 &= 0 \\ a_{52}x_2 + a_{54}x_4 + a_{55}x_5 &= 0 \end{aligned}$$

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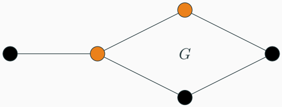

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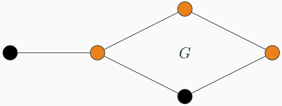

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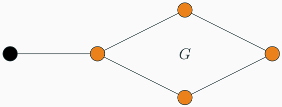
$$x_4 = 0$$

Relation of Z to maximum nullity



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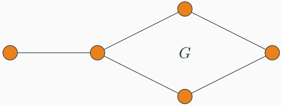
$$x_5 = 0$$

Relation of Z to maximum nullity



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$$x_1 = 0$$

Relation of Z to maximum nullity

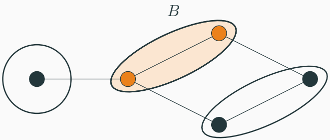
Lemma (AIM workshop 2008)
For any graph G and field $\mathbb{F}$,

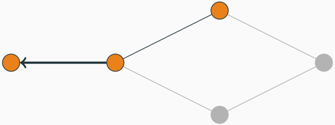
$$M^{\mathbb{F}}(G) \leq Z(G).$$

Variations of zero forcing

Connected zero forcing (Z_c): $G[B]$ is connected

Skew zero forcing (Z^+):



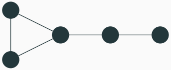


$$Z^+(G) \leq Z(G) \leq Z_c(G)$$

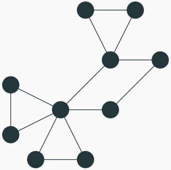
Some known cases

(AIM 2008), (Benson, Ferrero, Flagg, Hogben, Furst, Vasilevska, Wissman 2018) Paths, cycles, complete graphs and their products

(Brimkov, Hicks 2017)

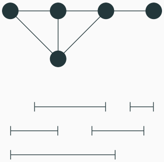


Unicyclic

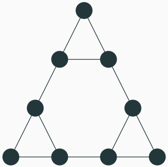


Cactus

(Brešar, Gologranc, Kos 2016)



Interval



Sierpinski

Our contribution

- Diameter: generalized Grassmann graphs
- Zero forcing: Hamming graphs and generalized Johnson, Grassmann graphs

Zero forcing in generalized Johnson graphs

Generalized Johnson graphs

Let $S \subseteq \{0, 1, \ldots, k-1\}$

	Johnson	Grassmann	Hamming
Notation	$J_S(n,k)$	$J_{q,S}(n,k)$	$H(n,q)$
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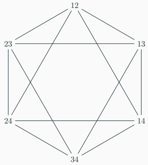
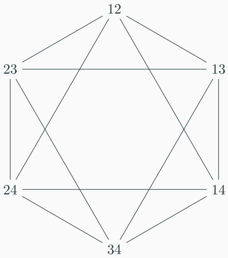


$$J_S(n,k) \simeq J_{\{s+n-2k \mid s \in S\}}(n, n-k)$$

$\rightarrow$ assume $n \geq 2k$

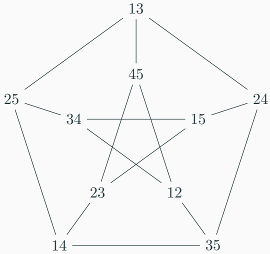
Important subfamilies

Johnson graphs



$$S = \{k - 1\}$$

Kneser graphs



$$S = \{0\}$$

Known results

(Fallat, Meagher, Soltani, Yang 2016)

$$Z^+(J(n,2)) = Z(J(n,2)) = \binom{n}{2} - n + 2$$

(Brešar, Gologranc, Kos 2016)

$$Z(K(n,k)) = \binom{n}{k} - \binom{2k}{k}$$

if $n \geq 3k+1$; upper bound for $n \leq 3k$

Extensions of Johnson and Kneser graphs

Natural generalizations:

Johnson $\quad S = \{k-1\} \quad \rightarrow \quad S = \{s, s+1, \ldots, k-1\}$

Kneser $\quad S = \{0\} \quad \rightarrow \quad S = \{0, 1, \ldots, s\}$

We study the more general cases $\min(S) = s$ and $\max(S) = s$.

The case $\min(S) = s$

Theorem

Let $S \subseteq \{0, 1, \dots, k-1\}$ with $s := \min(S)$ and $n \geq 2k - s$. Then

$$Z(J_S(n,k)) \leq Z_c(J_S(n,k)) \leq \binom{n}{k} - \binom{n-2(k-s)}{s}.$$

If $S = \{s, s+1, \dots, k-1\}$, equality holds throughout.

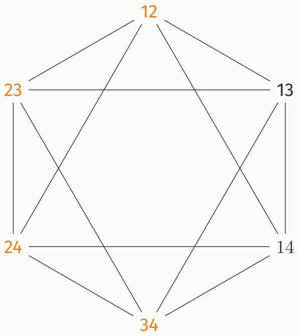
Proof sketch for Johnson graphs

- Upper bounds are ‘easy’: find a construction
- Lower bounds are hard: maximum nullity, etc.

  We use Grundy domination

Upper bound

$$B = V \backslash \{v \in V \mid 1 \in v,\ 2 \notin v\} \quad \rightarrow \quad Z(J(n,k)) \leq |B| = \binom{n}{k} - \binom{n-2}{k-1}$$



Force v with $(v \backslash 1) \cup 2$: $24 \rightarrow 14,\ 23 \rightarrow 13$

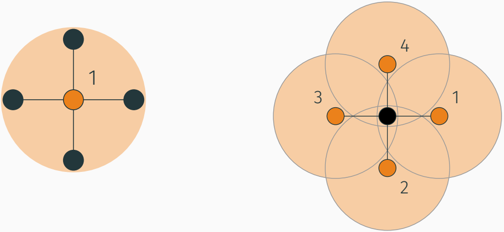
Upper bound

Johnson $V \setminus \{A \in V \mid \underline{1} \in A,\ \underline{2} \notin A\}$

$\downarrow$

gen. Johnson $V \setminus \{A \in V \mid \underline{[k-s]} \subset A,\ \underline{k-s+1, \dots, 2(k-s)} \notin A\}$

Grundy domination



Grundy domination number (γ_{gr}): longest sequence of vertices such that every element dominates something new

$\rightarrow$ How bad can a greedy algorithm be for the dominating set problem?

Grundy domination

Dual:

Z-Grundy domination number (γ_{gr}^{Z}): longest sequence of vertices such that every element dominates something new *other than itself*

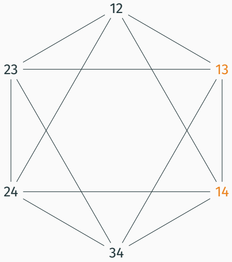
Theorem (Brešar et al. 2017)

$$Z(G) \geq |V| - \gamma_{gr}(G), \quad Z(G) = |V| - \gamma_{gr}^{Z}(G)$$

Lower bound using Grundy domination

If a vertex is dominated for the first time, it is *footprinted*

Pair element u_i in Grundy dominating sequence with v_i footprinted by it



{1,3} — {1,2}, {1,4} — {2,4}

↓

{1,3} — {3,4}, {1,4} — {1,3}

Sequence $\{(u_i, v_i)\} \quad \rightarrow$ sets $A_i = u_i,\ B_i = [n] \backslash v_i$

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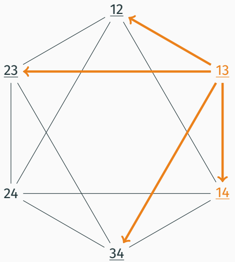

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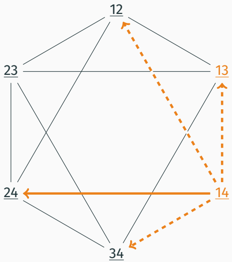

$\{1,3\} - \{1,2\},\ \{1,4\} - \{2,4\}$

$\downarrow$

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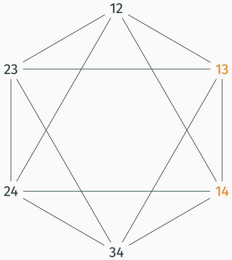


{1,3} — {1,2}, {1,4} — {2,4}

↓

{1,3} — {3,4}, {1,4} — {1,3}

- $|A_i \cap B_i| \leq 1$
- $|A_i \cap B_j| \geq 2$ for $j > i$

Bollobás' theorem: $Z(J(n,k)) \geq |S| = \binom{n}{k} - \binom{n-2}{k-1}$

The case $\max(S) = s$

Theorem
Let $S \subseteq \{0, 1, \dots, k-3\}$ and $n \geq \max(3k-2s, 2k+1)$, where $s := \max(S)$. Then

$$Z(J_S(n,k)) \leq Z_c(J_S(n,k)) \leq \binom{n}{k} - \binom{2k-2s}{k-s}.$$

If $S = \{0, 1, \dots, s\}$, equality holds throughout.

Note: independent of n

The case $\max(S) = s$

Theorem
Let $S \subseteq \{0, 1, \dots, k-3\}$ and $n \geq \max(3k - 2s, 2k + 1)$, where $s := \max(S)$. Then

$$Z(J_S(n,k)) \leq Z_c(J_S(n,k)) \leq \binom{n}{k} - \binom{2k-2s}{k-s}.$$

If $S = \{0, 1, \dots, s\}$, equality holds throughout.

What about $(n, k, s) = (9, 4, 1)$?

Conjecture

Conjecture
Let $S \subseteq \{0, 1, \dots, k-3\}$ and $n \geq 2k+1$, where $s := \max(S)$. Then

$$Z(J_S(n,k)) \leq Z_c(J_S(n,k)) \leq \binom{n}{k} - \binom{2k-2s}{k-s}.$$

Computational evidence suggests this is true

Construction found for $s = 1$

Generalized Grassmann graphs

Generalized Grassmann graphs

Let $S \subseteq \{0, 1, \dots, k-1\}$

	Johnson	Grassmann	Hamming
Notation	$J_S(n, k)$	$J_{q,S}(n, k)$	$H(n, q)$
Vertices	$\binom{[n]}{k}$	k-dim. subspaces of $\mathbb{F}_q^n$	$\{0, 1, \dots, q-1\}^n$
Edges	$\lvert u \cap v \rvert \in S$	$\dim u \cap v \in S$	$q-1$ entries same

$J_{q,S}(n, k) \simeq J_{q,\{s+n-2k \mid s \in S\}}(n, n-k) \quad \to \quad$ assume $n \geq 2k$

Diameter and girth

(Agong, Amarra, Caughman, Herman, Terada 2018) Diameter and girth of generalized Johnson graphs

(Caughman, Herman, Terada 2023) Distance function and odd girth of generalized Johnson graphs

Diameter and girth

Nice fact: #trivially intersecting k-subspaces of $\mathbb{F}_q^n$ $>>$ #disjoint k-subsets of $[n]$.

Lemma (Bose, Burton 1966)

Let $n \geq k + m$. Given at most $q^{n-k-m+1}$ k-spaces in $\mathbb{F}_q^n$, we can always find an m-space that intersects them trivially.

Diameter and girth

Theorem

Let $n \geq 2k$ and $s = \min S$. Then

$$\operatorname{diam}\left(J_{q,S}(n,k)\right) = \begin{cases} 2 & \text{if } s = 0 \\ \left\lceil \frac{k}{k-s} \right\rceil & \text{if } s \neq 0. \end{cases}$$

Theorem

Every generalized Grassmann graph with $S \neq \emptyset$ has girth 3.

Zero forcing number

Theorem

Let $S \subseteq \{0, 1, \dots, k-3\}$ with $s := \max(S)$, and $n \geq \max(3k - 2s + 1, 2k + 1)$. Then

$$Z(J_{q,S}(n,k)) \leq Z_c(J_{q,S}(n,k)) = \begin{bmatrix} n \\ k \end{bmatrix}_q - \binom{2k-2s}{k-s}.$$

If $S = \{0, 1, \dots, s\}$, equality holds throughout.

Proof similar to generalized Johnson, with set elements replaced by basis vectors

The case $s := \min(S)$?

What about classic Grassmann graphs?

Sets:	$u \cap v = x$	$\Rightarrow$	$u \cap [n] \backslash v = k - x$
Subspaces:	$\dim u \cap v = x$	$\not\Rightarrow$	$\dim u \cap v^{\perp} = k - x$

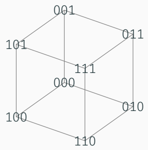
We need new proof techniques

Zero forcing in Hamming graphs

Hamming graphs

Let $S \subseteq \{0, 1, \dots, k-1\}$

	Johnson	Grassmann	Hamming
Notation	$J_S(n,k)$	$J_{q,S}(n,k)$	$H(n,q)$
Vertices	$\binom{[n]}{k}$	k-dim. subspaces of $\mathbb{F}_q^n$	$\{0,1,\dots,q-1\}^n$
Edges	$\lvert u \cap v \rvert \in S$	$\dim u \cap v \in S$	$q-1$ entries same



Main result

(AIM workshop 2008) $Z(H(2,q)) = q^2 - 2q + 2.$

(AIM workshop, Alon 2008) $Z(H(n,2)) = 2^{n-1}$

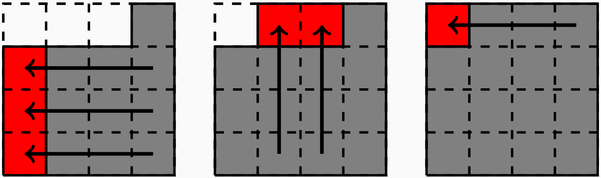
Theorem

For any $n, q \geq 2$,

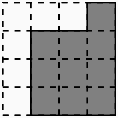
$$Z(H(n,q)) = \frac{1}{2}\left(q^n + (q-2)^n\right).$$

A constructive upper bound

If $n = 2$, the following is a zero forcing set:

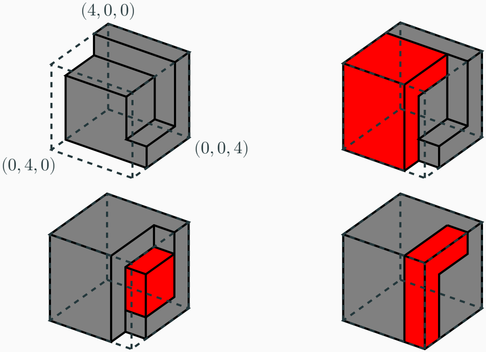


A constructive upper bound



$H(n,q) = H(n-1,q) \square K_q$

$\rightarrow$ take q copies of the zero forcing set of $H(n-1,q)$, remove core vertices from one



Maximum nullity lower bound

Over $\mathbb{F}_2$, consider $B_n = A(H(n,q)) + I$

$$B_n = \underbrace{J \otimes I \otimes \cdots \otimes I}_{n} + \cdots + \underbrace{I \otimes \cdots \otimes I \otimes J}_{n}$$

Odd q

$$B_n = J \otimes I^{\otimes(n-1)} + I \otimes B_{n-1},$$
$$B_1 = J$$

Even q

(Key difference: eigenvectors of J)

Maximum nullity lower bound: odd q

$n = 3$:

J has eigenvectors $\mathbf{1}, x_2, x_3$ with eigenvalues 1,0,0

Let $x = \mathbf{1} \otimes x_2 \otimes \mathbf{1}$

$$\begin{aligned} Bx &= (J \otimes I \otimes I + I \otimes J \otimes I + I \otimes I \otimes J)(\mathbf{1} \otimes x_2 \otimes \mathbf{1}) \\ &= (J\mathbf{1}) \otimes (Ix_2) \otimes (I\mathbf{1}) + (I\mathbf{1}) \otimes (Jx_2) \otimes (I\mathbf{1}) + (I\mathbf{1}) \otimes (Ix_2) \otimes (J\mathbf{1}) \\ &= \mathbf{1} \otimes x_2 \otimes \mathbf{1} + \mathbf{1} \otimes 0 \otimes \mathbf{1} + \mathbf{1} \otimes x_2 \otimes \mathbf{1} \\ &= 0 \cdot (\mathbf{1} \otimes x_2 \otimes \mathbf{1}) \end{aligned}$$

Maximum nullity lower bound: odd q

$n = 3$:

J has eigenvectors $\mathbf{1}, x_2, x_3$ with eigenvalues 1,0,0

Let $x = \mathbf{1} \otimes x_2 \otimes \mathbf{1}$

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$\mathbf{1}$ remains $\mathbf{1}$

Maximum nullity lower bound: odd q

$n = 3$:

J has eigenvectors $\mathbf{1}, x_2, x_3$ with eigenvalues 1,0,0

Let $x = \mathbf{1} \otimes x_2 \otimes \mathbf{1}$

$$
\begin{aligned}
Bx &= (J \otimes I \otimes I + I \otimes J \otimes I + I \otimes I \otimes J)(\mathbf{1} \otimes x_2 \otimes \mathbf{1}) \\
&= (J\mathbf{1}) \otimes (Ix_2) \otimes (I\mathbf{1}) + (I\mathbf{1}) \otimes (Jx_2) \otimes (I\mathbf{1}) + (I\mathbf{1}) \otimes (Ix_2) \otimes (J\mathbf{1}) \\
&= \mathbf{1} \otimes x_2 \otimes \mathbf{1} + \mathbf{1} \otimes 0 \otimes \mathbf{1} + \mathbf{1} \otimes x_2 \otimes \mathbf{1} \\
&= 0 \cdot (\mathbf{1} \otimes x_2 \otimes \mathbf{1})
\end{aligned}
$$

x_2 vanishes

Maximum nullity lower bound: odd q

$n = 3$:

J has eigenvectors $\mathbf{1}, x_2, x_3$ with eigenvalues 1,0,0

Let $x = \mathbf{1} \otimes x_2 \otimes \mathbf{1}$

$$\begin{aligned} Bx &= (J \otimes I \otimes I + I \otimes J \otimes I + I \otimes I \otimes J)(\mathbf{1} \otimes x_2 \otimes \mathbf{1}) \\ &= (J\mathbf{1}) \otimes (Ix_2) \otimes (I\mathbf{1}) + (I\mathbf{1}) \otimes (Jx_2) \otimes (I\mathbf{1}) + (I\mathbf{1}) \otimes (Ix_2) \otimes (J\mathbf{1}) \\ &= \mathbf{1} \otimes x_2 \otimes \mathbf{1} + \mathbf{1} \otimes 0 \otimes \mathbf{1} + \mathbf{1} \otimes x_2 \otimes \mathbf{1} \\ &= 0 \cdot (\mathbf{1} \otimes x_2 \otimes \mathbf{1}) \end{aligned}$$

$\rightarrow$ Even number of $\mathbf{1}$'s

Maximum nullity lower bound: even q

Problem: now $J\mathbf{1} = 0$

$$B_n = J \otimes I^{\otimes(n-1)} + I \otimes B_{n-1}, \quad B_1 = J$$

Induction: if X_{n-1} nullifies B_{n-1}, take

- $\boldsymbol{x}_i \otimes \boldsymbol{v}, \quad \boldsymbol{v} \in X_{n-1}$
- $\mathbf{1} \otimes \boldsymbol{w} + \boldsymbol{e}_1 \otimes (B_{n-1}\boldsymbol{w}), \quad \boldsymbol{w} \in \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_q\}^{\otimes(n-1)}$

Closing remarks

Open problems

- Get rid of lower bound on n for generalized Johnson graphs;

Theorem

Let $S \subseteq \{0, 1, \dots, k-3\}$ and $n \geq \max(3k-2s, 2k+1)$, where $s := \max(S)$. Then

$$Z(J_S(n,k)) \leq Z_c(J_S(n,k)) \leq \binom{n}{k} - \binom{2k-2s}{k-s}.$$

If $S = \{0, 1, \dots, s\}$, equality holds throughout.

Open problems

- Get rid of lower bound on n for generalized Johnson graphs;
- Zero forcing number of Grassmann graphs;
- Zero forcing on distance-regular graphs in general.