Association Schemes, Directed Strongly Regular Graphs and Partial Geometric Designs

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(based on works of O. Olmez, K. Nowak and T. Tranel)

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Objects

We discuss the characteristics of partial geometric designs whose concurrence matrices are circulant. If time permits we examine such partial geometric designs arising from association schemes.

Finite Incidence Structures & t-Designs

Characterization of Partial Geometric Designs (PGDs)

Association Schemes and PGDs

PGDs from Association Schemes

Links of incidence structures and graphs

- partial geometries $\longleftrightarrow$ (directed) strongly regular graphs
- partial geometric designs $\longleftrightarrow$ (directed) strongly regular graphs
- partial geometric designs $\longleftrightarrow$ relation graphs of association schemes

Study the characteristics of partial geometric designs and related incidence structures and combinatorial objects.

Finite incidence structure

A *finite incidence structure* is a triple $(P, \mathcal{B}, \mathcal{I})$ consisting of

- a finite set P of *points*,
- a finite set $\mathcal{B}$ of *blocks*, and
- an *incidence* relation $\mathcal{I} \subseteq P \times \mathcal{B}$.

t-design

In particular, when

$$\mathcal{B} \subseteq \{B : \emptyset \neq B \subset P\} \quad \text{and} \quad (p, B) \in \mathcal{I} \Leftrightarrow p \in B,$$

$(P, \mathcal{B}, \mathcal{I})$ is called a (non-trivial, simple) *design* denoted by $(P, \mathcal{B})$.

A design $(P, \mathcal{B})$ with $|P| = v$ and $|\mathcal{B}| = b$ is called a 1-(v, b, k, r) *design* if $^{\forall}B \in \mathcal{B},\ |B| = k$ and $^{\forall}p \in P,\ |\{B \in \mathcal{B} : B \ni p\}| = r$.

For $t \geq 2$, a t-(v, b, k, r, λ) *design* is a 1-(v, b, k, r) design such that every set of t points is contained in λ blocks.
Denote it by t-(v, k, λ) design: $\lambda\binom{v}{t} = b\binom{k}{t}$.

Partial geometry

A *partial geometry* $pg(r, k, \alpha)$ for $\alpha \geq 1$, is a 1-(v, b, k, r) design such that

1. it is geometric; i.e., any two points have at most one common incident block, and
2. for any antiflag (p, B) of the design there exist α blocks containing p and intersecting B.

- A $pg(r, k, \alpha)$ has

$$\begin{aligned} v &= k + \frac{1}{\alpha}k(k-1)(r-1) \\ b &= r + \frac{1}{\alpha}r(r-1)(k-1). \end{aligned}$$

- A $pg(r, k, \alpha)$ is a 2-$(v, k, 1)$ design if and only if $\alpha = k$.

Partial geometric design (PGD)

Given a 1-(v, b, k, r) design $(P, \mathcal{B})$, $^{\forall}(p, B) \in P \times \mathcal{B}$, let

$$s(p, B) := |\{(q, C) : \ q \in B \cap C, \ C \ni p\}|.$$

PGD$(v, b, k, r; \alpha, \beta)$

A PGD with parameters $(v, b, k, r; \alpha, \beta)$ is a 1-(v, b, k, r) design $(P, \mathcal{B})$ satisfying the property:

$^{\forall}(p, B) \in P \times \mathcal{B}$, *there exist constants* α *and* β *such that*

$$s(p, B) = \begin{cases} \beta & \text{if } p \in B, \\ \alpha & \text{if } p \notin B. \end{cases}$$

Parameters of a PGD

Given a PGD$(v, b, k, r; \alpha, \beta)$, let $n = \beta - \alpha$,

1. $(v - k)\alpha + k\beta = k^2 r$
2. $v = \frac{1}{\alpha} k(kr - n); \quad b = \frac{1}{\alpha} r(kr - n)$
3. $k + r \quad \leq \quad n + \alpha + 1 = \beta + 1 \quad \leq \quad kr.$

Incidence matrix

Let N be the incidence matrix of a PGD$(v, b, k, r; \alpha, \beta)$.
Let J denote the all-1 matrix (not necessarily square).
Then we have

1. $JN = kJ, \quad NJ = rJ,$ and
2. $NN^\top N \quad = \quad \beta N + \alpha(J - N).$

Concurrence λ_{pq} of points p and q (Neumaier)

Given a PGD$(v, b, k, r; \alpha, \beta)$ and for $p, q \in P$, let

$$\lambda_{pq} := |\{B \in \mathcal{B} : \; p, q \in B\}| \quad = \quad \left[NN^T\right]_{pq}.$$

1. $^\forall p \in P, \quad \sum_{q \in P - \{p\}} \lambda_{pq} = r(k-1), \quad NN^T J = rkJ = JNN^T.$

2. $$s(p, B) = \sum_{q \in B} \lambda_{pq} = \left[NN^T N\right]_{pB} \; = \; \left[\beta N + \alpha(J - N)\right]_{pB}.$$

3. $^\forall p, q \in P, \quad \sum_{z \in P} \lambda_{pz}\lambda_{qz} = \left[\left(NN^T\right)^2\right]_{pq} = \left[\left(NN^T N\right) N^T\right]_{pq}$

$$= \left[\left(nN + \alpha J\right) N^T\right]_{pq} = \sum_{B : B \ni q} s(p, B) = \begin{cases} \beta r & \text{if } p = q \\ n\lambda_{pq} + \alpha r & \text{if } p \neq q \end{cases}$$

Concurrence profiles of a PGD

A PGD$(v, b, k, r; \alpha, \beta)$ holds one, two or three concurrences:

$$\lambda_{pq} \in \{\lambda_1, (\lambda_2), ((\lambda_3))\}, \quad r \geq \lambda_1 > \lambda_2 > \lambda_3 \geq 0;$$

- 2-$(v, k, \lambda) \equiv$ PGD$\Big(v,\ \frac{\lambda v(v-1)}{k-1},\ k,\ \frac{\lambda(v-1)}{k-1};\ \lambda k,\ \lambda(k-1)+r\Big)$:
  $\lambda_{pq} \in \{\lambda\}$.
- $pg(k, r, \alpha) \ \equiv\ $ PGD$(v, b, k, r; \alpha, \beta)$: $\quad \lambda_{pq} \in \{1, 0\}$.
- A transversal design TD$_\lambda(k, u)$, (where $u = \frac{v}{k}$) is a
  PGD$(ku,\ \lambda u^2,\ k,\ \lambda u;\ \lambda(k-1),\ \lambda(k-1)+\lambda u)$.

$$\lambda_{pq} = \begin{cases} 0 & \text{if } p, q \text{ belong to the same group} \\ \lambda & \text{else} \end{cases}$$

Concurrences of a PGD and spectrum of NN^T (Neumaier)

Let $(P, \mathcal{B})$ be a PGD$(v, b, k, r; \alpha, \beta)$ and let N be its incidence matrix.

1. NN^T has two or three distinct eigenvalues; namely,

$$\mathrm{Spec}(NN^T) \quad = \quad \left[(kr)^1, \quad n^\sigma, \quad 0^{v-1-\sigma}\right]$$

2. If $\lambda_{pq} \in \{\lambda_1, \lambda_2\}$ for ${}^\forall p, q, (p \neq q) \in P$ with $r \geq \lambda_1 > \lambda_2 \geq 0$, then for each p,

$$k_1 := |\{q \in P : \lambda_{pq} = \lambda_1\}| \quad = \quad \frac{r(k-1) - (v-1)\lambda_2}{\lambda_1 - \lambda_2}.$$

Partitioning $P \times P$ for a PGD (Lei-Qu-Shan)

Let $(P, \mathcal{B})$ be a PGD$(v, b, k, r; \alpha, \beta)$. Suppose $\lambda_{xy} \in \{\lambda_1, \lambda_2, \lambda_3\}$ for $x, y \in P$, $x \neq y$. If relations R_i, for $i = 1, 2, 3$, are given by

$$R_i := \{(x, y) \in P \times P : \ \lambda_{xy} = \lambda_i\}$$

then $R_0 \cup R_1 \cup R_2 \cup R_3 = P \times P$ where $R_0 = \{(x, x) : \ x \in P\}$.

For $i \in \{1, 2, 3\}$, with $\{h, j\} = \{1, 2, 3\} - \{i\}$,

$$k_i = |R_i(x)| := |\{y \in P : \ \lambda_{xy} = \lambda_i\}|$$

$$= \frac{(n + \alpha - r)r - r(k-1)(\lambda_h + \lambda_j) + (v-1)\lambda_h \lambda_j}{(\lambda_i - \lambda_h)(\lambda_i - \lambda_j)}.$$

For k_i, use $v - 1 = k_1 + k_2 + k_3$ and

$$k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = \sum_{y \in P - \{x\}} \lambda_{xy} = r(k-1),$$

$$k_1\lambda_1^2 + k_2\lambda_2^2 + k_3\lambda_3^2 = \sum_{y \in P - \{x\}} \lambda_{xy}^2 = (n + \alpha - r)r.$$

Association scheme from a PGD (Lei-Qu-Shan)

Let $(P, \mathcal{B})$ be a PGD$(v, b, k, r; \alpha, \beta)$. Suppose $\lambda_{xy} \in \{\lambda_1, \lambda_2, \lambda_3\}$ for any $x \neq y$ and $\lambda_3 = r - n$. Let R_i be given by

$$R_i := \{(x, y) \in P \times P : \ \lambda_{xy} = \lambda_i\} \quad \text{for } i = 1, 2, 3.$$

Then $(P, \{R_i\}_{[3]})$ becomes an association scheme.

Q. Find such PGDs! (i) $\lambda_1 > \lambda_2 > \lambda_3 \geq 0$ and (ii) $\lambda_3 = r - n$

Qu-Lei's examples: PGD$(3r, \ \frac{3}{4}r^2, \ 4, \ r; \ 4, \ r+4)$, for r even

For example, when $r = 4$, take $V = \mathbb{Z}_{12}$, $\mathcal{B}$ consists of:

$$\begin{array}{cccc} \{0,1,3,4\} & \{0,1,5,6\} & \{0,2,7,8\} & \{0,2,9,10\} \\ \{1,7,8,11\} & \{1,9,10,11\} & \{2,3,4,11\} & \{2,5,6,11\} \\ \{3,5,7,9\} & \{3,5,8,10\} & \{4,6,7,9\} & \{4,6,8,10\} \end{array}$$

Example (Tranel-S.): PGD(8, 8, 4, 4; 6, 10)

$$P = \{1, 2, \ldots, 8\}$$

$$\mathcal{B} = \left\{ \begin{array}{cccc} \{1,2,3,4\}, & \{1,2,5,6\}, & \{1,3,5,7\}, & \{1,4,5,8\} \\ \{5,6,7,8\}, & \{3,4,7,8\}, & \{2,4,6,8\}, & \{2,3,6,7\} \end{array} \right\}$$

$$NN^T = \left[\begin{array}{c|c} 2I + 2J & 2I + J \\ \hline 2I + J & 2I + 2J \end{array} \right] = \begin{bmatrix} 4 & 2 & 2 & 2 & 3 & 1 & 1 & 1 \\ 2 & 4 & 2 & 2 & 1 & 3 & 1 & 1 \\ 2 & 2 & 4 & 2 & 1 & 1 & 3 & 1 \\ 2 & 2 & 2 & 4 & 1 & 1 & 1 & 3 \\ 3 & 1 & 1 & 1 & 4 & 2 & 2 & 2 \\ 1 & 3 & 1 & 1 & 2 & 4 & 2 & 2 \\ 1 & 1 & 3 & 1 & 2 & 2 & 4 & 2 \\ 1 & 1 & 1 & 3 & 2 & 2 & 2 & 4 \end{bmatrix}$$

Observe that:
NN^T is recognized as an association relation table for a scheme.
Notice that NN^T is not a circulant matrix

PGDs having circulant concurrence matrices (Tranel-S.)

1. PGD$(8, 10, 4, 5; 8, 12)$ with $NN^T = C[5, 2, 3, 2, 1, 2, 3, 2]$ and $\mathcal{B}$:
   $\{1, 2, 3, 4\}$, $\{1, 2, 3, 8\}$, $\{1, 3, 5, 7\}$, $\{1, 4, 6, 7\}$, $\{1, 6, 7, 8\}$
   $\{2, 4, 5, 7\}$, $\{2, 4, 6, 8\}$, $\{2, 5, 7, 8\}$, $\{3, 4, 5, 6\}$, $\{3, 5, 6, 8\}$.
   (N is found by a computer search.)

2. PGD$(12, 12, 4, 4; 4, 8)$ with $NN^T = C[4, 1, 1, 2, 1, 1, 0, 1, 1, 2, 1, 1]$:
   $P = \{0, 1, 2, \ldots, 9, a, b\}$ and
   $\mathcal{B} :=$
   $\{0, 1, 3, 4\}, \{0, 2, 3, 5\}, \{0, 7, 9, a\}, \{0, 8, 9, b\}, \{1, 2, a, b\}, \{1, 4, 6, 9\},$
   $\{1, 5, 8, a\}, \{2, 4, 7, b\}, \{2, 5, 6, 9\}, \{3, 6, 7, a\}, \{3, 6, 8, b\}, \{4, 5, 7, 8\}.$

3. PGD$(12, 14, 6, 7; 18, 24)$ with $NN^T = C[7, 3, 4, 3, 4, 3, 1, 3, 4, 3, 4, 3]$
   Via computer search we have the following blocks:
   $\{0, 1, 2, 3, 4, 5\}, \{0, 1, 2, 3, 4, b\}, \{0, 1, 5, 8, 9, a\}, \{0, 2, 4, 6, 8, a\},$
   $\{0, 2, 5, 7, 9, a\}, \{0, 3, 7, 8, a, b\}, \{0, 4, 7, 8, 9, b\}, \{1, 2, 6, 9, a, b\},$
   $\{1, 3, 5, 6, 8, a\}, \{1, 3, 5, 7, 9, b\}, \{1, 4, 6, 8, 9, b\}, \{2, 3, 6, 7, a, b\},$
   $\{2, 4, 5, 6, 7, 9\}, \{3, 4, 5, 6, 7, 8\}.$

Circulant matrix

An $n \times n$ matrix of the form

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & c_2 & \dots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & c_1 & \dots & c_{n-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ c_2 & \vdots & \vdots & \ddots & \ddots & c_1 \\ c_1 & c_2 & c_3 & \dots & c_{n-1} & c_0 \end{bmatrix}$$

where $c_0, c_1, c_2, \dots, c_{n-1}$ are complex numbers, is called a circulant matrix.

The eigenvalues of C are given by $f(\omega^k)$ for $k = 0, 1, \dots, n-1$ where ω is an n^{th}-root of unity and

$$f(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_{n-2}\lambda^{n-2} + c_{n-1}\lambda^{n-1}.$$

Example

Let $P = \{1,2,3,4,5,6\}$ and $\mathcal{B} = \{\{1,2,3\},\{1,5,6\},\{2,4,6\},\{3,4,5\}\}$. Then $(P,\mathcal{B})$ is a PGD with parameters $(6,4,3,2;2,4)$. The incidence and concurrence matrices are, respectively:

$$N = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \text{ and } NN^T = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

More specifically, the PGD is a $\mathrm{TD}_1(3,2)$.

Proposition: Tranel-S

For any $v \equiv 0 \bmod 2$ with $v \geq 6$, there is a

$$\mathrm{PGD}(v,\ \tfrac{1}{8}v(v\text{-}2),\ 4,\ \tfrac{1}{2}v\text{-}1;\ 4,\ v)$$

whose concurrence matrix is the circulant

$$NN^T = C\left[\tfrac{1}{2}v\text{-}1,\ \underbrace{1,\ \ldots,\ 1}_{\frac{1}{2}v-1},\ \tfrac{1}{2}v\text{-}1,\ \underbrace{1,\ \ldots, 1}_{\frac{1}{2}v-1}\right]$$

with the spectrum

$$\left\{(2v-4)^1,\ \ (v-4)^{\frac{1}{2}v-1},\ \ 0^{\frac{1}{2}v}\right\}.$$

Examples: Tranel-S.

There are at least 11 infinite families of PGDs of order 12 whose concurrence matrices are circulant.

A circulant matrix $C = C[c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_5, c_4, c_3, c_2, c_1]$ is feasible to be recognized as the concurrence matrix, NN^T, of a $\text{PGD}\left(12, \frac{12r}{k}, k, r; \frac{k(kr-n)}{12}, n + \frac{(kr-n)}{12}\right)$ when

$$\text{Spec}(NN^T) = \{kr^1, n^\sigma, 0^{11-\sigma}\} \text{ where } \sigma = \frac{r(12-k)}{n}.$$

Using the spectrum of NN^T along with the conditions on the eigenvalues of a circulant matrix, we use a linear program to find all PGDs arise for $k = 3$, 4, and 6 as in the following table.

PGDs of order 12 having circulant concurrence matrices

σ	b	k	r	(α, β)	NN^T	**Description**	#	IDs in [SvD]
9	$4r$	3	r	$(\frac{r}{2}, \frac{3}{2}r)$	$C[r, \frac{r}{4}, \frac{r}{4}, 0, \frac{r}{4}, \frac{r}{4}, 0]$	$\biguplus_{i=1}^{4} r_i \mathrm{TD}_i(3,4)$	*	N37
4	$3r$	4	r	$(\frac{2}{3}r, \frac{8}{3}r)$	$C[r, \frac{r}{3}, 0, \frac{r}{3}, 0, \frac{r}{3}, r]$	$\mathrm{TD}_1(2,3) \otimes J_{2,\frac{r}{3}}$	1	N48
5	$3r$	4	r	$(\frac{4}{5}r, \frac{12}{5}r)$	$C[r, \frac{r}{5}, \frac{r}{5}, \frac{r}{5}, \frac{r}{5}, \frac{r}{5}, r]$	$2\text{-}(6,2,1) \otimes J_{2,\frac{r}{5}}$	1	N33
8	$3r$	4	r	$(r, 2r)$	$C[r, \frac{r}{4}, \frac{r}{4}, \frac{r}{2}, \frac{r}{4}, \frac{r}{4}, 0]$	$N \otimes J_{1,\frac{r}{4}}$	≥ 1	N19
8	$3r$	4	r	$(r, 2r)$	$C[r, \frac{r}{3}, \frac{r}{3}, \frac{r}{3}, 0, \frac{r}{3}, \frac{r}{3}]$	$\mathrm{TD}_1(4,3) \otimes J_{1,\frac{r}{3}}$	≥ 1	N47
3	$2r$	6	r	$(2r, 4r)$	$C[r, \frac{r}{2}, \frac{r}{2}, 0, \frac{r}{2}, \frac{r}{2}, r]$	$D_{1,6,r} \otimes J_{2,1}$	*	N22, N41, N62
3	$2r$	6	r	$(2r, 4r)$	$C[r, \frac{r}{3}, \frac{r}{3}, \frac{r}{3}, r, \frac{r}{3}, \frac{r}{3}]$	$2\text{-}(4,2,1) \otimes J_{3,\frac{r}{3}}$	1	N22, N50
5	$2r$	6	r	$(\frac{12}{5}r, \frac{18}{5}r)$	$C[r, \frac{2}{5}r, \frac{2}{5}r, \frac{2}{5}r, \frac{2}{5}r, \frac{2}{5}r, r]$	$2\text{-}(6,3,2) \otimes J_{2,\frac{r}{5}}$	1	N61
6	$2r$	6	r	$(\frac{5}{2}r, \frac{7}{2}r)$	$C[r, \frac{r}{2}, \frac{r}{2}, \frac{r}{2}, \frac{r}{2}, \frac{r}{2}, 0]$	$\mathrm{TD}_{\frac{r}{2}}(6,2)$	≥ 1	N21, N39, N60
7	$2r$	6	r	$(\frac{18}{7}r, \frac{24}{7}r)$	$C[r, \frac{3}{7}r, \frac{4}{7}r, \frac{3}{7}r, \frac{4}{7}r, \frac{3}{7}r, \frac{r}{7}]$	$N \otimes J_{1,\frac{r}{7}}$	≥ 1	N27
9	$2r$	6	r	$(\frac{8}{3}r, \frac{10}{3}r)$	$C[r, \frac{r}{2}, \frac{r}{2}, \frac{r}{3}, \frac{r}{2}, \frac{r}{2}, \frac{r}{3}]$	$N \otimes J_{1,\frac{r}{6}}$	≥ 1	N20

*SvD: Labels of the designs listed in "van Dam, E. R., Spence, E.: Combinatorial designs with two singular values II. Partial geometric designs. *Linear Alg. Appl.*, **396**, 303–316 (2005)."

Q. So far all known PGDs have at most 3 distinct concurrences (aside from $c_0 = r$). Show that all PGDs have at most 3 distinct concurrences!

Q. Characterize PGDs that having circulant concurrence matrices!

Q. Classify all PGDs having circulant concurrence matrices!

PGDs from 3-class fusion of Hamming schemes $H(d,3)$, $d \geq 3$

Let X be the ternary linear $[7,5]$-code with generating matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Define relations R_is on X according to the (Hamming) distance δ.

$$\begin{aligned} R_0 &= \{(x,x) \mid x \in X\} \\ R_1 &= \{(x,y) \mid \delta(x,y) \in \{1,4,7\}\} \\ R_2 &= \{(x,y) \mid \delta(x,y) \in \{2,5\}\} \\ R_3 &= \{(x,y) \mid \delta(x,y) \in \{3,6\}\}, \end{aligned}$$

Then $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq 3})$ is a 3-class association scheme.

1. The intersection matrices $B_1, B_2,$ and B_3 of $\mathcal{X}$ are given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 72 & 21 & 24 & 18 \\ 0 & 30 & 24 & 27 \\ 0 & 20 & 24 & 27 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 30 & 24 & 27 \\ 90 & 30 & 33 & 36 \\ 0 & 30 & 32 & 27 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 20 & 24 & 27 \\ 0 & 30 & 32 & 27 \\ 80 & 30 & 24 & 25 \end{bmatrix}.$$

2. We have the following identities in its Bose-Mesner algebra:

$$\begin{aligned} A_1^3 &= 1593A_1 + 1512(J - A_1), \\ A_2^3 &= 3051A_2 + 2970(J - A_2), \\ (A_3 + I)^3 &= 2241(A_3 + I) + 2160(J - A_3 - I). \end{aligned}$$

3. If we take $N = A_1$, N becomes the incidence matrix of a PGD.

Nowak-Olmez-S, 2015

Let $\mathcal{Z}$ be a 3-class association scheme. Suppose that the character table P of $\mathcal{Z}$ is given by

$$P = \begin{bmatrix} 1 & m(m-1) & m(m+1) & (m-1)(m+1) \\ 1 & m & 0 & -m-1 \\ 1 & 0 & -m & m-1 \\ 1 & -m & m & -1 \end{bmatrix}.$$

Then the relation graphs A_1, A_2, and $A_3 + A_0$ of $\mathcal{Z}$ give rise to three PGDs with parameters $v = b$ and $(v, k; \alpha, \beta)$:

$$(3m^2,\ m(m-1);\ \frac{1}{3}m^2(m^2-3m+2),\ \frac{1}{3}m^2(m^2-3m+5)),$$

$$(3m^2,\ m(m+1);\ \frac{1}{3}m^2(m^2+3m+2),\ \frac{1}{3}m^2(m^2+3m+5)),$$

$$(3m^2,\ m^2;\ \frac{1}{3}m^2(m^2-1),\ \frac{1}{3}m^2(m^2+2)).$$

A PGD induced from a set of associate relations

Let $\mathcal{X} = (X, \{R_i\}_{[d]})$ be an association scheme and let $R_M = \bigcup_{i \in M} R_i$ where $\emptyset \neq M \subset [d] - \{0\}$. For each $x \in X$ define

$$B_x = \{y : (x, y) \in R_M\}$$

Suppose $\mathcal{X}$ has a fusion scheme containing a relation R_M. Then $(X, \{B_x : x \in X\})$ becomes a 1-(v, b, k, r)-design with $v = b = |X|, \quad k = r = \sum_{i \in M} k_i$.

Xu, '22

Suppose an association scheme $\mathcal{X} = (X, \{R_i\}_{[d]})$ contains a '*block*' of relations $\{R_i : i \in M\}$ where $\emptyset \neq M \subset [d] - \{0\}$ such that for each $x \in X, \quad |B_x| = |\{y : (x, y) \in \cup_{i \in M} R_i\}| = k.$ Then $(X, \{B_x : x \in X\})$ is a PGD$(|X|, |X|, k, k; \alpha, \beta)$ if and only if $A_M A_M^T A_M = \beta A_M + \alpha(J - A_M)$ where $A_M = \cup_{i \in M} A_i$.

Theorem (Xu, '23)

Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le 3})$ be a self-dual association scheme such that R_1, R_2, and $R_0 \cup R_3$ give three partial geometric designs. Then either (i) or (2) holds:

(i) $\mathcal{X}$ is primitive and there exists an integer m such that $m \equiv 0 \bmod 3$ and the character table of $\mathcal{X}$ is

$$\begin{bmatrix} 1 & m(m-1) & m(m+1) & (m-1)(m+1) \\ 1 & m & 0 & -m-1 \\ 1 & 0 & -m & m-1 \\ 1 & -m & m & -1 \end{bmatrix}.$$

(ii) $\mathcal{X}$ is imprimitive and there exists an odd integer m such that the character table is either

$$\begin{bmatrix} 1 & \frac{m(m+1)}{2} & \frac{m(m+1)}{2} & m \\ 1 & \frac{m+1}{2} & \frac{-m-1}{2} & -1 \\ 1 & \frac{-m-1}{2} & \frac{m+1}{2} & -1 \\ 1 & \frac{-m-1}{2} & \frac{-m-1}{2} & m \end{bmatrix} \text{ or } \begin{bmatrix} 1 & \frac{m(m+1)}{2} & \frac{m(m+1)}{2} & m \\ 1 & \frac{m+1}{-2i} & \frac{m+1}{2i} & -1 \\ 1 & \frac{m+1}{2i} & \frac{m+1}{-2i} & -1 \\ 1 & \frac{-m-1}{2} & \frac{-m-1}{2} & m \end{bmatrix}.$$

Conversely, if the character table of $\mathcal{X}$ is one of the above, then R_1, R_2, and $R_0 \cup R_3$ of $\mathcal{X}$ induce partial geometric designs.

Q. Is there such a primitive association schemes of order $3m$ where $m \neq 3^p$?

Q. Find all such imprimitive association schemes!

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