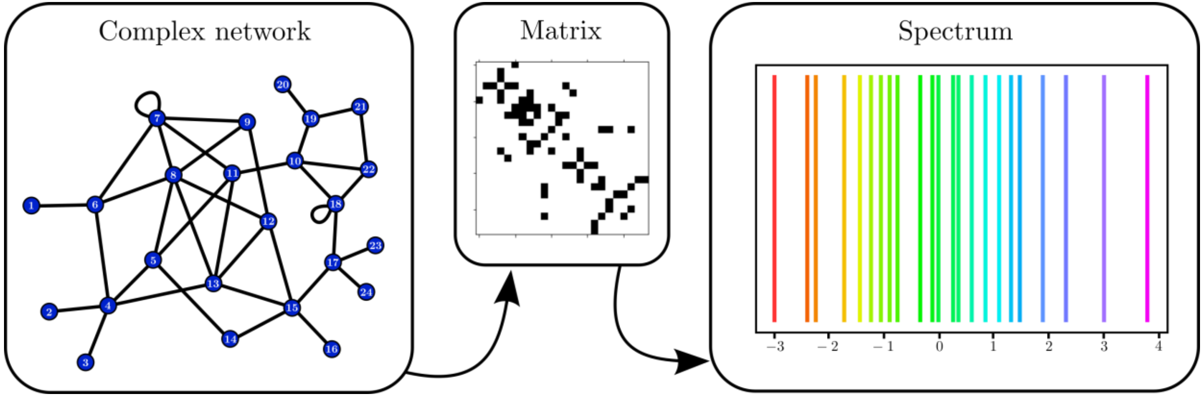
Towards generalized spectral determinacy of random graphs



Alexander Van Werde, Algebraic Graph Theory Seminar (2025)

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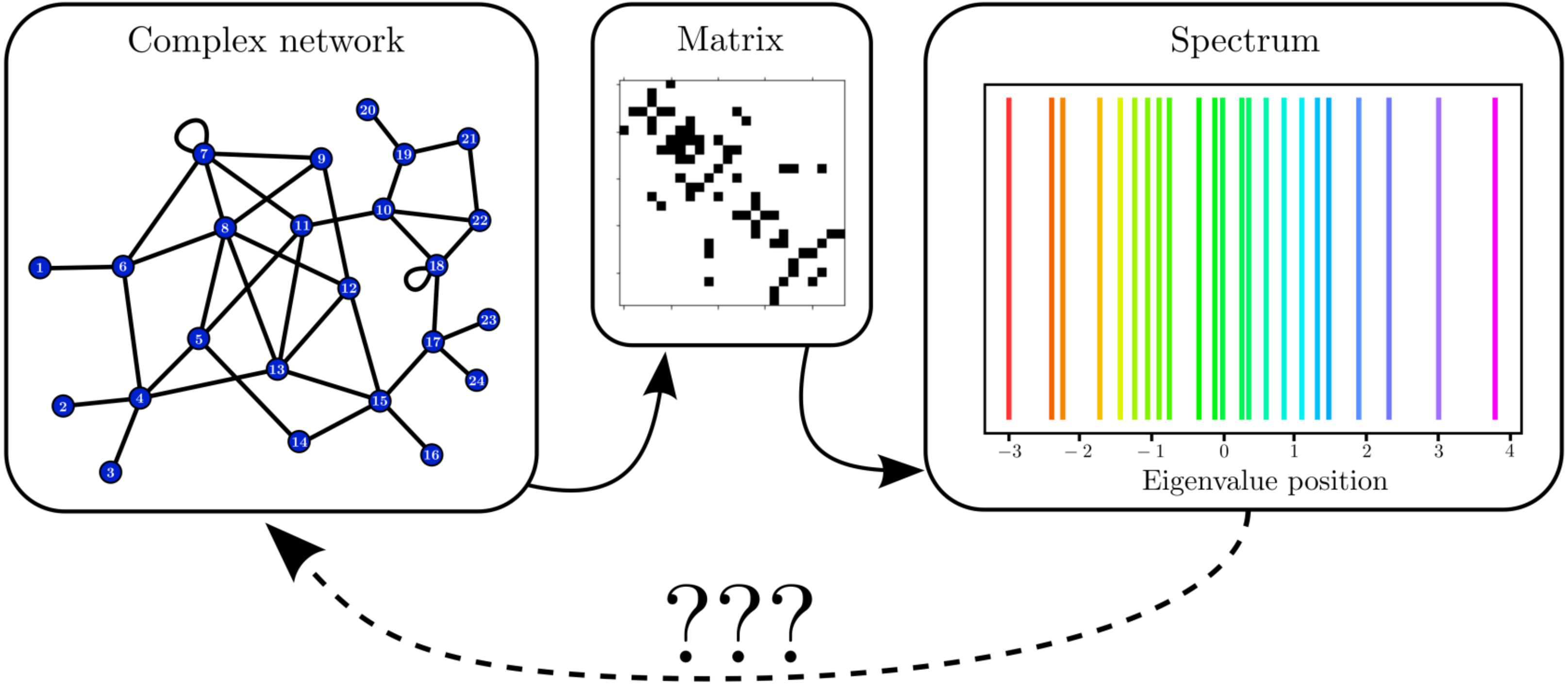
TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

Can *all* information be recovered from spectrum?



Some history: spectral determinacy of graphs

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No, non-isomorphic graphs can have the same adjacency spectrum.

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"It is an open question whether almost all graphs are characterized by their characteristic polynomials. It is not even clear if we should seek to prove this, or to disprove it."

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Conjecture: almost all graphs are determined by spectrum.

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2023 (Koval and Kwan)
At least $\exp(cn)$ graphs are determined by spectrum.

We lack flexible proof techniques…

Some history: generalized spectral determinacy

1980 (Johnson and Newman)

"It is our view, however, that to some extent these examples are algebraic accidents due to the interpretation of the formal symbols 0 and 1 as real numbers."

Definition. (Generalized cospectral)

Graphs G, H are said to be *generalized cospectral if*

$$\mathrm{spec}\left(\mathrm{A}_{\mathrm{G}}^{x,y}\right) = \mathrm{spec}\left(\mathrm{A}_{\mathrm{H}}^{x,y}\right) \qquad \forall x, y \in \mathbb{R}.$$

where $A_G^{x,y}$ is the variant on the adjacency matrix with $1 \to x$ and $0 \to y$.

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Conjecture (Wang): Satisfied with non-vanishing probability! 😮

Sufficient condition for generalized spectral determinacy

Definition (Walk matrix)

Given an integer matrix $\boldsymbol{X} \in \mathbb{Z}^{n\times n}$, consider the matrix

$$\boldsymbol{W} := [e, \boldsymbol{X}e, , \boldsymbol{X}^2 e, \dots, \boldsymbol{X}^{n-1} e]$$

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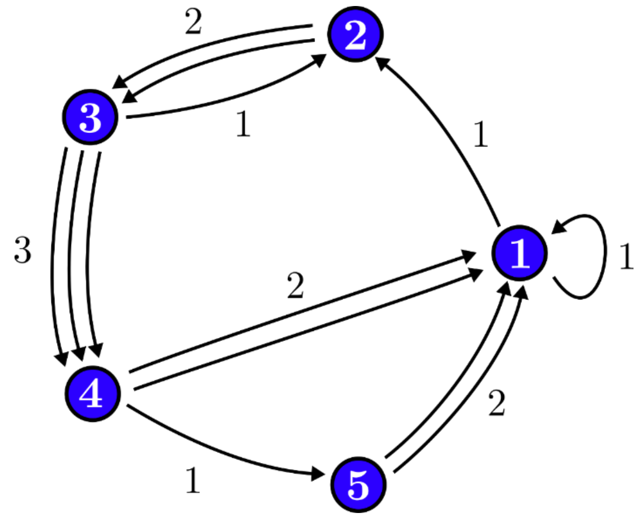
Interpret $\boldsymbol{X}_{i,j}$ as edge multiplicity.

Then, $\boldsymbol{W}_{i,j}$ counts walks of length $j-1$ ending in i.

Example.

$$\boldsymbol{W}_{5,3} = 3 \cdot 1 = 3$$



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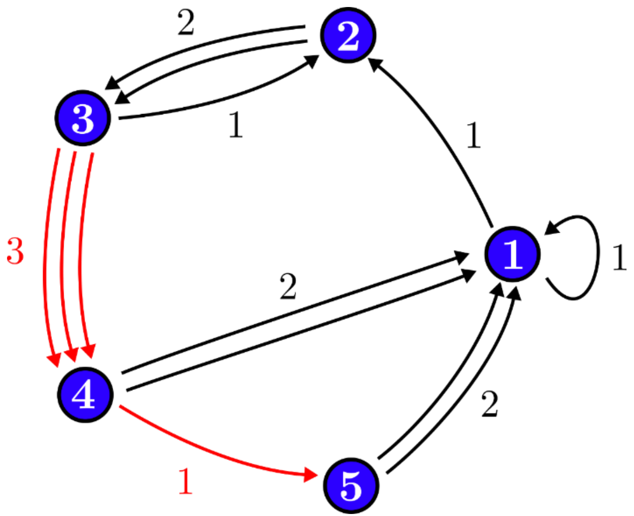

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Notation

$$\boldsymbol{W}(\mathbb{Z}^n) := \{\boldsymbol{W}v : v \in \mathbb{Z}^n\} = \{p(\boldsymbol{X})e : p \in \mathbb{Z}[x]\},$$

and

$$\operatorname{coker}(\boldsymbol{W}) := \mathbb{Z}^n / \boldsymbol{W}(\mathbb{Z}^n)$$

Given an Abelian group G and a prime power p^m, let $G_{p^m} := G/p^m G$.

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Given an Abelian group G and a prime power p^m, let $G_{p^m} := G/p^m G$.

Theorem. (Wang 2017; see also Qui, Wang, and Zhang 2023)

Consider a simple graph G and set $\boldsymbol{X} := \boldsymbol{A}_G$. Assume that $\mathrm{coker}(\mathbf{W})_{2^2} \cong (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}$ and $\mathrm{coker}(\mathbf{W})_{p^2} \in \{0, \mathbb{Z}/p\mathbb{Z}\}$ for odd primes p.

Then, G is determined by generalized spectrum.

Suppose $\boldsymbol{X}$ is random.

How can we study the distribution of $\mathbf{coker}(\boldsymbol{W})$?

Results

Disclaimer.

For technical reasons, all results in this talk assume that $\boldsymbol{X}$ has independent entries.

This implies that we can not (yet) deal with the adjacency matrices of *simple* random graphs: those have dependent entries due to the symmetry constraint $\boldsymbol{X} = \boldsymbol{X}^{\mathrm{T}}$.

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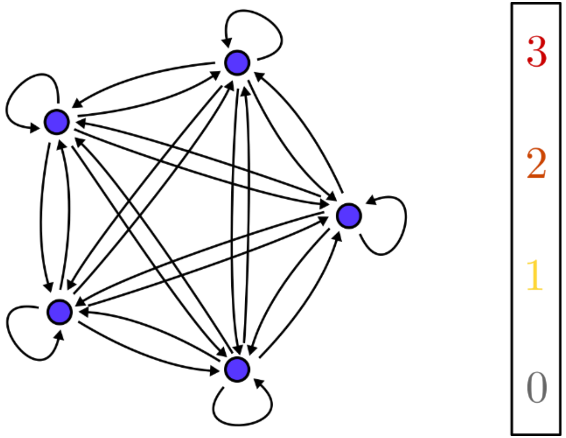
Assumption 1st result

Fix a prime p and integer $m \geq 0$.

Assume that the entries of $\boldsymbol{X}$ are independent and $\mathrm{Unif}\{0,1,\dots,p^m - 1\}$-distributed.



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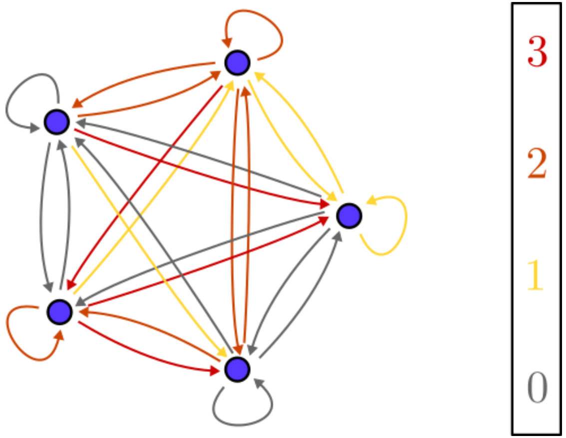

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Theorem 1.

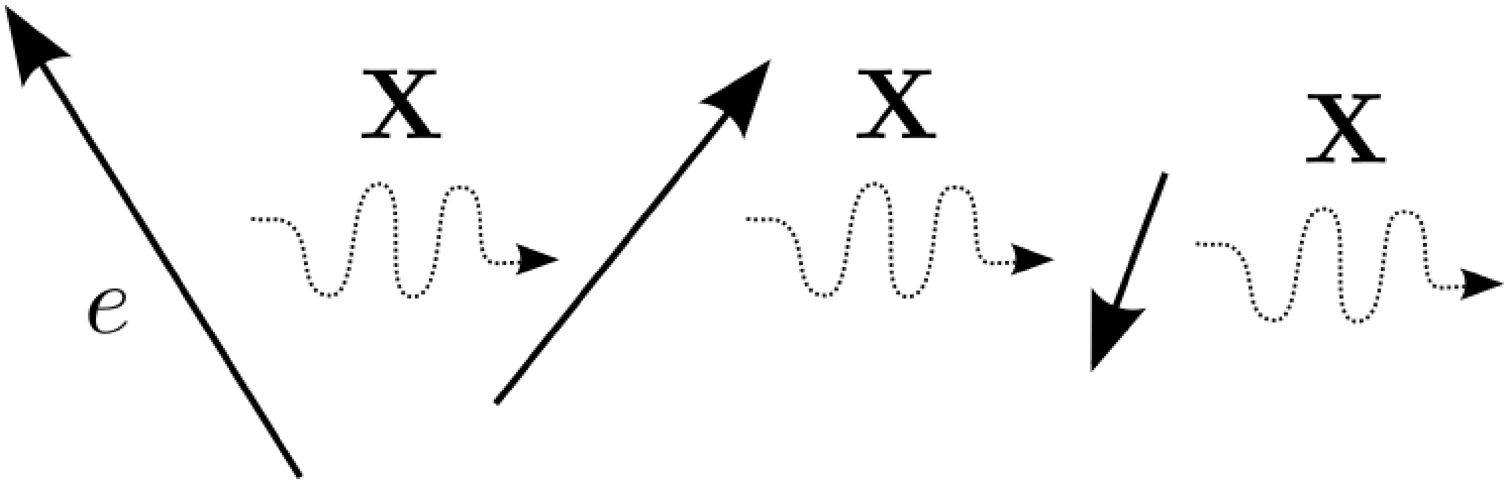
We have

$$\lim_{n\to\infty} \mathbb{P}\left(\mathrm{coker}(\boldsymbol{W})_{p^m} \cong \bigoplus_{i=1}^{\ell} \frac{\mathbb{Z}}{p^{\lambda_i}\mathbb{Z}}\right) = \prod_{i=i_0}^{\infty}\left(1-p^{-(i+1)}\right) \prod_{j=1}^{\ell} p^{-j\delta_j}$$

for every $0=\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_\ell \leq m$.

Here, $i_0 := \#\{1 \leq i \leq \ell : \lambda_i = m\}$ and $\delta_j = \lambda_{\ell-j+1} - \lambda_{\ell-j}$.

Proof idea



Proof idea

There is dependence!

Observe that if

$$\boldsymbol{X}^j e \in \operatorname{span}_{\mathbb{Z}}\left(e, \boldsymbol{X}e, \ldots, \boldsymbol{X}^{j-1}e\right) + p^k\mathbb{Z}^n$$

then also

$$\boldsymbol{X}^{j+1} e \in \operatorname{span}_{\mathbb{Z}}\left(e, \boldsymbol{X}e, \ldots, \boldsymbol{X}^{j-1}e, \boldsymbol{X}^j e\right) + p^k\mathbb{Z}^n.$$

Key observation. (Informally)

Aside from the obstruction above, there is independence.

Interpretable proof!

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Sadly, the technique is not robust.

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How can we study *unweighted* graphs?

Results

Assumption simplified 2nd result

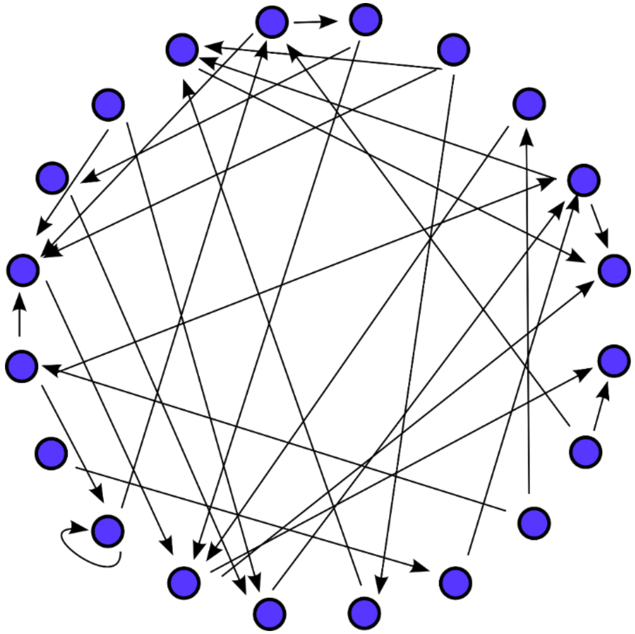
Suppose $\boldsymbol{X}$ has independent $\{0,1\}$-valued entries. (Not necessarily identically distributed.)

Further, consider a sparse setting:

$$\mathbb{P}(\boldsymbol{X}_{i,j} = 1) \leq \mathbb{P}(\boldsymbol{X}_{i,j} = 0)$$

But not *too* sparse:

$$\mathbb{P}(\boldsymbol{X}_{i,j} = 1) \gg \ln(n)/n$$



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Assumption simplified 2nd result

$\boldsymbol{X}$ has independent $\{0,1\}$-valued entries with $\mathbb{P}(\boldsymbol{X}_{i,j} = 1) \leq \mathbb{P}(\boldsymbol{X}_{i,j} = 0)$ and

$$\mathbb{P}\big(\boldsymbol{X}_{i,j} = 1\big) \gg \ln(n)/n$$

Technical condition.

Additionally assume tightness:

$$\lim_{C\to\infty} \liminf_{n\to\infty} \mathbb{P}\big(\#\mathrm{coker}(\boldsymbol{W})_{p^m} \leq C\big) = 1.$$

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Theorem 2. (Simplified)

Fix a finite collection of primes $\mathcal{P}$.

Then, given the conditions above,

1. The same limiting law applies to $\mathrm{coker}(\boldsymbol{W})_{p^m}$ for every $p \in \mathcal{P}$.
2. We have asymptotic independence for different primes $p \in \mathcal{P}$.

Robust proof technique:

category-theoretic moment method.

Category-theoretic moment method

Definition. (Category-theoretic moment)

Consider a ring R, a deterministic R-module N, and a random R-module Y.

Then, the N-moment of Y is $\mathbb{E}[\#\mathrm{Sur}_R(Y, N)]$.

Theorem. (Sawin and Wood, 2022)

Consider a random R-module Y and a sequence of random R-modules Y_n.

Then, under certain conditions, to prove that $Y_n \to Y$ in distribution it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}[\#\mathrm{Sur}_R(Y_n, N)] = \mathbb{E}[\#\mathrm{Sur}_R(Y, N)]$$

for every fixed finite R-module N.

Category-theoretic moment method

We show that $\mathbb{E}\left[\#\mathrm{Sur}_{\mathbb{Z}[x]}(\mathrm{coker}(\boldsymbol{W}), N)\right] = (\#N)^{-1}$ for every finite $\mathbb{Z}[x]$-module N.

Related problems were studied by e.g., Nguyen and Wood (2022) and Cheong and Yu (2023).

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Proof sketch.

Using that group morphism $F: \mathbb{Z}^n \to N$ descends to $\mathbb{Z}[x]$-module morphism from $\mathrm{coker}(\boldsymbol{W})$ if and only if $F(e) = 0$ and $F\boldsymbol{X} = xF$,

$$\mathbb{E}[\#\mathrm{Sur}_{\mathbb{Z}[x]}(\mathrm{coker}(\boldsymbol{W}), N)] = \sum_{F \in \mathrm{Sur}_{\mathbb{Z}}(\mathbb{Z}^n, N):\, F(e)=0} \mathbb{P}(F\boldsymbol{X} = xF).$$

There are approximately $(\#N)^{n-1}$ summands since

$$\#\{F \in \mathrm{Sur}_{\mathbb{Z}}(\mathbb{Z}^n, N):\ F(e) = 0\} \approx \#\{F \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, N):\ F(e) = 0\}.$$

For typical F, one has $\mathbb{P}(F\boldsymbol{X} = xF) \approx (\#N)^{-n}$.

Thank you!

Key reference related to this talk are as follows:

Generalized spectral determinacy:

W. Wang and C.-X. Xu. *A sufficient condition for a family of graphs being determined by their generalized spectra*. European Journal of Combinatorics, 2006.

W. Wang. *A simple arithmetic criterion for graphs being determined by their generalized spectra.* Journal of Combinatorial Theory, Series B, 2017.

L. Qiu, W. Wang, and H. Zhang. *Smith normal form and the generalized spectral characterization of graphs.* Discrete Mathematics, 2023

Category-theoretic moment method:

W. Sawin and M.M. Wood. *The moment problem for random objects in a category.* arXiv:2210.06279v1, 2022.

The current work:

A. Van Werde. *Cokernel statistics for walk matrices of directed and weighted random graphs.* Combinatorics, Probability and Computing, 2025

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