Balanced splittable Hadamard matrices: restrictions and constructions

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Joint work with Jonathan Jedwab and Samuel Simon

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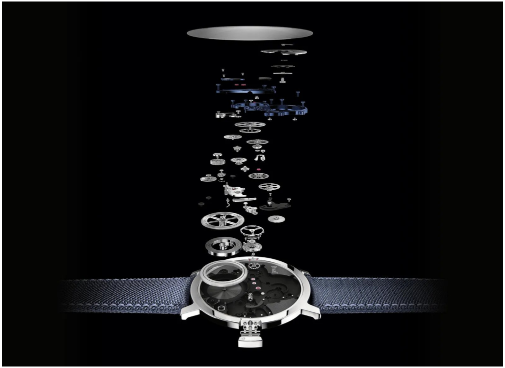

Internal structure matters!

Example (Hadamard matrix)

$$H_1 = \begin{bmatrix} 1 \end{bmatrix} \quad H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

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Remark

A Hadamard matrix of order n exists $\Leftrightarrow$ there exists an orthogonal basis of $\mathbb{R}^n$ containing only $\{1, -1\}$ entries

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A complete solution is by far elusive.

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Construction idea: imposing internal structures

Let A, B, C, D be $n \times n$ $\{1, -1\}$ matrices satisfying

- A, B, C, D are symmetric and circulant
- $AA^T + BB^T + CC^T + DD^T = 4nI_n$

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$$W = \begin{bmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{bmatrix}$$

W is a $4n \times 4n$ *Williamson matrix*, which is a special type of Hadamard matrices.

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There is no Williamson matrix of order $4 \cdot 35$.

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Remark (Equivalence of Hadamard matrices)

Two Hadamard matrices are equivalent if they are identical up to permutation and negation of rows and columns.

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$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
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- $\ell \leq \frac{n}{2}$, (n, ℓ, a, b)-BSHM w.r.t H_1 $\Leftrightarrow$ $(n, n-\ell, -a, -b)$-BSHM w.r.t H_2 (switching transformation)

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When $2 < \ell < n-2$*, the balanced splittable property reflects an in-depth internal structure of Hadamard matrices*

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$\ell > a$: repeated columns in H_1 prohibited

$\ell = a$: repeated columns in H_1 guaranteed

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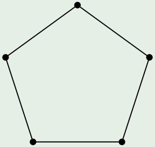
Example (Strongly regular graph (SRG))



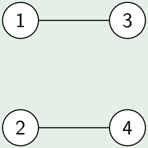
$(5, 2, 0, 1)$-SRG

- regular
- edge regular
- non-edge regular

Example (BSHM and associated SRG)

H is a BSHM$(4, 2, 2, 0)$ w.r.t H_1

$$H = \begin{bmatrix} H_1 \\ \\ H_2 \end{bmatrix} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$



$(4, 1, 0, 0)$-SRG

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2. *H can be transformed to a $BSHM(n, \ell, a, -a)$ $H' = \begin{bmatrix} H'_1 \\ H'_2 \end{bmatrix}$ with respect to H'_1, and $H'_1\mathbf{1} = \mathbf{0}$. The associated SRG has parameters*
$$(v, k', \lambda', \mu') = (n, \frac{(n-1)a - \ell}{2a}, \frac{n-4}{4} + \frac{n-4\ell}{4a}, \frac{n(a-1)}{4a})$$

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4. *a is even and $\frac{\ell}{a}$ is an odd integer and $\frac{n}{4a}$ is an integer.*

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Suppose $H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ *is a* $BSHM(n, \ell, a, b)$ *w.r.t.* H_1, $b \neq -a$. *The matrix* H *has exactly one of Types 1 and 2.*

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3. $\frac{\ell - b}{b - a}$ *and* $\frac{n}{b-a}$ *and* $\frac{n(b+1)}{2(b-a)}$ *and* $\frac{nb(b+1)}{(b-a)^2}$ *are integers*

Theorem (Kharaghani and Suda (2019), continued)

H has Type 2, i.e., $H_2\mathbf{1} = \mathbf{0}$

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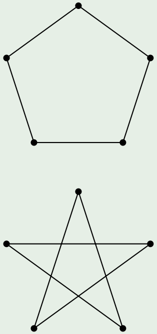
1. $n(\ell + ab - a - b) = (\ell - a)(\ell - b)$ *and* $ab \leq 0$
2. *The associated SRG has parameters*

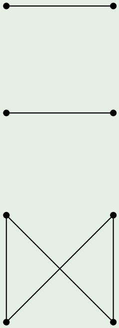
$$(n, \frac{\ell - b}{b - a} + \frac{n(b-1)}{b-a}, \frac{nb(b-1)}{(b-a)^2} + \frac{2(\ell - b)}{b - a} - \frac{n}{b-a}, \frac{nb(b-1)}{(b-a)^2}).$$

3. $\frac{\ell - b}{b - a}$ *and* $\frac{n}{b-a}$ *and* $\frac{n(b-1)}{2(b-a)}$ *and* $\frac{nb(b-1)}{(b-a)^2}$ *are integers*

Example (primitive and imprimitive SRGs)



primitive SRG



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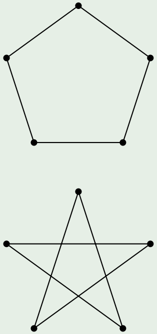

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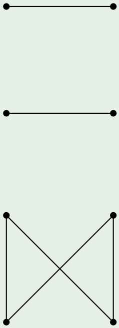

imprimitive SRG

We call a BSHM primitive or imprimitive if the associated SRG is primitive or imprimitive.

Table: Five classes for a BSHM(n, ℓ, a, b) satisfying $2 < \ell \le \frac{n}{2}$ (Jedwab, Li, Simon (2023))

$b = -a$	$b \neq -a$			
	Type 1		Type 2	
primitive	imprimitive	primitive	imprimitive	primitive

	$b \neq -a$	
	Type 2	
	imprimitive	primitive
parameter relations	$(n, \ell, a, b) = (8rs, 4s, 4s, 0)$ for $r, s \geq 1$	$n = \frac{(\ell-a)(\ell-b)}{\ell+ab-a-b}$, $\ell \equiv a \equiv b \pmod 4$, $a > 0 \geq b$
G	$4sK_{2r}$	$v = n$, $k = \frac{\ell-b+n(b-1)}{b-a}$, $\lambda = \mu + \frac{2(\ell-b)-n}{b-a}$, $\mu = \frac{nb(b-1)}{(b-a)^2}$
integers		$\frac{\ell-b}{b-a}$, $\frac{n}{b-a}$, $\frac{n(b-1)}{2(b-a)}$, $\frac{nb(b-1)}{(b-a)^2}$

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Suppose there exists a BSHM(n, ℓ, a, b) *with* $2 < \ell < n-2$. *Then* $\ell \equiv a \equiv b \pmod 4$.

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Remark

Using the above theorem, we can show there exists no $(36, \ell, a, b)$ *BSHM with* $2 < \ell < 34$.

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Using the above theorem, we can show there exists no $(36, \ell, a, b)$ *BSHM with* $2 < \ell < 34$.

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Among more than 15 million inequivalent Hadamard matrices of order 36, none of them is balanced splittable.

	$b \neq -a$	
	Type 2	
	imprimitive	primitive
parameter relations	$(n, \ell, a, b) = (8rs, 4s, 4s, 0)$ for $r, s \geq 1$	$n = \frac{(\ell-a)(\ell-b)}{\ell+ab-a-b}$, $\ell \equiv a \equiv b \pmod 4$, $a > 0 \geq b$
G	$4sK_{2r}$	$v = n$, $k = \frac{\ell-b+n(b-1)}{b-a}$, $\lambda = \mu + \frac{2(\ell-b)-n}{b-a}$, $\mu = \frac{nb(b-1)}{(b-a)^2}$
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Kronecker product

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_4 = H_2 \otimes H_2 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right]$$

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Repeatedly applying the Kronecker product, Hadamard matrix of order 2^m can be constructed for each $m \geq 1$.

Theorem (Jedwab, Li, Simon (2023))

There exists a $BSHM(8rs, 4s, 4s, 0)$ *in each of the following cases:*

1. *there exist Hadamard matrices of order* $2r$ *and* $4s$
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Remark

Following the first construction above, fix s *such that a Hadamard matrix of order* $4s$ *exists. Set* $r = 2^m$ *for some* $m \geq 1$. *Note that* $n = 8rs = 2^{m+1}\ell$ *is not bounded by* ℓ^2 *as* m *can be arbitrarily large.*

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This observation follows from incorporating the primitive/imprimitive notation of SRG into BSHM.

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parameter relations	$(n, \ell, a, b) = (8rs, 4s, 4s, 0)$ for $r, s \geq 1$	$n = \frac{(\ell-a)(\ell-b)}{\ell+ab-a-b}$, $\ell \equiv a \equiv b \pmod 4$, $a > 0 \geq b$
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Theorem (Jedwab, Li, Simon (2023))

Suppose $H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ *is a* $BSHM(8rs, 4s, 4s, 0)$ *with respect to* H_1 *(Type II and imprimitive). Then the associated SRG is* $4sK_{2r}$*. There exists a Hadamard matrix* L *of order* $4s$*, and the columns of* H *can be reordered so that* $H_1 = \underbrace{\begin{bmatrix} L & L & \dots & L \end{bmatrix}}_{2r}$.

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Question

For primitive BSHM, what structural information is contained the associated SRG?

Result (Known constructions)

Suppose there exist Hadamard matrices of orders n and s. Then there exists:

1. $BSHM(n^2, 2n-2, n-2, -2)$ *for* $n \geq 2$
2. $BSHM(n^2, 2n-1, n-1, -1)$ *for* $n \geq 4$
3. $BSHM(ns, n, n, 0)$ *for* $n \geq 2$
4. $BSHM(2^{2m}, 2^{m-1}(2^m-1), 2^{m-1}, -2^{m-1})$ *for* $m \geq 2$
5. $BSHM(q(q+1), q, q, -1)$ *for* $q \geq 3$, $q \equiv 3 \mod 4$, *where* $q+1$ *is the order of a skew-type Hadamard matrix*
6. $BSHM(4n^2, 2n^2-n, n, -n)$

Most known BSHMs are constructed via Kronecker product. We want to find "primary constructions" that do not depend on Kronecker product.

We proposed a primary construction based on the character table of elementary abelian 2-groups.

Example (Character of elementary abelian 2-groups)

$G = \mathbb{Z}_2 \times \mathbb{Z}_2$,

Each $(a, b) \in G$ induces a character $\chi_{(a,b)}$, for instance

$$\chi_{(1,1)}((0,1)) = (-1)^{0\cdot 1} \cdot (-1)^{1\cdot 1} = 1 \cdot (-1) = -1.$$

Each character $\chi_{(a,b)}$ induces a group homomorphism

$$\begin{aligned} \chi_{(a,b)} : G &\mapsto \{1, -1\} \\ (c, d) &\mapsto (-1)^{ac} \cdot (-1)^{bd} = (-1)^{ac+bd} \end{aligned}$$

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character group: $\widehat{G} = \{\chi_g \mid g \in G\} \cong G$.

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The character table of G is a $|G| \times |G|$ matrix H with rows indexed by elements of G and columns by $\widehat{G}$.

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The character table of an elementary abelian 2-group serves as the underlying Hadamard matrix. To construct a BSHM, it remains to properly split the matrix.

Example (Partial difference set)

Let $G = \mathbb{Z}_2^4$ and $D = \{(0,0,0,1),(0,0,1,0),(0,0,1,1)\}$. The multiset $\{\{x - y \mid x, y \in D, x \neq y\}\}$ contains

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- each element of D exactly 2 times
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D is a $(16,3,2,0)$ partial difference set in G.

Note that $D = \{(0,0,0,1),(0,0,1,0),(0,0,1,1)\}$ is a $(16,3,2,0)$ partial difference set in $G = \mathbb{Z}_2^4$. Let H be the character table of G.

BSHM$(16,3,3,-1)$ H w.r.t. red submatrix

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1\\
1 & .-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1\\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1\\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1\\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1\\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1\\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1\\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1\\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1\\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1\\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1\\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1\\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1
\end{bmatrix}
$$

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- For $n \in \{16, 64, 256\}$ and each plausible parameter set (n, ℓ, a, b), there is an BSHM(n, ℓ, a, b) derived from the partial difference set construction.

5 disjoint partial difference sets in $\mathbb{Z}_2^4$:

$D_1 = \{(0,0,0,1),(0,0,1,0),(0,0,1,1)\}$, $D_2 = \{(0,1,0,0),(1,0,0,0),(1,1,0,0)\}$
$D_3 = \{(0,1,0,1),(1,0,1,0),(1,1,1,1)\}$, $D_4 = \{(0,1,1,0),(1,0,1,1),(1,1,0,1)\}$
$D_5 = \{(0,1,1,1),(1,0,0,1),(1,1,1,0)\}$

BSHM$(16,3,3,-1)$ H w.r.t. multiple submatrices

$$\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
1&1&-1&-1&1&1&-1&-1&1&1&-1&-1&1&1&-1&-1\\
1&.-1&-1&1&1&-1&-1&1&1&-1&-1&1&1&-1&-1&1\\
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1&-1&1&-1&-1&1&-1&1&1&-1&1&-1&-1&1&-1&1\\
1&1&-1&-1&1&1&-1&-1&-1&-1&1&1&-1&-1&1&1\\
1&-1&-1&1&-1&1&1&-1&-1&1&1&-1&1&-1&-1&1\\
1&1&-1&-1&-1&-1&1&1&1&1&-1&-1&-1&-1&1&1\\
1&-1&-1&1&1&-1&-1&1&-1&1&1&-1&-1&1&1&-1\\
1&-1&1&-1&-1&1&-1&1&-1&1&-1&1&1&-1&1&-1\\
1&-1&-1&1&-1&1&1&-1&1&-1&-1&1&-1&1&1&-1\\
1&-1&1&-1&1&-1&1&-1&-1&1&-1&1&-1&1&-1&1\\
1&1&-1&-1&-1&-1&1&1&-1&-1&1&1&1&1&-1&-1
\end{bmatrix}$$

BSHM w.r.t multiple submatrices

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
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1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
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Future Problems

Parameter range of ℓ

For BSHM(n, ℓ, a, b) with $\ell > a$, further narrow down the range $\sqrt{2n} \leq \ell \leq \frac{n}{2}$ or prove the tightness.

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upper bound is nearly tight: BSHM$(4n^2, 2n^2 - n, n, -n)$ exists whenever a Hadamard matrix of order n exists.

Future Problems

Parameter range of ℓ

For BSHM(n, ℓ, a, b) with $\ell > a$, further narrow down the range $\sqrt{2n} \leq \ell \leq \frac{n}{2}$ or prove the tightness.

upper bound is nearly tight: BSHM$(4n^2, 2n^2 - n, n, -n)$ exists whenever a Hadamard matrix of order n exists.

lower bound is less clear: derived from equiangular tight frames and two-distance tight frames in $\mathbb{R}^\ell$.

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- Identify BSHMs from known constructions: there have been plenty of constructions of Hadamard matrices, some of them may already give BSHMs.

Main References

- Kharaghani and Suda, *Discrete Mathematics*, 2019.
- Fickus, Jasper, Mixon, and Peterson, *Applied and Computational Harmonic Analysis*, 2021.
- Jedwab, Li, and Simon, *Electronic Journal of Combinatorics*, 2023.
- Kharaghani and Suda, *Electronic Journal of Combinatorics*, 2023.