Finite bivariate Tratnik functions

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University of Waterloo Virtual Algebraic Graph Theory Seminar
March 31, 2025

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Association schemes

Symmetric association scheme $\mathscr{Z} = (X, \{R_i\}_{i=0}^{N})$.

Adjacency matrices $\{A_i\}_{i=0}^{N}$ are $|X| \times |X|$ 01-matrices.

- $A_0 = \mathbb{I}$
- $\displaystyle\sum_{i=0}^{N} A_i = \mathbb{J}$
- $A_i^t = A_i$
- $\displaystyle A_i A_j = A_j A_i = \sum_{k=0}^{N} p_{ij}^k A_k$
- $A_i \circ A_j = \delta_{ij} A_i$

Principal idempotents $\{E_i\}_{i=0}^N$.

- $E_0 = \frac{1}{|X|}\mathbb{J}$
- $\displaystyle\sum_{i=0}^{N} E_i = \mathbb{I}$
- $E_i^t = E_i$
- $\displaystyle E_i \circ E_j = \frac{1}{|X|}\sum_{k=0}^{N} q_{ij}^k E_k$
- $E_i E_j = \delta_{ij} E_i$

$$A_i = \sum_{j=0}^{N} p_i(j) E_j, \qquad E_i = \frac{1}{|X|}\sum_{j=0}^{N} q_i(j) A_j$$

P-polynomial and Q-polynomial properties

$\mathcal{Z}$ is P-polynomial with respect to the ordering $A_0, A_1, \ldots, A_N$ if there exist polynomials $v_i(x)$ of degree i such that

$$A_i = v_i(A_1) \quad \text{for } i = 0, 1, \ldots, N.$$

Equivalently:

- $\mathcal{Z}$ is metric: $p_{1j}^{j\pm 1} \neq 0$ and $(p_{ij}^k \neq 0) \Rightarrow (|i-j| \leq k \leq i+j)$

$$A_1 A_i = p_{1i}^{i-1} A_{i-1} + p_{1i}^{i} A_i + p_{1i}^{i+1} A_{i+1}$$

- Eigenvalues $p_i(j) = v_i(\theta_j)$, with $\theta_j = p_1(j)$

$\mathcal{Z}$ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_N$ if there exist polynomials $v_i^\star(x)$ of degree i such that (under Hadamard product)

$$|X|E_i = v_i^\star(|X|E_1) \quad \text{for } i = 0, 1, \ldots, N.$$

Equivalent to: cometric, $q_i(j) = v_i^\star(\theta_j^\star)$.

Distance-regular graphs

Connected, undirected graph $G = (X, \Gamma)$.

$\mathrm{dist}(x, y) =$ length of a shortest path between x and y in G.

G is distance-regular if for fixed i and j, the number of vertices $z \in X$ at distance i from a vertex x and at distance j from a vertex y is a non-negative integer p_{ij}^k which depends only on the distance k between x and y.

One-to-one correspondence with P-polynomial association schemes:

$\mathscr{Z}$ is P-polynomial iff (X, R_1) defines a distance-regular graph with distance matrices A_i.

Polynomials

Wilson duality:

$$\frac{p_i(j)}{p_i(0)} = \frac{q_j(i)}{q_j(0)} \quad \xRightarrow{(P \text{ and } Q)\text{-poly}} \quad v_j^\star(\theta_i^\star) = \frac{m_j}{k_i} v_i(\theta_j)$$

Recurrence relation:

$$\theta_x v_i(\theta_x) = b_{i-1} v_{i-1}(\theta_x) + a_i v_i(\theta_x) + c_{i+1} v_{i+1}(\theta_x)$$

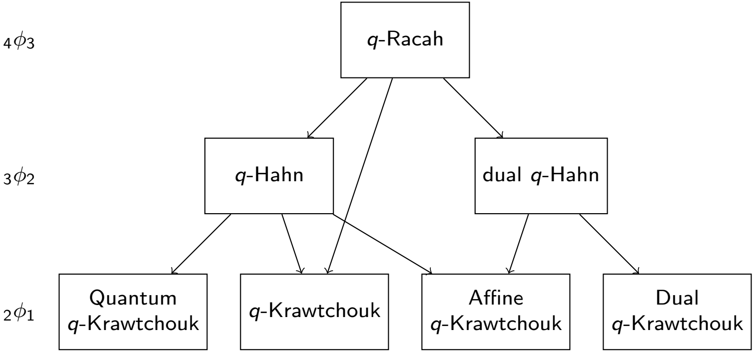
Difference equation:

$$\theta_i^\star v_i(\theta_x) = c_x^\star v_i(\theta_{x-1}) + a_x^\star v_i(\theta_x) + b_x^\star v_i(\theta_{x+1})$$

Orthogonality relation:

$$\sum_{x=0}^{N} m_x v_i(\theta_x) v_j(\theta_x) = |X| k_i \delta_{ij}$$

Leonard's theorem: these polynomials are finite families of the (q-)Askey scheme or Bannai-Ito polynomials.



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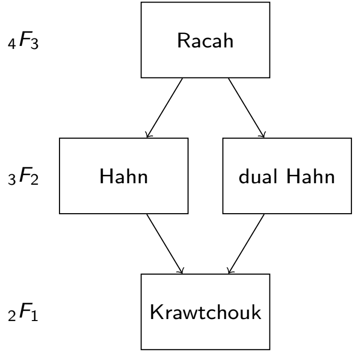


Leonard pairs

Leonard pair $(A, A^\star)$ on vector space V of dimension $N+1$:

- $\exists$ basis $\{u_x\}_{x=0}^N$ such that A is diagonal and $A^\star$ is irreducible tridiagonal:

$$\begin{aligned} Au_x &= \theta_x u_x, \\ A^\star u_x &= b^\star_{x-1} u_{x-1} + a^\star_x u_x + c^\star_{x+1} u_{x+1}, \end{aligned}$$

with $b^\star_{x-1} c^\star_{x+1} \neq 0$.

- $\exists$ basis $\{u^\star_i\}_{i=0}^N$ such that A is irreducible tridiagonal and $A^\star$ is diagonal:

$$\begin{aligned} Au^\star_i &= b_{i-1} u^\star_{i-1} + a_i u^\star_i + c_{i+1} u^\star_{i+1}, \\ A^\star u^\star_i &= \theta^\star_i u^\star_i, \end{aligned}$$

with $a_{i-1} c_{i+1} \neq 0$.

Polynomials arise as coefficients of change of basis.
Terwilliger algebra, Askey–Wilson algebra.

Multivariate generalizations

- Several multivariate generalizations of the polynomials of the (q-)Askey scheme.
- Higher rank algebras (Bannai-Ito, Racah, Askey–Wilson).
- Higher rank Leonard pairs.
  Iliev, Terwilliger
- Bivariate P- and/or Q-polynomial association schemes.
  Bernard, Crampé, Poulain d'Andecy, Vinet, Zaimi
- Multivariate P- and/or Q-polynomial association schemes.
  Bannai, Kurihara, Zhao, Zhu
- Multivariate distance-regular graphs.
  Bernard, Crampé, Vinet, Zaimi, Zhang
- Factorized A_2-Leonard pairs.
  Crampé, Zaimi

Important constituants of m-variate structures:

- Domain $\mathcal{D} \subset \mathbb{N}^m$ with certain properties.

$$\epsilon_1 = (1, 0, 0, \ldots, 0), \quad \epsilon_2 = (0, 1, 0, \ldots, 0), \ldots$$

- Total monomial order $\leq$ on $\mathbb{N}^m$.
- $A_\alpha = v_\alpha(A_{\epsilon_1}, A_{\epsilon_2}, \ldots, A_{\epsilon_m})$, multidegree $\alpha \in \mathcal{D}$.
- Conditions on the boundary of $\mathcal{D}$.
- For graphs, partition of edges and notion of an m-distance.

Multivariate metric condition (for symmetric schemes):

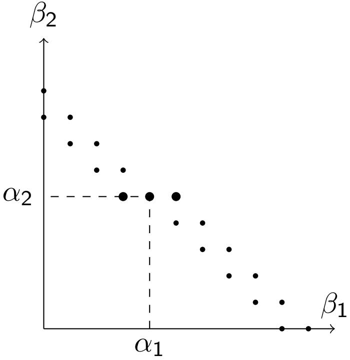
$$p_{\epsilon_i,\alpha}^{\alpha+\epsilon_i} \neq 0 \quad \text{if} \quad \alpha + \epsilon_i \in \mathcal{D} \quad \text{and} \quad p_{\epsilon_i,\alpha}^{\alpha-\epsilon_i} \neq 0 \quad \text{if} \quad \alpha - \epsilon_i \in \mathcal{D},$$
$$p_{\epsilon_i,\alpha}^{\beta} \neq 0 \quad \Rightarrow \quad \beta \leq \alpha + \epsilon_i \quad \text{and} \quad \alpha \leq \beta + \epsilon_i.$$

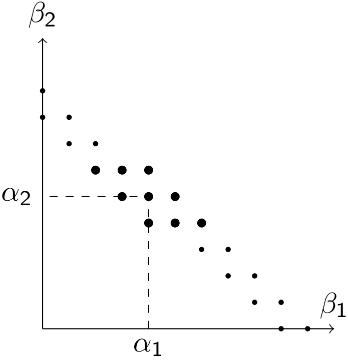
$$x_i v_\alpha(\boldsymbol{x}) = \sum_{\substack{\beta \in \mathcal{D} \\ \beta \leq \alpha + \epsilon_i, \alpha \leq \beta + \epsilon_i}} p_{\epsilon_i,\alpha}^{\beta} v_\beta(\boldsymbol{x}), \qquad i = 1, 2, \ldots, m.$$

(Caution near the boundary!)

$$\alpha = (\alpha_1, \alpha_2), \qquad \beta = (\beta_1, \beta_2)$$

$$p^{\beta}_{\epsilon_i,\alpha} \neq 0 \quad \Rightarrow \quad \beta \leq \alpha + \epsilon_i \quad \text{and} \quad \alpha \leq \beta + \epsilon_i$$





Some examples

1. Direct product.

$A_i = v_i(A_1)$ for $i = 0, \ldots, N$, and $\tilde{A}_j = \tilde{v}_j(\tilde{A}_1)$ for $j = 0, \ldots, \tilde{N}$.
$\mathcal{D} = \{0, 1, \ldots, N\} \times \{0, 1, \ldots, \tilde{N}\}$

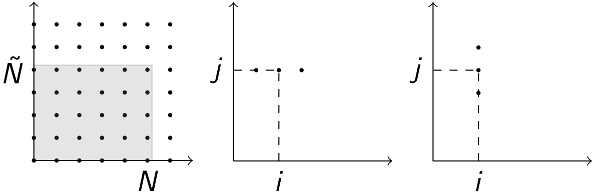
$$A_{ij} := A_i \otimes \tilde{A}_j = v_{ij}(A_{10}, A_{01}), \quad (i,j) \in \mathcal{D}$$

$$v_{ij}(x,y) = v_i(x)\tilde{v}_j(y).$$

Two recurrence relations with three terms each:

$$\theta_x v_{ij}(\theta_x, \tilde{\theta}_y) = p_{1i}^{i-1} v_{i-1,j}(\theta_x, \tilde{\theta}_y) + p_{1i}^{i} v_{ij}(\theta_x, \tilde{\theta}_y) + p_{1i}^{i+1} v_{i+1,j}(\theta_x, \tilde{\theta}_y),$$
$$\tilde{\theta}_y v_{ij}(\theta_x, \tilde{\theta}_y) = \tilde{p}_{1j}^{j-1} v_{i,j-1}(\theta_x, \tilde{\theta}_y) + \tilde{p}_{1j}^{j} v_{ij}(\theta_x, \tilde{\theta}_y) + \tilde{p}_{1j}^{j+1} v_{i,j+1}(\theta_x, \tilde{\theta}_y).$$

Similarly for the difference equations.

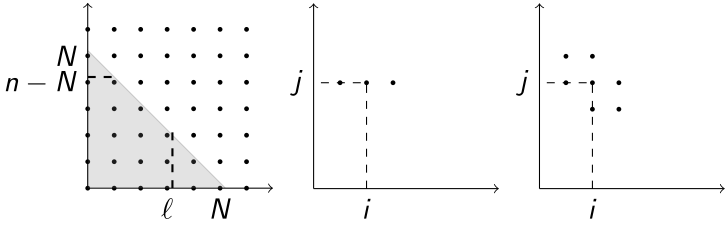


2. Association schemes based on attenuated spaces $\mathcal{A}_q(n, \ell, N)$
Eigenvalues are:

$$T_{ij}(x,y) \propto \begin{bmatrix} N-x \\ j \end{bmatrix}_q \underbrace{{}_3\phi_2\left(\begin{matrix} q^{-i},\ q^{-x},\ 0 \\ q^{-\ell},\ q^{-N+j} \end{matrix} \,\middle|\, q, q\right)}_{\text{affine } q\text{-Krawtchouk}} \underbrace{{}_3\phi_2\left(\begin{matrix} q^{-j},\ q^{-y},\ q^{x+y-n-1} \\ q^{-n+N},\ q^{-N+x} \end{matrix} \,\middle|\, q, q\right)}_{\text{dual } q\text{-Hahn}}$$

3. Non-binary Johnson schemes $J_r(n, N)$
Eigenvalues are:

$$T_{ij}(x,y) \propto \binom{N-x}{j} \underbrace{{}_2F_1\left(\begin{matrix} -i,\ -x \\ -N+j \end{matrix} \,\middle|\, \frac{r-1}{r-2}\right)}_{\text{Krawtchouk}} \underbrace{{}_3F_2\left(\begin{matrix} -j,\ -y,\ x+y-n-1 \\ -n+N,\ -N+x \end{matrix} \,\middle|\, 1\right)}_{\text{dual Hahn}}$$



Plan

Tratnik functions

Two finite families of polynomials $\{R_i(x;\boldsymbol{\rho})\}_i^N$ and $\{\mathcal{R}_j(y;\boldsymbol{r})\}_j^M$, with respective sets of parameters $\boldsymbol{\rho}$ and $\boldsymbol{r}$.

$$T_{i,j}(x,y) = \nu_j(x)\ R_i(x;\boldsymbol{\rho}_j)\ \mathcal{R}_j(y;\boldsymbol{r}_x) = \nu_j(x)\ R_i(x;\boldsymbol{\rho}_j)\ S_y(j;\boldsymbol{\sigma}_x).$$

Bispectrality of $R_i(x;\boldsymbol{\rho})$:

$$\lambda_{x,\boldsymbol{\rho}}\ R_i(x;\boldsymbol{\rho}) = A_{i,\boldsymbol{\rho}}\ R_{i+1}(x;\boldsymbol{\rho}) - (A_{i,\boldsymbol{\rho}} + C_{i,\boldsymbol{\rho}})\ R_i(x;\boldsymbol{\rho}) + C_{i,\boldsymbol{\rho}}\ R_{i-1}(x;\boldsymbol{\rho}),$$
$$\mu_{i,\boldsymbol{\rho}}\ R_i(x;\boldsymbol{\rho}) = B_{x,\boldsymbol{\rho}}\ R_i(x+1;\boldsymbol{\rho}) - (B_{x,\boldsymbol{\rho}} + D_{x,\boldsymbol{\rho}})\ R_i(x;\boldsymbol{\rho}) + D_{x,\boldsymbol{\rho}}\ R_i(x-1;\boldsymbol{\rho}),$$

with $C_{0,\boldsymbol{\rho}} = D_{0,\boldsymbol{\rho}} = 0$.

Bispectrality of $S_y(j;\boldsymbol{\sigma})$:

$$\ell_{j,\boldsymbol{\sigma}}\ S_y(j;\boldsymbol{\sigma}) = \mathcal{A}_{y,\boldsymbol{\sigma}}\ S_{y+1}(j;\boldsymbol{\sigma}) - (\mathcal{A}_{y,\boldsymbol{\sigma}} + \mathcal{C}_{y,\boldsymbol{\sigma}})\ S_y(j;\boldsymbol{\sigma}) + \mathcal{C}_{y,\boldsymbol{\sigma}}\ S_{y-1}(j;\boldsymbol{\sigma}),$$
$$m_{y,\boldsymbol{\sigma}}\ S_y(j;\boldsymbol{\sigma}) = \mathcal{B}_{j,\boldsymbol{\sigma}}\ S_y(j+1;\boldsymbol{\sigma}) - (\mathcal{B}_{j,\boldsymbol{\sigma}} + \mathcal{D}_{j,\boldsymbol{\sigma}})\ S_y(j;\boldsymbol{\sigma}) + \mathcal{D}_{j,\boldsymbol{\sigma}}\ S_y(j-1;\boldsymbol{\sigma}),$$

with $\mathcal{C}_{0,\boldsymbol{\sigma}} = 0$ and $\mathcal{D}_{0,\boldsymbol{\sigma}} = 0$.

First recurrence relation

$$T_{i,j}(x,y) = \nu_j(x)\ R_i(x;\boldsymbol{\rho}_j)\ S_y(j;\boldsymbol{\sigma}_x)$$

From the recurrence relation of $R_i(x;\boldsymbol{\rho}_j)$,

$$\lambda_{x,\boldsymbol{\rho}_j}\ T_{i,j}(x,y) = A_{i,\boldsymbol{\rho}_j}\ T_{i+1,j}(x,y) - (A_{i,\boldsymbol{\rho}_j} + C_{i,\boldsymbol{\rho}_j})\ T_{i,j}(x,y) + C_{i,\boldsymbol{\rho}_j}\ T_{i-1,j}(x,y)\,.$$

In order for this to be considered as a recurrence relation for $T_{ij}(x,y)$,

$$\lambda_{x,\boldsymbol{\rho}_j} = \tau_j\,\widetilde{\lambda}_x + \tau_j'.$$

With suitable renormalizations, we can chose

$$\lambda_{x,\boldsymbol{\rho}_j} = \lambda_x, \qquad \lambda_0 = 0.$$

We get a first recurrence relation for $T_{i,j}(x,y)$ (with 3 terms):

$$\lambda_x\ T_{i,j}(x,y) = A_{i,\boldsymbol{\rho}_j}\ T_{i+1,j}(x,y) - (A_{i,\boldsymbol{\rho}_j} + C_{i,\boldsymbol{\rho}_j})\ T_{i,j}(x,y) + C_{i,\boldsymbol{\rho}_j}\ T_{i-1,j}(x,y)\,.$$

First (q-)difference equation

$$T_{i,j}(x,y) = \nu_j(x)\ R_i(x;\boldsymbol{\rho}_j)\ S_y(j;\boldsymbol{\sigma}_x)$$

From the recurrence relation of $S_y(j;\boldsymbol{\sigma}_x)$,

$$\ell_{j,\boldsymbol{\sigma}_x}\ T_{i,j}(x,y) = \mathcal{A}_{y,\boldsymbol{\sigma}_x}\ T_{i,j}(x,y+1) - (\mathcal{A}_{y,\boldsymbol{\sigma}_x} + \mathcal{C}_{y,\boldsymbol{\sigma}_x})\ T_{i,j}(x,y) + \mathcal{C}_{y,\boldsymbol{\sigma}_x}\ T_{i,j}(x,y-1)\,.$$

Similarly, we take

$$\ell_{j,\boldsymbol{\sigma}_x} = \ell_j, \qquad \ell_0 = 0,$$

and we get a first (q-)difference equation for $T_{i,j}(x,y)$ (with 3 terms):

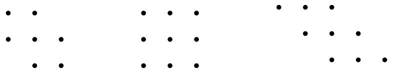
$$\ell_j\ T_{i,j}(x,y) = \mathcal{A}_{y,\boldsymbol{\sigma}_x}\ T_{i,j}(x,y+1) - (\mathcal{A}_{y,\boldsymbol{\sigma}_x} + \mathcal{C}_{y,\boldsymbol{\sigma}_x})\ T_{i,j}(x,y) + \mathcal{C}_{y,\boldsymbol{\sigma}_x}\ T_{i,j}(x,y-1)\,.$$

Second recurrence relation

We want:

$$w_{x,y}\, T_{i,j}(x,y) = \sum_{(\epsilon,\epsilon')\in\mathcal{S}} \Phi_{i,j}^{\epsilon,\epsilon'}\; T_{i+\epsilon,j+\epsilon'}(x,y)\,.$$

- $\mathcal{S}_{A_2} = \{(0,0),(0,1),(0,-1),(1,0),(-1,0),(-1,1),(1,-1)\}$.
- $\mathcal{S}_{B_2} = \{(0,0),(0,1),(0,-1),(1,0),(-1,0),(-1,1),(1,-1),(1,1),(-1,-1)\}$.
- $\mathcal{S}_{B_2'} = \{(0,0),(0,1),(0,-1),(1,0),(-1,0),(-1,1),(2,-1),(1,-1),(-2,1)\}$.



A_2 $\qquad$ B_2 $\qquad$ B_2'

We consider

$$w_{x,y} = k_{x,y}(m_{y,\boldsymbol{\sigma}_x} + k'_{x,y}).$$

From the (q-)difference equation of $S_y(j;\boldsymbol{\sigma}_x)$,

$$w_{x,y}\; T_{i,j}(x,y) = k_{x,y}\nu_j(x) R_i(x;\boldsymbol{\rho}_j)$$
$$\times\Big(\mathcal{B}_{j,\boldsymbol{\sigma}_x} S_y(j+1;\boldsymbol{\sigma}_x) - (\mathcal{B}_{j,\boldsymbol{\sigma}_x} + \mathcal{D}_{j,\boldsymbol{\sigma}_x} - k'_{x,y})\; S_y(j;\boldsymbol{\sigma}_x) + \mathcal{D}_{j,\boldsymbol{\sigma}_x}\; S_y(j-1;\boldsymbol{\sigma}_x)\Big).$$

We need:

$$k_{x,y}\frac{\nu_j(x)}{\nu_{j+1}(x)}\mathcal{B}_{j,\boldsymbol{\sigma}_x} R_i(x;\boldsymbol{\rho}_j) = \sum_{\epsilon} \Phi_{i,j}^{\epsilon,1}\; R_{i+\epsilon}(x;\boldsymbol{\rho}_{j+1})\,,$$
$$-k_{x,y}(\mathcal{B}_{j,\boldsymbol{\sigma}_x} + \mathcal{D}_{j,\boldsymbol{\sigma}_x} - k'_{x,y}) R_i(x;\boldsymbol{\rho}_j) = \sum_{\epsilon=0,1,-1} \Phi_{i,j}^{\epsilon,0}\; R_{i+\epsilon}(x;\boldsymbol{\rho}_j)\,,$$
$$k_{x,y}\frac{\nu_j(x)}{\nu_{j-1}(x)}\mathcal{D}_{j,\boldsymbol{\sigma}_x} R_i(x;\boldsymbol{\rho}_j) = \sum_{\epsilon} \Phi_{i,j}^{\epsilon,-1} R_{i+\epsilon}(x;\boldsymbol{\rho}_{j-1})\,.$$

Can see that $k_{x,y} = k_x$ and $k'_{x,y} = k'_x$.

Second (q-)difference equation

We want

$$\omega_{i,j}\; T_{i,j}(x,y) = \sum_{(\epsilon,\epsilon')\in\mathcal{S}'} \Psi_{y,x}^{\epsilon,\epsilon'}\; T_{i,j}(x+\epsilon', y+\epsilon),$$

with an eigenvalue

$$\omega_{i,j} = \kappa_{j,i}(\mu_{i,\boldsymbol{\rho}_j} + \kappa'_{j,i}).$$

Note that $\mathcal{S}'$ can be of different type than $\mathcal{S}$.

From the (q-)difference relation of $R_i(x;\boldsymbol{\rho}_j)$,

$$\omega_{i,j}\ T_{i,j}(x,y) = \kappa_{j,i}\nu_j(x)S_y(s;\boldsymbol{\sigma}_x)$$
$$\times\Big(B_{x,\boldsymbol{\rho}_j}\ R_i(x+1;\boldsymbol{\rho}_j) - (B_{x,\boldsymbol{\rho}_j} + D_{x,\boldsymbol{\rho}_j} - \kappa'_{j,i})\ R_i(x;\boldsymbol{\rho}_j) + D_{x,\boldsymbol{\rho}_j}\ R_i(x-1;\boldsymbol{\rho}_j)\Big).$$

We need:

$$\kappa_{j,i}\frac{\nu_j(x)}{\nu_j(x+1)}B_{x,\boldsymbol{\rho}_j}S_y(j;\boldsymbol{\sigma}_x) = \sum_{\epsilon}\Psi^{\epsilon,1}_{y,x}S_{y+\epsilon}(j;\boldsymbol{\sigma}_{x+1})\,,$$
$$-\kappa_{j,i}(B_{x,\boldsymbol{\rho}_j} + D_{x,\boldsymbol{\rho}_j} - \kappa'_{j,i})S_y(j;\boldsymbol{\sigma}_x) = \sum_{\epsilon=0,\pm1}\Psi^{\epsilon,0}_{y,x}S_{y+\epsilon}(j;\boldsymbol{\sigma}_x)\,,$$
$$\kappa_{j,i}\frac{\nu_j(x)}{\nu_j(x-1)}D_{x,\boldsymbol{\rho}_j}S_y(j;\boldsymbol{\sigma}_x) = \sum_{\epsilon}\Psi^{\epsilon,-1}_{y,x}S_{y+\epsilon}(j;\boldsymbol{\sigma}_{x-1})\,.$$

Can see that $\kappa_{j,i} = \kappa_j$ and $\kappa'_{j,i} = \kappa'_j$.

Middle equations should correspond to the recurrence relations of $R_i(x;\boldsymbol{\rho}_j)$ and $S_y(s;\boldsymbol{\sigma}_x)$, up to global normalizations and some additive terms.

$$\underbrace{-k_x(\mathcal{B}_{j,\boldsymbol{\sigma}_x}+\mathcal{D}_{j,\boldsymbol{\sigma}_x}-k'_x)}_{=\varphi_j\lambda_x+\psi_j} R_i(x;\boldsymbol{\rho}_j) = \sum_{\epsilon=0,1,-1} \Phi^{\epsilon,0}_{i,j}\; R_{i+\epsilon}(x;\boldsymbol{\rho}_j)\,,$$

$$\underbrace{-\kappa_j(B_{x,\boldsymbol{\rho}_j}+D_{x,\boldsymbol{\rho}_j}-\kappa'_j)}_{=f_x\ell_j+g_x} S_y(j;\boldsymbol{\sigma}_x) = \sum_{\epsilon=0,\pm1} \Psi^{\epsilon,0}_{y,x} S_{y+\epsilon}(j;\boldsymbol{\sigma}_x).$$

Other relations are contiguity relations.

$$k_x\frac{\nu_j(x)}{\nu_{j+1}(x)}\mathcal{B}_{j,\boldsymbol{\sigma}_x} R_i(x;\boldsymbol{\rho}_j) = \sum_{\epsilon} \Phi^{\epsilon,1}_{i,j}\; R_{i+\epsilon}(x;\boldsymbol{\rho}_{j+1})\,,$$
$$k_x\frac{\nu_j(x)}{\nu_{j-1}(x)}\mathcal{D}_{j,\boldsymbol{\sigma}_x} R_i(x;\boldsymbol{\rho}_j) = \sum_{\epsilon} \Phi^{\epsilon,-1}_{i,j} R_{i+\epsilon}(x;\boldsymbol{\rho}_{j-1})\,.$$

Contiguity relations

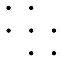
Classify solutions of equations of the form

$$\lambda^{+}_{x,\boldsymbol{\rho}}\, R_i(x;\boldsymbol{\rho}) = \sum_{\epsilon\in\mathscr{S}} \Phi_i^{\epsilon,+}\, R_{i+\epsilon}(\overline{x};\overline{\boldsymbol{\rho}}),$$
$$\lambda^{-}_{x,\boldsymbol{\rho}}\, R_i(\overline{x};\overline{\boldsymbol{\rho}}) = \sum_{-\epsilon\in\mathscr{S}} \Phi_i^{\epsilon,-}\, R_{i+\epsilon}(x;\boldsymbol{\rho}).$$

$\overline{x} = x + \eta$, $\eta \in \{0, +1, -1\}$,
$\overline{\boldsymbol{\rho}}$ is a set of modified parameters,
$\mathscr{S}$ is one of: $\{0, -1\}$, $\{0, -1, 1\}$ or $\{0, -1, -2\}$,
$\overline{N}$ is chosen among $\{N, N+1, N-1, N-2, N+2\}$.
We will require $\lambda_{x,\boldsymbol{\rho}} = \zeta\lambda_{\overline{x},\overline{\boldsymbol{\rho}}} - \xi$.

Related to Christoffel and Geronimus transformations.
Some previous work done by R. Oste and J. Van der Jeugt (2016).
Ongoing work with N. Crampé, L. Morey and L. Vinet.

Type A_2



$$X_{i,\rho} := A_{i-1,\rho} C_{i,\rho}\,, \qquad Y_{i,\rho} := -(A_{i,\rho} + C_{i,\rho}).$$

Definition

The constraints $(\mathfrak{C}_{A_2})$ are defined by the requirement that the following expressions

$$\zeta^{2i} \frac{\zeta Y_{i,\overline{\rho}} - Y_{i,\rho} - \xi}{X_{i+1,\rho} - \zeta^2 X_{i,\overline{\rho}}} \prod_{k=1}^{i} \frac{X_{k,\overline{\rho}}}{X_{k,\rho}}\,,$$

$$\zeta^{2i} \frac{X_{i+1,\rho} - \zeta^2 X_{i+1,\overline{\rho}}}{(\zeta Y_{i,\overline{\rho}} - Y_{i+1,\rho} - \xi) X_{i+1,\rho}} \prod_{k=1}^{i} \frac{X_{k,\overline{\rho}}}{X_{k,\rho}}\,,$$

are equal and independent of i for any $i \geq 0$.

Proposition

The polynomials $R_i(x;\boldsymbol{\rho})$ chosen such that $\lambda_{x,\boldsymbol{\rho}} = \zeta\lambda_{\overline{x},\overline{\boldsymbol{\rho}}} - \xi$ satisfy the contiguity relation of type A_2

$$\lambda^+_{x,\boldsymbol{\rho}} R_i(x;\boldsymbol{\rho}) = \Phi_i^{0,+} R_i(\overline{x};\overline{\boldsymbol{\rho}}) + \Phi_i^{-1,+} R_{i-1}(\overline{x};\overline{\boldsymbol{\rho}}), \qquad i \geq 0,$$

(with the convention $\Phi_0^{-1,+} = 0$ and with $\Phi_0^{0,+} \neq 0$), if and only if the coefficients are given by (up to a global normalization)

$$\lambda^+_{x,\boldsymbol{\rho}} = 1,$$

$$\Phi_i^{0,+} = \zeta^i \prod_{k=0}^{i-1} \frac{A_{k,\overline{\boldsymbol{\rho}}}}{A_{k,\boldsymbol{\rho}}}, \qquad i \geq 0,$$

$$\Phi_i^{-1,+} = \zeta^{-i+1} \left(1 - \frac{\zeta A_{0,\overline{\boldsymbol{\rho}}} + \xi}{A_{0,\boldsymbol{\rho}}}\right) \prod_{k=1}^{i-1} \frac{C_{k+1,\boldsymbol{\rho}}}{C_{k,\overline{\boldsymbol{\rho}}}}, \qquad i \geq 1,$$

and the constraints $(\mathfrak{C}_{A_2})$ are satisfied.

Proposition

The polynomials $R_i(x;\boldsymbol{\rho})$ chosen such that $\lambda_{x,\boldsymbol{\rho}} = \zeta\lambda_{\overline{x},\overline{\boldsymbol{\rho}}} - \xi$ satisfy the contiguity relation of type A_2

$$\lambda^-_{x,\boldsymbol{\rho}} R_i(\overline{x};\overline{\boldsymbol{\rho}}) = \Phi^{1,-}_i R_{i+1}(x;\boldsymbol{\rho}) + \Phi^{0,-}_i R_i(x;\boldsymbol{\rho}), \qquad i \geq 0,$$

(with $\Phi^{0,-}_0 \neq 0$) if and only if the coefficients are given by (up to a global normalization)

$$\lambda^-_{x,\boldsymbol{\rho}} = (A_{0,\boldsymbol{\rho}} - \zeta A_{0,\overline{\boldsymbol{\rho}}} - \xi)\left(\frac{\lambda_{x,\boldsymbol{\rho}}}{A_{0,\boldsymbol{\rho}}} + 1\right) + C_{1,\boldsymbol{\rho}}\,,$$

$$\Phi^{1,-}_i = (A_{0,\boldsymbol{\rho}} - \zeta A_{0,\overline{\boldsymbol{\rho}}} - \xi)\zeta^{-i}\prod_{k=1}^{i}\frac{A_{k,\boldsymbol{\rho}}}{A_{k-1,\overline{\boldsymbol{\rho}}}}\,, \qquad i \geq 0,$$

$$\Phi^{0,-}_i = \zeta^i C_{1,\boldsymbol{\rho}}\prod_{k=1}^{i}\frac{C_{k,\overline{\boldsymbol{\rho}}}}{C_{k,\boldsymbol{\rho}}}\,, \qquad i \geq 0,$$

and the constraints $(\mathfrak{C}_{A_2})$ are satisfied.

Type B_2

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

Similar results, but more involved expressions...

From two relations of type A_2

$$\begin{aligned} \lambda^{+}_{x,\boldsymbol{\rho}} R_i(x;\boldsymbol{\rho}) &= \Phi^{0,+}_i R_i(\overline{x};\overline{\boldsymbol{\rho}}) + \Phi^{-1,+}_i R_{i-1}(\overline{x};\overline{\boldsymbol{\rho}})\,, \\ \widetilde{\lambda}^{-}_{\widetilde{x},\widetilde{\boldsymbol{\rho}}} R_i(\overline{x};\overline{\boldsymbol{\rho}}) &= \widetilde{\Phi}^{1,-}_i R_{i+1}(\widetilde{x};\widetilde{\boldsymbol{\rho}}) + \widetilde{\Phi}^{0,-}_i R_i(\widetilde{x};\widetilde{\boldsymbol{\rho}})\,, \end{aligned}$$

one can obtain a relation of type B_2

$$\begin{aligned} &\widetilde{\lambda}^{-}_{\widetilde{x},\widetilde{\boldsymbol{\rho}}} \lambda^{+}_{x,\boldsymbol{\rho}} R_i(x;\boldsymbol{\rho}) \\ &= \Phi^{0,+}_i \widetilde{\Phi}^{1,-}_i R_{i+1}(\widetilde{x};\widetilde{\boldsymbol{\rho}}) + (\Phi^{0,+}_i \widetilde{\Phi}^{0,-}_i + \Phi^{-1,+}_i \widetilde{\Phi}^{1,-}_{i-1}) R_i(\widetilde{x};\widetilde{\boldsymbol{\rho}}) + \Phi^{-1,+}_i \widetilde{\Phi}^{0,-}_{i-1} R_{i-1}(\widetilde{x};\widetilde{\boldsymbol{\rho}}). \end{aligned}$$

Type B_2'

$$\begin{matrix} \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \end{matrix}$$

Similar results, but more involved expressions...

From two relations of type A_2

$$\begin{aligned} \lambda^{+}_{x,\boldsymbol{\rho}} R_i(x;\boldsymbol{\rho}) &= \Phi_i^{0,+} R_i(\overline{x};\overline{\boldsymbol{\rho}}) + \Phi_i^{-1,+} R_{i-1}(\overline{x};\overline{\boldsymbol{\rho}})\,, \\ \widetilde{\lambda}^{+}_{\overline{x},\overline{\boldsymbol{\rho}}} R_i(\overline{x};\overline{\boldsymbol{\rho}}) &= \widetilde{\Phi}_i^{0,+} R_i(\widetilde{x};\widetilde{\boldsymbol{\rho}}) + \widetilde{\Phi}_i^{-1,+} R_{i-1}(\widetilde{x};\widetilde{\boldsymbol{\rho}})\,, \end{aligned}$$

one can obtain a relation of type B_2'

$$\widetilde{\lambda}^{+}_{\overline{x},\overline{\boldsymbol{\rho}}} \lambda^{+}_{x,\boldsymbol{\rho}} R_i(x;\boldsymbol{\rho}) =$$

$$\Phi_i^{0,+}\widetilde{\Phi}_i^{0,+} R_i(\widetilde{x};\widetilde{\boldsymbol{\rho}}) + (\Phi_i^{0,+}\widetilde{\Phi}_i^{-1,+} + \Phi_i^{-1,+}\widetilde{\Phi}_{i-1}^{0,+}) R_{i-1}(\widetilde{x};\widetilde{\boldsymbol{\rho}}) + \Phi_i^{-1,+}\widetilde{\Phi}_{i-1}^{-1,+} R_{i-2}(\widetilde{x};\widetilde{\boldsymbol{\rho}}).$$

q-Racah case

$$R_i^{(qR)}(x;\boldsymbol{\rho}=\alpha,\beta,\gamma,N,q) = {}_4\phi_3\left(\begin{matrix} q^{-i},\ \alpha\beta q^{i+1},\ q^{-x},\ \gamma q^{x-N} \\ \alpha q,\ \beta\gamma q,\ q^{-N}\end{matrix}\ \middle|\ q;q\right).$$

$\overline{x} = x + \eta$.

Type A_2 solutions:

(I) $\eta = 0\,,\quad \overline{\alpha} = \alpha\,,\quad \overline{\beta} = q\beta\,,\quad \overline{\gamma} = \gamma/q\,,\quad \overline{N} = N-1\,,\quad q^{\overline{x}} = q^x;$

(II) $\eta = 0\,,\quad \overline{\alpha} = \alpha\,,\quad \overline{\beta} = q\beta\,,\quad \overline{\gamma} = \gamma\,,\quad \overline{N} = N\,,\quad q^{\overline{x}} = q^x;$

(III) $\eta = -1\,,\quad \overline{\alpha} = q\alpha\,,\quad \overline{\beta} = \beta\,,\quad \overline{\gamma} = q\gamma\,,\quad \overline{N} = N-1\,,\quad q^{\overline{x}} = q^{x-1};$

(IV) $\eta = 0\,,\quad \overline{\alpha} = q\alpha\,,\quad \overline{\beta} = \beta\,,\quad \overline{\gamma} = \gamma\,,\quad \overline{N} = N\,,\quad q^{\overline{x}} = q^x;$

All solutions of type B_2 or B_2' are obtained from the solutions of type A_2.
Similar results for other families of (q-)Askey scheme.
Bannai-Ito: no A_2 type relations, unless we consider “complementary” BI.

Back to Tratnik functions

$$T_{i,j}(x,y) = \nu_j(x)\ R_i(x;\boldsymbol{\rho}_j)\ S_y(j;\boldsymbol{\sigma}_x).$$

$$\begin{aligned}
&-k_x(\mathcal{B}_{j,\boldsymbol{\sigma}_x} + \mathcal{D}_{j,\boldsymbol{\sigma}_x} - k'_x) = \varphi_j \lambda_x + \psi_j\,,\\
&-\kappa_j(B_{x,\boldsymbol{\rho}_j} + D_{x,\boldsymbol{\rho}_j} - \kappa'_j) = f_x \ell_j + g_x\,,
\end{aligned}$$

$$\begin{aligned}
k_x \frac{\nu_j(x)}{\nu_{j+1}(x)} \mathcal{B}_{j,\boldsymbol{\sigma}_x} R_i(x;\boldsymbol{\rho}_j) &= \sum_\epsilon \Phi^{\epsilon,1}_{i,j}\ R_{i+\epsilon}(x;\boldsymbol{\rho}_{j+1})\,,\\
k_x \frac{\nu_j(x)}{\nu_{j-1}(x)} \mathcal{D}_{j,\boldsymbol{\sigma}_x} R_i(x;\boldsymbol{\rho}_j) &= \sum_\epsilon \Phi^{\epsilon,-1}_{i,j} R_{i+\epsilon}(x;\boldsymbol{\rho}_{j-1})\,,\\
\kappa_j \frac{\nu_j(x)}{\nu_j(x+1)} B_{x,\boldsymbol{\rho}_j} S_y(j;\boldsymbol{\sigma}_x) &= \sum_\epsilon \Psi^{\epsilon,1}_{y,x} S_{y+\epsilon}(j;\boldsymbol{\sigma}_{x+1})\,,\\
\kappa_j \frac{\nu_j(x)}{\nu_j(x-1)} D_{x,\boldsymbol{\rho}_j} S_y(j;\boldsymbol{\sigma}_x) &= \sum_\epsilon \Psi^{\epsilon,-1}_{y,x} S_{y+\epsilon}(j;\boldsymbol{\sigma}_{x-1})\,.
\end{aligned}$$

Strategy

- Choose a pair (P, Q) of families of polynomials in the (q-)Askey scheme so that

$$T_{i,j}(x,y) = \nu_j(x)\ R_i^{(P)}(x;\boldsymbol{\rho}_j)\ R_y^{(Q)}(j;\boldsymbol{\sigma}_x)\,.$$

- Constraints due to middle equations (recurrence relations).
- Use classification of contiguity recurrence relations to get constraints on parameters $\boldsymbol{\rho}_j$ and $\boldsymbol{\sigma}_x$, and expressions for the coefficients.
- Compatibility constraints for $\nu_j(x)$.

Preliminary results: A_2/A_2 Tratnik functions (polynomials)

Triangular domains ($q = 1$)
$i + j \leq N$ and $x + y \leq N$:

$$T_{i,j}(x,y) = \frac{(x-N)_j}{(-N)_j}\ R_i^{(H)}(x;\alpha,\beta+j,N-j)R_j^{(dH)}(y;a,b+x,N-x)$$

$$T_{i,j}(x,y) = \frac{(x-N)_j}{(-N)_j}\ R_i^{(K)}(x;\alpha,N-j)R_j^{(dH)}(y;a,b+x,N-x)$$

$$T_{i,j}(x,y) = \frac{(x-N)_j}{(-N)_j}\ R_i^{(H)}(x;\alpha,\beta+j,N-j)R_j^{(K)}(y;a,N-x)$$

$$T_{i,j}(x,y) = \frac{(x-N)_j}{(-N)_j}\ R_i^{(K)}(x;\alpha,N-j)R_j^{(K)}(y;a,N-x)$$

Consistent with polynomials appearing in factorized A_2-Leonard pairs.

Rectangular domains ($q = 1$)
$i, x \leq N$ and $j, y \leq M$:

$$T_{i,j}(x,y) = \frac{(\alpha+1+x)_j}{(\alpha+1)_j}\ R_i^{(H)}(x;\alpha+j,\beta,N)R_j^{(dH)}(y;\alpha+x,b,M)$$
$$T_{i,j}(x,y) = R_i^{(H)}(x;\alpha,\beta+j,N)R_j^{(dH)}(y;a,x-M-1-\beta,M)$$

Truncated triangular domains ($q = 1$)
$i + j \leq M$, $x + y \leq M$ and $i, x \leq N$:

$$T_{i,j}(x,y) = \frac{(x-M)_j}{(-M)_j}\ R_i^{(H)}(x;-M-1+j,\beta,N)R_j^{(dH)}(y;a,b+x,M-x)$$
$$T_{i,j}(x,y) = \frac{(x-N)_j}{(-N)_j}\ R_i^{(H)}(x;\alpha,\beta+j,N-j)R_j^{(dH)}(y;-N+x-1,b,M)$$
$$T_{i,j}(x,y) = \frac{(x-M)_j}{(-M)_j}\ R_i^{(H)}(x;-M-1+j,\beta,N)R_j^{(K)}(y;a,M-x)$$
$$T_{i,j}(x,y) = \frac{(x-N)_j}{(-N)_j}\ R_i^{(K)}(x;\alpha,N-j)R_j^{(dH)}(y;-N+x-1,b,M)$$

Conclusion

- Classify (finite) bivariate functions of Tratnik type which satisfy certain bispectral properties.
- Consider all combinations of univariate polynomials, and combinations of types A_2, B_2 or B_2' for the bispectrality.
- Solve constraints in order for the bispectrality to be satisfied.
- Possibilities with $\overline{x} \neq x$?
- Boundaries of domains.
- Study the structure of the associated Leonard pairs of rank 2.