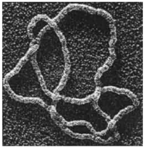
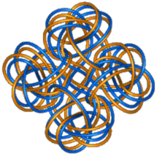
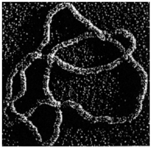
Parameter constraints for distance-regular graphs that afford spin models







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September 15, 2023

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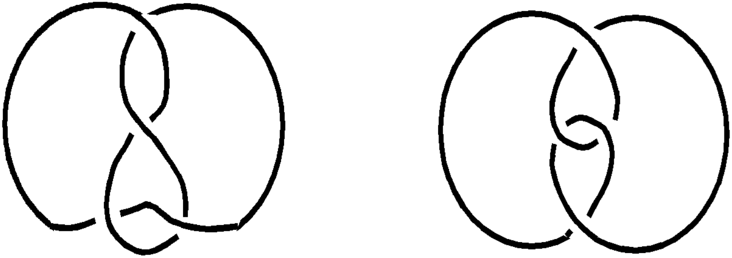
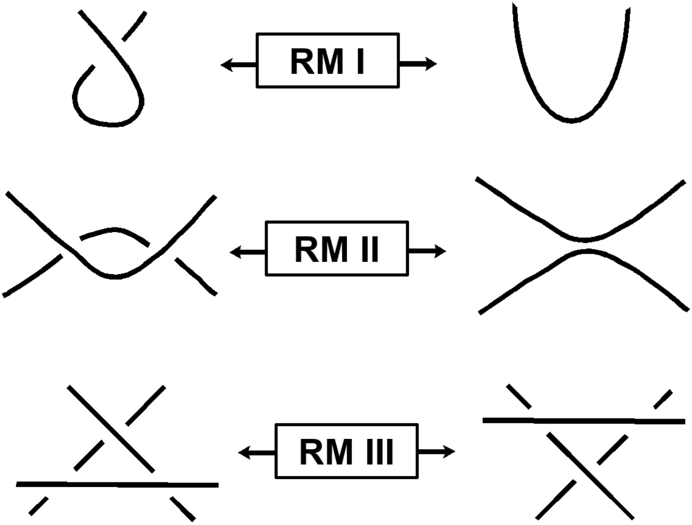
$$-1/q^{(D-1)/2} < \eta < -1/q^{D/2} \qquad \text{or} \qquad -1/q^{D-1} < \eta < 0. \tag{1}$$

Overview - How to Tell if Two Diagrams are Same Knot?



Overview - Do They Differ by Reidemeister Moves?

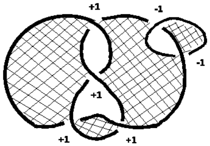


Overview - Associate the Diagrams with Graphs!

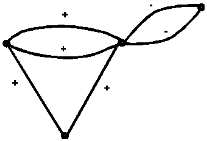
Construction of Tait graph:

- Given a link diagram with signed crossings

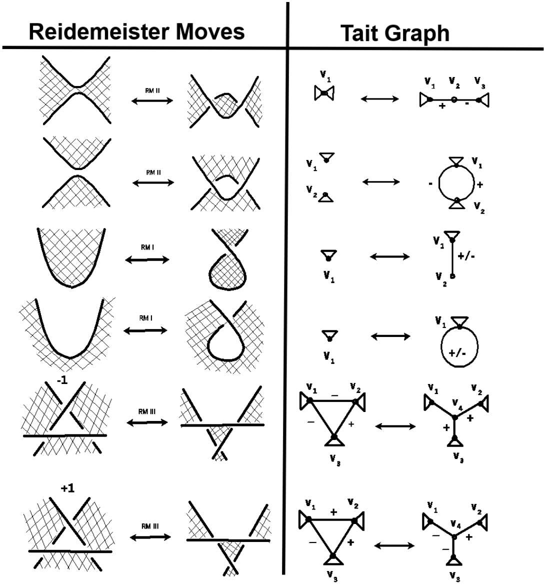
- Two-color the diagram



- Construct graph



Overview - How Do Reidemeister Moves Affect Graph?



Overview - Use a Special Kind of Matrix W

A **spin model** is a symmetric *n x n* matrix W with entries in $\mathrm{Mat}_X(\mathbb{C})$ that satisfies the following **invariance equations** $\forall\ a,b,c \in X$:

Type II :

$$\sum_{x \in X} W^{+}_{a,x}\, W^{-}_{b,x} = n\, \delta_{a,b}$$

Type III:

$$\sum_{x \in X} W^{+}_{a,x}\, W^{+}_{b,x}\, W^{-}_{c,x} = \sqrt{n}\; W^{+}_{a,b}\, W^{-}_{a,c}\, W^{-}_{b,c}$$

Overview - Use W to Compute Z_W for Each Diagram

Given:

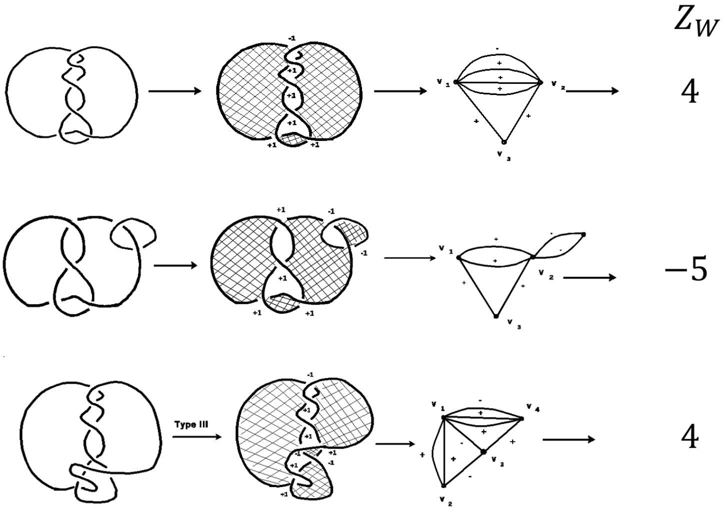
- W, a spin model in $\mathrm{Mat}_X(\mathbb{C})$ where $n = |X|$.
- L be a link diagram, $\mathcal{L}_L$ the Tait graph with vertices V.

Then:

- a **state** is a function σ: V → X.
- the **partition function** is defined to be

$$Z_W = \left(\frac{1}{\sqrt{n}}\right)^{|V|-1} \sum_{\substack{states \\ \sigma : V \to X}} \prod_{\substack{edges \\ v, v' \in \mathcal{L}_L}} W^{\pm}_{\sigma(v), \sigma(v')}$$

Overview - If Z_W different, not same! If Z_W same...?



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• Recently (very), Terwilliger & Nomura announced new results! Using Leonard pairs, they show that whether a DRG to afford a spin model is equivalent to the existence of a certain central element Z in the Terwilliger algebra, and they show how to construct W from Z.

Let's Begin! Define Spin models

Let X be a nonempty finite set.

A **spin model** on X is a symmetric matrix $W \in \mathrm{Mat}_X(\mathbb{C})$ with non-zero entries such that for all $a, b, c \in X$:

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$$\sum_{y \in X} W_{yb}(W_{yc})^{-1} = |X|\delta_{bc}, \tag{2}$$

$$\sum_{y \in X} W_{ya}W_{yb}(W_{yc})^{-1} = L\, W_{ab}(W_{ac})^{-1}(W_{cb})^{-1}, \tag{3}$$

for some $L \in \mathbb{R}$ such that $L^2 = |X|$.

Nomura Algebra

Let W denote a spin model on X.

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$N(W)$ is a subalgebra of $\mathrm{Mat}_X(\mathbb{C})$. Jaeger showed in 1998 that $W \in N(W)$. We refer to $N(W)$ as the **Nomura algebra** of W.

Distance-regular graphs (DRGs)

Let Γ denote a finite, connected, undirected simple graph, with vertex set X, distance function ∂, and diameter D. For each $x \in X$ and $i \in \mathbb{Z}$, set

$$\Gamma_i(x) := \{y \in X \mid \partial(x, y) = i\}.$$

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We say Γ is **distance-regular**, with **intersection numbers** p^h_{ij}, whenever for all integers h, i, j and all $x, y \in X$ with $\partial(x, y) = h$,

$$|\Gamma_i(x) \cap \Gamma_j(y)| = p^h_{ij}.$$

Note $p^h_{ij} = 0$ if $h > i + j$ (or $i > h + j$ or $j > h + i$).

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$$c_i := p^i_{1\,i-1}, \qquad a_i := p^i_{1\,i}, \qquad b_i := p^i_{1\,i+1}$$

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$$c_i + a_i + b_i = k \qquad (0 \le i \le D).$$

Bose-Mesner algebra of a DRG Γ

For each i $(0 \le i \le D)$, let A_i be the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with x, y-entry

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in X).$$

We call A_i the i^{th} **distance matrix** of Γ. Observe that $A_0 = I$ and $A_1 = A$, the adjacency matrix of Γ.

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So $A_0, A_1, \ldots, A_D$ form a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. M is closed under the entry-wise product $\circ$. Each A_i is a polynomial of degree i in A, so A generates M.

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We call M the **Bose-Mesner algebra** of Γ.

Primitive Idempotents for M

It can be shown that M has a second basis $E_0, E_1, ..., E_D$ such that:

$$E_0 = |X|^{-1} J, \qquad E_i^t = \overline{E}_i = E_i, \qquad E_i E_j = \delta_{ij} E_i, \qquad \sum_{h=0}^{D} E_h = I,$$

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The graph Γ is said to be Q-**polynomial** (for $E_0, E_1, ..., E_D$) when each primitive idempotent E_i is a $\circ$-polynomial of degree i in E_1.

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$$\begin{array}{lll} \mathcal{NW}^h_{ij}(z) = |\Gamma(z) \cap \Omega^h_{i,j+1}| & \mathcal{N}^h_{ij}(z) = |\Gamma(z) \cap \Omega^h_{i+1,j+1}| & \mathcal{NE}^h_{ij}(z) = |\Gamma(z) \cap \Omega^h_{i+1,j}| \\ \mathcal{W}^h_{ij}(z) = |\Gamma(z) \cap \Omega^h_{i-1,j+1}| & \mathcal{H}^h_{ij}(z) = |\Gamma(z) \cap \Omega^h_{i,j}| & \mathcal{E}^h_{ij}(z) = |\Gamma(z) \cap \Omega^h_{i+1,j-1}| \\ \mathcal{SW}^h_{ij}(z) = |\Gamma(z) \cap \Omega^h_{i-1,j}| & \mathcal{S}^h_{ij}(z) = |\Gamma(z) \cap \Omega^h_{i-1,j-1}| & \mathcal{SE}^h_{ij}(z) = |\Gamma(z) \cap \Omega^h_{i,j-1}| \end{array}$$

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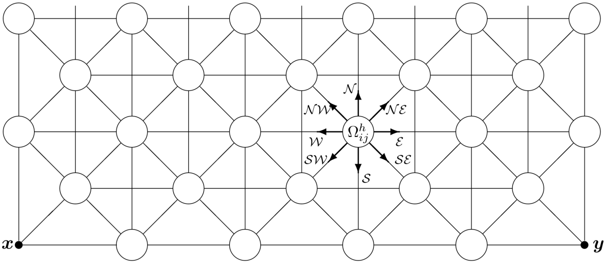

Note: The function $\mathcal{H}$ is not depicted, but it counts edges from Ω^h_{ij} into itself.

Distance distribution diagrams (DDDs for DRGs)

Lemma

Let Γ be a DRG with diameter $D \geq 3$. Pick any $x, y \in X$ and let $h = \partial(x, y)$. For $0 \leq i, j \leq D$ and for $z \in \Omega_{ij}^h$,

$$\mathcal{W}_{ij}^h(z) + \mathcal{SW}_{ij}^h(z) + \mathcal{S}_{ij}^h(z) = c_i, \tag{5}$$

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\end{aligned}$$

Equations (5)-(10) are not independent. The sum of (5)-(7) is identical to the sum of (8)-(10). Any five of the six equations, however, is independent.

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$$W \in M \subseteq N(W).$$

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- When Γ affords W, there exist complex scalars t_i $(0 \leq i \leq D)$ such that

$$W = \sum_{i=0}^{D} t_i A_i, \tag{11}$$

where $A_0, A_1, \ldots, A_D$ are the distance matrices of Γ.

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- We say a DRG Γ **affords** a spin model W whenever

$$W \in M \subseteq N(W).$$

- When Γ affords W, there exist complex scalars t_i $(0 \le i \le D)$ such that

$$W = \sum_{i=0}^{D} t_i A_i, \tag{11}$$

where $A_0, A_1, \ldots, A_D$ are the distance matrices of Γ.

- Since the entries of W are nonzero,

$$t_i \neq 0 \qquad 0 \le i \le D.$$

Distance-regular graphs that support a spin model

Assume Γ affords a spin model W.

For $B \in M \subseteq N(W)$, let $\Psi(B) \in \mathrm{Mat}_X(\mathbb{C})$ be the matrix with bc-entry defined by

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By Curtin, there exists an ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents of Γ such that

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By Curtin, Γ is Q-polynomial with respect to the standard order. (In fact, Γ is **self-dual**.)

Global Definition

In this talk we are interested in DRGs that afford a spin model, so we make the following definition.

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Let Γ be a DRG with vertex set X, diameter $D \geq 3$, and valency $k \geq 3$.

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The parameters of Γ

Curtin and Nomura determined the eigenvalues and intersection numbers of Γ in terms of the diameter D and the scalars q and η.

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$$c_i = \frac{h\eta q^{i-1-D}(1-q^i)(1+\eta q^{D-i})(1-\eta^2 q^{D+i-1})}{(1-\eta^2 q^{2i-1})(1-\eta q^{i-1})} \qquad (1 \le i \le D),$$

$$a_i = \frac{h(q^i-1)(q^D\eta - 1)(q-\eta^2 q^i)(q^D\eta^2 + q)}{q^{D+1}(\eta-1)(q^i\eta - q)(q^i\eta - 1)} \qquad (0 \le i \le D),$$

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where the scalar $h = \frac{q^D(1-\eta^2 q)(\eta-1)}{\eta(q-1)(1-\eta^2 q^D)(1+\eta q^{D-1})}$.

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With reference to Definition 3, the following hold.

$$q^i \neq 1 \qquad (1 \leq i \leq D), \tag{12}$$

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1. describe how the adjacency matrix A acts on the irreducible modules in terms of the parameters q,η
2. find multiplicities of irreducible T-modules in terms of q,η
3. prove q is real and, if Γ is not bipartite, then $q > 0$ and η is real.

Multiplicities of T-modules in terms of q,η

Theorem (C+W '05)

With the notation above, the following are nonnegative integers:

(i) $\mathrm{mult}(0, D) = 1$.

(ii) $$\mathrm{mult}(1, D-1) = -\frac{(\eta+1)(q^D-1)(q^{D-1}\eta^2+1)(q^D\eta^3+1)}{\eta(q-1)(q^{D-1}\eta+1)(q^{2D-1}\eta^3+1)}.$$

(iii) $$\mathrm{mult}(1, D-2) = \frac{q(\eta+1)(q^{D-1}-1)(q^D\eta-1)(q^{D-1}\eta^3+1)}{(q-1)(q^D\eta^2-1)(q^{2D-1}\eta^3+1)}.$$

(iv) $\mathrm{mult}(2, D-2)$

$$= \frac{(\eta+1)(q^D-1)(q^{D-1}-1)(q^{2D-1}\eta^4-1)(q\eta+1)(q^{D-1}\eta^2+1)(q^{D-1}\eta^3+1)(q^{D+2}\eta^3+1)}{\eta^2(q^2-1)(q-1)(q^D\eta^2-1)(q^{D-1}\eta+1)(q^{D-2}\eta+1)(q^{2D-1}\eta^3+1)(q^{2D}\eta^3+1)}.$$

(v) $\mathrm{mult}(2, D-3)$

$$= -\frac{(\eta+q)(q^D-1)(q^{D-2}-1)(q^D\eta-1)(q\eta+1)(q^{D-1}\eta^2+1)(q^{D-1}\eta^3+1)(q^{D+1}\eta^3+1)}{\eta(q-1)^2(q^{D+1}\eta^2-1)(q^{D-2}\eta+1)(q^{2D-2}\eta^3+1)(q^{2D}\eta^3+1)}.$$

Constraints on q and η from 2005 T-algebra paper

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With the notation above, suppose Γ is bipartite. Then

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Theorem (C+W '05)

With the notation above, suppose Γ *is not bipartite. Then* $a_1 \neq 0$ *and*

$$q, \eta \in \mathbb{R} \qquad \text{and} \qquad q > 1.$$

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Lemma (Curtin+Nomura '99)

Pick any $x, y \in X$ and let $\partial(x,y) = h$. For any $0 \le i, j \le D$ and $z \in \Omega^h_{ij}$,

$$\begin{aligned}
\theta_h \frac{t_i}{t_j} &= \mathcal{SW}^h_{ij}(z)\frac{t_{i-1}}{t_j} + \mathcal{W}^h_{ij}(z)\frac{t_{i-1}}{t_{j+1}} + \mathcal{NW}^h_{ij}(z)\frac{t_i}{t_{j+1}} + \mathcal{N}^h_{ij}(z)\frac{t_{i+1}}{t_{j+1}} \\
&\quad + \mathcal{NE}^h_{ij}(z)\frac{t_{i+1}}{t_j} + \mathcal{E}^h_{ij}(z)\frac{t_{i+1}}{t_{j-1}} + \mathcal{SE}^h_{ij}(z)\frac{t_i}{t_{j-1}} + \mathcal{S}^h_{ij}(z)\frac{t_{i-1}}{t_{j-1}} + \mathcal{H}^h_{ij}(z)\frac{t_i}{t_j} \\
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$$\mathcal{H}_{ij}^{h}(z) = -\frac{(q^i - 1)(q^j - 1)(q\eta^2 - 1)(q^D\eta - 1)(q^D\eta^2 + q)}{(q - 1)(q^i\eta - 1)(q^j\eta - 1)(q^D\eta^2 - 1)(q^D\eta + q)}. \tag{15}$$

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For such boundary cells, the other 8 functions in the DDD also depend only on h, i, j, not on vertices x, y, z. They can be derived in terms of $\mathcal{H}_{ij}^h(z)$.

Constraints on q and η

We now deduce new constraints on q,η when Γ is not bipartite.

Lemma (1)

With the notation above, the following hold.

1. *If $h > 0$ then $\eta^2 < 1/q^{D-1}$.*
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With the notation above, suppose Γ is not bipartite. The following hold.

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$$\frac{z_i}{a_1} = \frac{(q^i - q)\eta}{(\eta+1)(q^i\eta - q)} > 0 \quad \text{and} \quad 1 - \frac{z_i}{a_1} = \frac{(q^i\eta^2 - q)}{(\eta+1)(q^i\eta - q)} > 0.$$

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Since $q > 1$, the result follows by induction on i. □

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Proof.

For each i $(2 \le i \le D)$, the scalar

$$\mathcal{S}_{i,2}^{i-1}(z) = L_i = \frac{c_2 q(q^i\eta - 1)}{(q+1)(q^i\eta - q)}$$

is a positive integer.

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is a positive integer. Since c_2, q, and $q+1$ are all positive, the result follows by induction on i. □

Main Results

Lemma (4)

With the notation above, suppose Γ *is not bipartite. The following hold.*

1. *If* $\eta > 0$ *then* $\eta > 1/q$.
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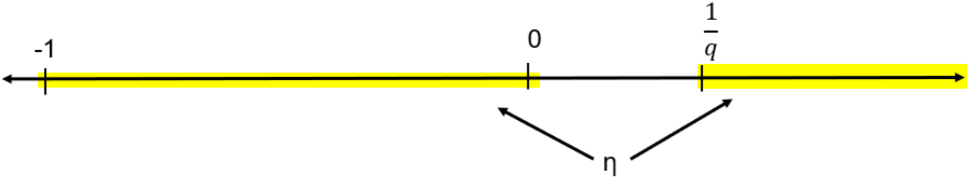

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$$a_1 = -\frac{(\eta+1)(q^D\eta - 1)(q\eta^2 - 1)(q^D\eta^2 + q)}{\eta(q\eta - 1)(q^D\eta + q)(q^D\eta^2 - 1)} > 0.$$

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Note $(q^D\eta^2 + q)$ is positive since $q > 1$. So $(q^D\eta + q)(q^D\eta^2 - 1)$ has the same sign as $-(\eta + 1)$, which is negative by Lemma (4). □

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With the notation above, suppose Γ is not bipartite. The following hold.

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But $\eta < 0$ so $(\eta + \frac{1}{q^{D-1}})(\eta + \frac{1}{q^{D/2}}) > 0$.

(2). Since $\eta < 0$, Lemma (2) at $i = D$ says $q^D \eta^2 - q < 0$. □

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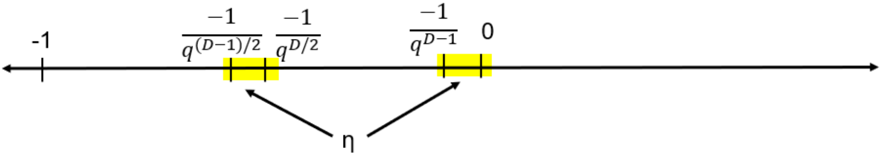

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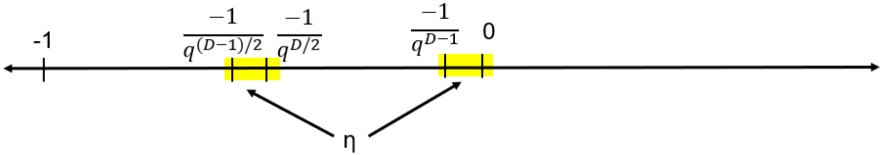

- What's next? Use integrality!

The End

Thank you!

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