Deza graphs and vertex connectivity

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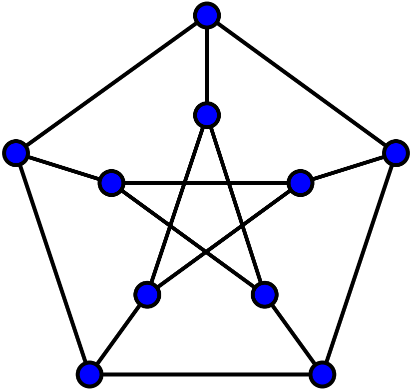
Algebraic Graph Theory Seminar

Waterloo, 29 April 2024

Strongly regular graphs

A k-regular graph on v vertices is called a strongly regular graph (SRG for short) with parameters (v, k, λ, μ) if any pair of adjacent vertices has exactly λ common neighbors, and any pair of non-adjacent vertices has exactly μ common neighbors.

Strongly regular graphs



$$(10, 3, 0, 1)$$

Deza graphs

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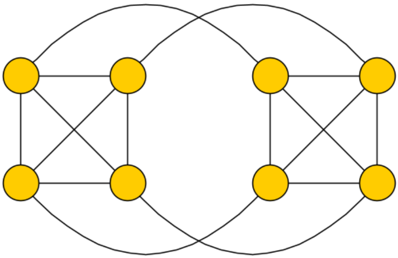
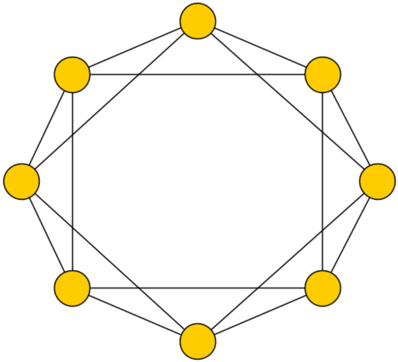
A Deza graph is called a strictly Deza graph if it has diameter 2 and is not strongly regular.

[1] M. Erickson, S. Fernando, W.H. Haemers, D. Hardy, J. Hemmeter, Deza graphs: A generalization of strongly regular graphs, *J. Comb. Des.*, **7** no. 6, (1999) 359–405

Deza graphs



$(8, 4, 2, 0)$



$(8, 4, 2, 1)$

Parameters of Deza graphs

Let N_w be the neighbourhood of vertex w: $N_w = \{u : u \sim w\}$.
For fixed vertex u of Deza graph Γ we define
$\alpha = |\{w \in \Gamma : |N_u \cap N_w| = a\}|, \quad \beta = |\{w \in \Gamma : |N_u \cap N_w| = b\}|.$

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Proposition 1 ([1, Proposition 1.1])

The values of α and β can be calculated directly from the parameters

$$\alpha = \begin{cases} \dfrac{b(v-1) - k(k-1)}{b-a}, \textit{if } a \neq b, \\ \dfrac{k(k-1)}{a}, \textit{if } a = b; \end{cases}$$

$$\beta = \begin{cases} \dfrac{a(v-1) - k(k-1)}{a-b}, \textit{if } a \neq b, \\ \dfrac{k(k-1)}{a}, \textit{if } a = b. \end{cases}$$

Parameters of Deza graphs

Proposition 2 ([1, Corollary 1.2])

Let Γ be Deza graph with parameters (v, k, b, a), β, α. Then
(1) $b - a$ divides $b(v-1) - k(k-1)$;
(2) if $\alpha \neq 0$, then $v \geq 2k - a$;
(3) if $\alpha, \beta \neq 0$, then $a(v-1) < k(k-1) < b(v-1)$.

Enumeration of strictly Deza graphs

In 1999 the complete list of strictly Deza graphs with at most 13 vertices was presented (see [1]).
In 2011 an algorithm for enumeration of Deza graphs was presented by S. Goryainov and L. Shalaginov and this list was extended up to 16 vertices (see [2]).
In 2019 the algorithm for enumerating Deza graphs from [2] was reworked and a complete list of strictly Deza graphs up to 21 vertices was obtained (see [3]).

[2] S. Goryainov, L. Shalaginov, On Deza graphs with 14, 15, and 16 vertices, *Sib. Electron. Mat. Izv.*, **8**, (2011) 105–115

[3] S. Goryainov, D. Panasenko, L. Shalaginov, Enumeration of strictly Deza graphs with at most 21 vertices, *Sib. Electron. Mat. Izv.*, **18(2)**, (2021) 1423–1432

Enumeration of strictly Deza graphs

There are 139 strictly Deza graphs with no more than 21 vertices.
For 74 graphs the construction is not determined yet.

The results of the enumeration are available at
`http://alg.imm.uran.ru/dezagraphs/deza.php`.

Divisible design graphs

A k-regular graph on v vertices is called a divisible design graph (DDG for short) with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if its vertex set can be partitioned into m classes of size n, such that two distinct vertices from the same class have exactly λ_1 common neighbors, and two vertices from different classes have exactly λ_2 common neighbors.
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The definition implies that divisible design graphs are Deza graphs.

A DDG with $m = 1$, $n = 1$, or $\lambda_1 = \lambda_2$ is called improper, otherwise it is called proper. We will consider only proper DDGs.

Canonical partition

Let $V_1 \cup V_2 \cup \ldots \cup V_t$ be the partition of the vertex set of a graph Γ with the property that every vertex of V_i has exactly r_{ij} neighbours in V_j. Then $V_1 \cup V_2 \cup \ldots \cup V_t$ is called an equitable t-partition of Γ. Matrix $R = (r_{ij})_{t \times t}$ is called the quotient matrix of equitable partition.

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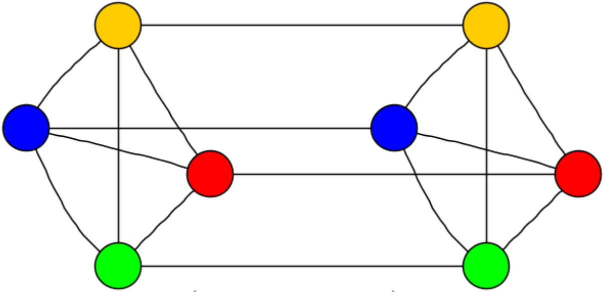
Matrix $R = (r_{ij})_{t\times t}$ is called the quotient matrix of equitable partition.

Equitable partition from the definition of DDGs is called a canonical partition.

Divisible design graphs



$$(8, 4, 0, 2, 4, 2)$$

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Divisible designs

An incidence structure with constant block size k is called a divisible design whenever the set of points can be partitioned into m classes of size n, such that two points from one class occur together in exactly λ_1 blocks, and two points from different classes occur together in exactly λ_2 blocks.

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A divisible design D is called symmetric (SDD for short) if the dual of D (that is, the design with the transposed incidence matrix) is again a divisible design with the same parameters as D.

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A divisible design graph is a graph whose adjacency matrix is the incidence matrix of a symmetric divisible design.

Divisible design graphs

Divisible design graphs were first studied in master's thesis by M.A. Meulenberg [4] and the list of feasible parameters of divisible design graphs up to 50 vertices was presented.

In two following papers [5, 6] feasible parameters of divisible design graphs up to 27 vertices were studied in more details and the existence of graphs was resolved in all but one cases. Moreover, all DDGs with $\lambda_1 = k$, $\lambda_1 = k-1$, $\lambda_2 = 0$ or $\lambda_2 = 2k - v$ were described.

[4] M.A. Meulenberg, Divisible Design Graphs, Master's thesis, *Tilburg University*, 2008.

[5] W.H. Haemers, H. Kharaghani, M.A. Meulenberg, Divisible Design Graphs, *J. of Comb. Th., Series A*, **118** (2011) 978–992

[6] D. Crnković, W.H. Haemer, Walk-regular divisible design graphs.Designs,*Codes and Cryptography*, **72** (2014) 165–175

Eigenvalues of divisible design graphs

Divisible Design Graphs graphs have at most 5 distinct eigenvalues. Formulas for eigenvalues and their multiplicities are presented in the table:

Eigenvalues	Multiplicity	
$\theta_0 = k$	1	
$\theta_1 = \sqrt{k-\lambda_1}$	f_1	
$\theta_2 = -\sqrt{k-\lambda_1}$	f_2	$f_1 + f_2 = m(n-1)$
$\theta_3 = \sqrt{k^2-\lambda_2 v}$	g_1	
$\theta_4 = -\sqrt{k^2-\lambda_2 v}$	g_2	$g_1 + g_2 = m - 1$

Enumeration of divisible design graphs

In 2020 [7] the extension of the algorithm for enumeration of Deza graphs was considered and applied for divisible design graphs with at most 39 vertices.

The results of the enumeration are available at
`http://alg.imm.uran.ru/dezagraphs/ddgtab.php`.

[7] D. Panasenko, L. Shalaginov, Classification of divisible design graphs with at most 39 vertices, *J. of Comb. Des.*, **30(4)** (2022) 205–219

Enumeration results

v	k	λ_1	λ_2	m	n	$\theta_1^{f_1}$	$\theta_2^{f_2}$	$\theta_3^{g_1}$	$\theta_4^{g_2}$	Total
8	4	0	2	4	2	2^1	-2^3	0^3	–	1!
12	5	0	2	6	2	$\sqrt{5}^3$	$-\sqrt{5}^3$	–	-1^5	1!
12	5	1	2	4	3	2^2	-2^6	1^3	–	1!
12	6	2	3	3	4	2^3	-2^6	0^2	–	1!
12	7	3	4	4	3	2^2	-2^6	1^2	-1^1	1!
15	4	0	1	5	3	2^5	-2^5	–	-1^4	1!
18	9	6	4	6	3	$\sqrt{3}^6$	$-\sqrt{3}^6$	3^1	-3^4	1!
20	7	3	2	4	5	2^4	-2^{12}	3^3	–	1!
20	9	0	4	10	2	3^4	-3^6	1^3	-1^6	1!
20	9	0	4	10	2	3^5	-3^5	–	-1^9	1!
20	13	9	8	4	5	2^4	-2^{12}	3^2	-3^1	1!
24	6	2	1	3	8	2^9	-2^{12}	$\sqrt{12}^1$	$-\sqrt{12}^1$	1!
24	7	0	2	8	3	$\sqrt{7}^8$	$-\sqrt{7}^8$	–	-1^7	1!
24	8	4	2	4	6	2^5	-2^{15}	4^3	–	1!
24	8	4	2	4	6	2^7	-2^{13}	4^2	-4^1	1!
24	8	4	2	4	6	2^9	-2^{11}	4^1	-4^2	6!
24	10	2	4	12	2	$\sqrt{8}^6$	$-\sqrt{8}^6$	2^3	-2^8	5!
24	10	3	4	8	3	$\sqrt{7}^8$	$-\sqrt{7}^8$	2^1	-2^6	2!
24	10	6	3	3	8	2^8	-2^{13}	$\sqrt{28}^1$	$-\sqrt{28}^1$	1!
24	14	6	8	12	2	$\sqrt{8}^6$	$-\sqrt{8}^6$	2^2	-2^9	1!
24	14	7	8	8	3	$\sqrt{7}^8$	$-\sqrt{7}^8$	–	-2^7	1!
24	16	12	10	4	6	2^5	-2^{15}	4^2	-4^1	1!
24	16	12	10	4	6	2^7	-2^{13}	4^1	-4^2	1!
24	16	12	10	4	6	2^9	-2^{11}	–	-4^3	4!

Enumeration results

v	k	λ_1	λ_2	m	n	$\theta_1^{f_1}$	$\theta_2^{f_2}$	$\theta_3^{g_1}$	$\theta_4^{g_2}$	Total
27	8	4	2	9	3	2^7	-2^{11}	$\sqrt{10}^4$	$-\sqrt{10}^4$	1!
27	18	9	12	9	3	3^6	-3^{12}	0^8	–	2!
28	6	2	1	7	4	2^9	-2^{12}	$\sqrt{8}^3$	$-\sqrt{8}^3$	1!
28	9	5	2	4	7	2^6	-2^{18}	5^3	–	1!
28	13	0	6	14	2	$\sqrt{13}^7$	$-\sqrt{13}^7$	–	-1^{13}	1!
28	13	4	6	7	4	3^9	-3^{12}	1^1	-1^5	16!
28	15	6	8	7	4	3^7	-3^{14}	1^6	–	56!
28	15	6	8	7	4	3^8	-3^{13}	1^3	-1^3	4!
28	19	15	12	4	7	2^6	-2^{18}	5^2	-5^1	1!
32	10	2	3	8	4	$\sqrt{8}^{12}$	$-\sqrt{8}^{12}$	2^1	-2^6	2!
32	10	6	2	4	8	2^7	-2^{21}	6^3	–	1!
32	10	6	2	4	8	2^{10}	-2^{18}	6^2	-6^1	1!
32	10	6	2	4	8	2^{13}	-2^{15}	6^1	-6^2	15!
32	14	2	6	16	2	$\sqrt{12}^8$	$-\sqrt{12}^8$	2^4	-2^{11}	15!
32	15	6	7	4	8	3^{12}	-3^{16}	–	-1^3	2+
32	16	0	8	16	2	4^6	-4^{10}	0^{15}	–	1!
32	17	8	9	4	8	3^{17}	-3^{11}	1^2	-1^1	?
32	18	6	10	16	2	$\sqrt{12}^8$	$-\sqrt{12}^8$	2^3	-2^{12}	1!
32	22	18	14	4	8	2^7	-2^{21}	6^2	-6^1	1!
32	22	18	14	4	8	2^{10}	-2^{18}	6^1	-6^2	1!
32	22	18	14	4	8	2^{13}	-2^{15}	–	-6^3	9!

Enumeration results

v	k	λ_1	λ_2	m	n	$\theta_1^{f_1}$	$\theta_2^{f_2}$	$\theta_3^{g_1}$	$\theta_4^{g_2}$	Total
35	12	3	4	7	5	3^{12}	-3^{16}	2^3	-2^3	2!
35	12	3	4	7	5	3^{14}	-3^{14}	–	-2^6	3854!
36	9	3	2	12	3	$\sqrt{6}^{12}$	$-\sqrt{6}^{12}$	3^4	-3^7	3!
36	9	4	2	18	2	$\sqrt{5}^9$	$-\sqrt{5}^9$	3^7	-3^{10}	7!
36	11	7	2	4	9	2^8	-2^{24}	7^3	–	1!
36	17	0	8	18	2	$\sqrt{17}^9$	$-\sqrt{17}^9$	–	-1^{17}	1!
36	24	15	16	4	9	3^{12}	-3^{20}	0^3	–	3+
36	25	21	16	4	9	2^8	-2^{24}	7^2	-7^1	1!
36	27	21	20	12	3	$\sqrt{6}^{12}$	$-\sqrt{6}^{12}$	3^1	-3^{10}	1!
38	9	0	2	19	2	3^8	-3^{11}	$\sqrt{5}^9$	$-\sqrt{5}^9$	2!

New constructions

Construction 1

Suppose Γ is an antipodal distance-regular graph of diameter 3 with antipodal classes of size r. Denote by A_i the matrix of a relation 'to be at distance i' on the vertices of Γ. If $\lambda = \mu + 2$, then matrix $A = A_1 + A_3$ is a matrix of a DDG with parameters $(r(2\mu+4)$, $2\mu+r+2$, $r-2$, $\mu+2$, $2\mu+4$, $r)$.

New constructions

Example

Halved 6-cube is an antipodal distance-regular graph with antipodal classes of size 2 and $\lambda = \mu + 2$. Connecting antipodal vertices gives DDG with parameters $(32, 16, 0, 8, 16, 2)$.

New constructions

Let G be a k-regular graph on v vertices with the smallest eigenvalue λ_{min}. A Hoffman-coclique is a coclique meeting the Hoffman upper bound $c = v\lambda_{min}/(\lambda_{min} - k)$.
A Hoffman coloring in G is a partition of the vertices into Hoffman-cocliques.

New constructions

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A Hoffman coloring in G is a partition of the vertices into Hoffman-cocliques.

Construction 2

Suppose Γ is a strongly regular graph with parameters $(v, k, \mu+2, \mu)$ and Γ has Hoffman coloring with Hoffman-cocliques of size n $(v = mn)$. Let A be an adjacency matrix of Γ, in which Hoffman-cocliques are located on the main diagonal, $K = K_{(m,n)}$, $I = I_v$. Then matrix $A+K-I$ is the adjacency matrix of a DDG with parameters $(mn,\ k+n-1,\ n+\mu-2,\ \mu+\frac{2k}{m-1},\ m,\ n)$.

New constructions

Example

Consider triangular graph T(8) and three Chang graphs with parameters $(28, 12, 6, 4)$. With construction 2 we can obtain 56 DDGs with parameters $(28, 15, 6, 8, 7, 4)$.

Construction 3

Let Γ be a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ and quotient matrix $R = aI_m + b(J_m - I_m)$. Take s copies of Γ and label all blocks of canonical partition in each copy with numbers $1, \ldots, m$. Then connect all vertices from blocks with the same label (adjacency inside the block does not change). The resulting graph is a DDG with parameters $(vs, k + (s-1)n, \lambda_1 + (s-1)n, \lambda_2, ms, n)$, if $\lambda_2 = 2b = 2a + (s-2)n$.

Construction 4

Let D be a symmetric divisible design with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, such that every block contains the same number of points $\frac{k}{m}$ from each class. Let Γ be the incidence graphs of D. Construct new graph Γ^* with the same vertex set as Γ and vertices x, y are adjacent in Γ^* when x, y are adjacent in Γ, or x, y are points from different classes of D, or x, y are blocks from different classes of dual design of D. Γ^* is a DDG with parameters $(2v, k + (m-1)n$, $\lambda_1 + (m-1)n$, $\lambda_2 + (m-2)n$, $2m$, $n)$, if $2(m-1)\frac{k}{m} = \lambda_2 + (m-2)n$.

Construction 5

Let D be a symmetric divisible design with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ with even m, such that every block contains the same number of points $\frac{k}{m}$ from each class. Γ is the incidence graphs of D. Split the set of classes of blocks and the set of classes of points into pairs. Construct a new graph Γ^* with the same vertex set as Γ and vertices x, y are adjacent in Γ^* when x, y are adjacent in Γ or x, y are from different classes of the same pair of classes partition. Γ^* is a DDG with parameters $(2v, k+n,$ $\lambda_1 + n$, λ_2, $2m$, $n)$, if $\frac{2k}{m} = \lambda_2$.

New constructions

A weighing matrix $W(n,k)$ of order n and weight k is an $n \times n$ $(0,1,-1)$-matrix, such that $WW^T = kI_n$.

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Construction 6

Let W be a $(4t, 4(t-1))$-weighing matrix, such that the main diagonal of W contains blocks of zeros of size 4. Construct matrix A' by replacing each 0 with O_2, each 1 with I_2 and each -1 with $J_2 - I_2$. Then matrix $A = A' + I_t \otimes ((J_4 - I_4) \otimes J_2)$ is the adjacency matrix of a DDG Γ with parameters $(8t, 4t+2, 6, 2t+2, 4t, 2)$.

Construction 7

Let Γ be a DDG obtained from Construction 20 with adjacency matrix A. The main diagonal of the A consists of $I_t \otimes ((J_4 - I_4) \otimes J_2)$, which gives a partition of Γ into complete multipartite graphs with 4 parts of size 2. Construct new graph Γ' by removing the edges of the complete bipartite subgraph $K_{4,4}$ from each part of this partition. Then Γ' is a DDG with parameters $(8t, 4t-2, 2, 2t-2, 4t, 2)$.

Corrections

In 2022 the tuple of parameters $(35, 12, 3, 4, 7, 5)$ with spectrum $3^{14}, -3^{14}, -2^{6}$ was revised, and the number of non-isomorphic graphs with such parameters turns out to be 35.

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In 2023 [8] a construction that allows to get all graphs with parameters $(36, 24, 15, 16, 4, 9)$ was proposed.

[8] V.V. Kabanov, A new construction of strongly regular graphs with parameters of the complement symplectic graph, *The Elect. J. of Comb.*, **30(1)** (2023) #P1.25

Unclassified DDGs

v	k	λ_1	λ_2	m	n	$\theta_1^{f_1}$	$\theta_2^{f_2}$	$\theta_3^{g_1}$	$\theta_4^{g_2}$	Total	Comment
27	8	4	2	9	3	2^7	-2^{11}	$\sqrt{10}^4$	$-\sqrt{10}^4$	1!	sporadic
28	6	2	1	7	4	2^9	-2^{12}	$\sqrt{8}^3$	$-\sqrt{8}^3$	1!	sporadic
28	13	4	6	7	4	3^9	-3^{12}	1^1	-1^5	16!	–
32	10	2	3	8	4	$\sqrt{8}^{12}$	$-\sqrt{8}^{12}$	2^1	-2^6	2!	structure
32	15	6	7	4	8	3^{12}	-3^{16}	–	-1^3	2+	Cayley
32	17	8	9	4	8	3^{17}	-3^{11}	1^2	-1^1	?	–
35	12	3	4	7	5	3^{12}	-3^{16}	2^3	-2^3	2!	–
36	9	3	2	12	3	$\sqrt{6}^{12}$	$-\sqrt{6}^{12}$	3^4	-3^7	3!	1 unclassified
38	9	0	2	19	2	3^8	-3^{11}	$\sqrt{5}^9$	$-\sqrt{5}^9$	2!	–

Vertex connectivity

The vertex connectivity $\varkappa(\Gamma)$ of the graph Γ is the minimum number of vertices whose deletion from Γ disconnects it.

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The vertex connectivity $\varkappa(\Gamma)$ of the graph Γ is the minimum number of vertices whose deletion from Γ disconnects it.
For example, the vertex connectivity of a strongly regular graph is equal to the degree of a vertex, which was proved by Brouwer and Mesner in 1985 ([9]). A similar statement for distance-regular graphs was proved by Brouwer and Koolen in 2009 ([10]).

[9] A. E. Brouwer and D. M. Mesner, The connectivity of strongly regular graphs, *Europ. J. Combin.*, **6** (1985) 215–216
[10] A. E. Brouwer and J. H. Koolen, The vertex-connectivity of a distance-regular graph, *Europ. J. Combin.*, **30(3)** (2009) 668–673

Vertex connectivity

A set of vertices S disconnects x and y if x and y belong to different connected components of the graph $\Gamma \setminus S$.

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A set of vertices S disconnects x and y if x and y belong to different connected components of the graph $\Gamma \setminus S$.
A set S of vertices of a graph Γ is called disconnecting if it disconnects some two of its vertices.

Vertex connectivity of Deza graphs

Theorem 1 ([11, 12])

Let Δ be a Deza graph obtained from a strongly regular graph Γ with restricted eigenvalues $r > 0$ and $s < 0$ by dual Seidel switching. Then, we have one of the following two cases:
(1) if $r \neq 2$, the vertex connectivity of Δ equals to the degree of a vertex, except for the case when Γ is a complement of $n \times n$-lattice, in that case the vertex connectivity is one less then the degree of a vertex;
(2) $r = 2$.

[11] A.L. Gavrilyuk, S. Goryainov, V.V. Kabanov, On the vertex connectivity of Deza graphs, *Proc. Steklov Inst. Math.*, **285 (Suppl 1)** (2014) 68–77.
[12] S. Goryainov, D. Panasenko, On vertex connectivity of Deza graphs with parameters of the complements to Seidel graphs, *Eur. J. of Comb.*, **80** (2019) 143–150.

Vertex connectivity of DDGs

Lemma 1

The vertex connectivity of a connected DDG Γ with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, where $\lambda_2 = 0$, equals k.

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Lemma 2

The vertex connectivity of a connected DDG Γ with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, where $\lambda_1 = k$, equals k.

Lemma 3

Let Γ be a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, where $\lambda_2 = 2k - v$. Then the following statements hold.
(1) If $\lambda_1 \neq k - 1$, then the vertex connectivity of Γ equals k.
(2) If $\lambda_1 = k - 1$, then the vertex connectivity of Γ equals $k - 1$.

Vertex connectivity of DDGs

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Lemma 4

The vertex connectivity of a DDG Γ with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, where $\lambda_1 = k - 1$ and $\lambda_2 \notin \{0, 2k - v\}$, equals $k - 1$.

DDGs from Hadamard matrices

An $m \times m$ matrix H is a Hadamard matrix if every entry is 1 or -1 and $HH^\top = mI$. A Hadamard matrix H is called graphical if H is symmetric with constant diagonal, and regular if all row and column sums are equal.

DDGs from Hadamard matrices

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Construction 8

Consider a regular graphical Hadamard matrix H of order $l^2 \geqslant 4$ with diagonal entries -1 and row sum l. The graph with adjacency matrix

$$A = \begin{bmatrix} M & N & O \\ N & O & M \\ O & M & N \end{bmatrix},$$

where

$$M = \frac{1}{2}\begin{bmatrix} J+H & J+H \\ J+H & J+H \end{bmatrix} \text{ and } N = \frac{1}{2}\begin{bmatrix} J+H & J-H \\ J-H & J+H \end{bmatrix},$$

is a DDG with parameters $(6l^2,\ 2l^2+l,\ l^2+l,\ (l^2+l)/2,\ 3,\ 2l^2)$.

Vertex connectivity of DDGs

Let Γ be a DDG with parameters $(6l^2, 2l^2 + l, l^2 + l, (l^2 + l)/2, 3, 2l^2)$ obtained with Construction 8 with positive l.

Lemma 5

The vertex connectivity of Γ is at most $2l^2$.

Hadamard matrices

If H_1 and H_2 are Hadamard matrices, then so is the Kronecker product $H_1 \otimes H_2$. Moreover, if H_1 and H_2 are regular with row sums l_1 and l_2, respectively, then $H_1 \otimes H_2$ is regular with row sum $l_1 l_2$. Similarly, the Kronecker product of two graphical Hadamard matrices is graphical again.

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Consider regular graphical Hadamard matrices H and H', where

$$H = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \text{ and } H' = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Denote by H_1 the matrix H. For any integer t such that $t > 1$, denote by H_t the Kronecker product $H_{t-1} \otimes H'$. The matrix H_t is a regular graphical Hadamard matrices of order 4^t with diagonal entries -1 and row sum 2^t.

Applying Construction 8 to H_t, we obtain a DDG with parameters $(6 \cdot 4^t,$ $2 \cdot 4^t + 2^t,\ 4^t + 2^t,\ 2 \cdot 4^{t-1} + 2^{t-1},\ 3,\ 2 \cdot 4^t)$. The smallest example is a DDG with parameters $(24, 10, 6, 3, 3, 8)$ and adjacency matrix

$$\begin{bmatrix} D & D & D & I & O & O \\ D & D & I & D & O & O \\ D & I & O & O & D & D \\ I & D & O & O & D & D \\ O & O & D & D & D & I \\ O & O & D & D & I & D \end{bmatrix},$$

where $D = J - I, J = J_4, I = I_4$ and $O = O_4$.

Hadamard matrices

By replacing

$$D \to \begin{bmatrix} D & D & D & I \\ D & D & I & D \\ D & I & D & D \\ I & D & D & D \end{bmatrix}, \; I \to \begin{bmatrix} I & I & I & D \\ I & I & D & I \\ I & D & I & I \\ D & I & I & I \end{bmatrix}, \; O \to \begin{bmatrix} O & O & O & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & O \end{bmatrix},$$

we get a recursive construction for a DDG with parameters $(6{\cdot}4^t,\ 2{\cdot}4^t{+}2^t,\ 4^t + 2^t,\ 2 \cdot 4^{t-1} + 2^{t-1},\ 3,\ 2 \cdot 4^t)$. Denote by Γ^t this DDG.

Hadamard matrices

By replacing

$$D \to \begin{bmatrix} D & D & D & I \\ D & D & I & D \\ D & I & D & D \\ I & D & D & D \end{bmatrix},\ I \to \begin{bmatrix} I & I & I & D \\ I & I & D & I \\ I & D & I & I \\ D & I & I & I \end{bmatrix},\ O \to \begin{bmatrix} O & O & O & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & O \end{bmatrix},$$

we get a recursive construction for a DDG with parameters $(6{\cdot}4^t,\ 2{\cdot}4^t{+}2^t,\ 4^t + 2^t,\ 2 \cdot 4^{t-1} + 2^{t-1},\ 3,\ 2 \cdot 4^t)$. Denote by Γ^t this DDG.

Consider the subgraph formed by the first $2 \cdot 4^t$ vertices of Γ^t (in terms of Construction 8 this subgraph has adjacency matrix M) and the subgraph formed by the last $2 \cdot 4^t$ vertices of Γ^t (in terms of Construction 8 this subgraph has adjacency matrix N). Let Γ_1^t denote the first subgraph and Γ_2^t denote the second subgraph.

Structural description of Γ_1^t and Γ_2^t

Lemma 6

The graph Γ_1^t is a DDG with parameters $(2 \cdot 4^t, 4^t + 2^t, 4^t + 2^t,\ 2 \cdot (4^{t-1} + 2^{t-1}),\ 4^t, 2)$ and its vertex connectivity equals $4^t + 2^t$.

Structural description of Γ_1^t and Γ_2^t

Lemma 6

The graph Γ_1^t is a DDG with parameters $(2\cdot 4^t, 4^t+2^t, 4^t+2^t,\ 2\cdot(4^{t-1}+2^{t-1}),\ 4^t, 2)$ and its vertex connectivity equals 4^t+2^t.

Lemma 7

The graph Γ_2^t is a DDG with parameters $(2\cdot 4^t, 4^t, 0,\ 2\cdot 4^{t-1},\ 4^t, 2)$ and its vertex connectivity equals 4^t.

Structural description of Γ_1^t and Γ_2^t

Lemma 6

The graph Γ_1^t is a DDG with parameters $(2 \cdot 4^t, 4^t + 2^t, 4^t + 2^t,\ 2 \cdot (4^{t-1} + 2^{t-1}),\ 4^t, 2)$ and its vertex connectivity equals $4^t + 2^t$.

Lemma 7

The graph Γ_2^t is a DDG with parameters $(2 \cdot 4^t, 4^t, 0,\ 2 \cdot 4^{t-1},\ 4^t, 2)$ and its vertex connectivity equals 4^t.

There are two known DDGs of diameter 2 with $\lambda_1 = 0$ (more generally, there are two known strictly Deza graphs with $a = 0$), one on 8 vertices and one on 32 vertices. The series Γ_2^t covers both cases and gives an infinite series of DDGs of diameter 2 with $\lambda_1 = 0$ (more generally, strictly Deza graphs with $a = 0$).

Vertex connectivity of Γ^t

Lemma 8

The vertex connectivity of Γ^t equals $2 \cdot 4^t$.

[13] D. Panasenko, The vertex connectivity of some classes of divisible design graphs, *Sib. Elect. Math. Izv.*, **19(2)** (2022) 426–438

Thanks for your attention!