# Eigenvalues of high dimensional Laplacian operators

University of Waterloo
Virtual Algebraic Graph Theory Seminar
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The dimension of the complex X = the maximal dimension of a simplex in X

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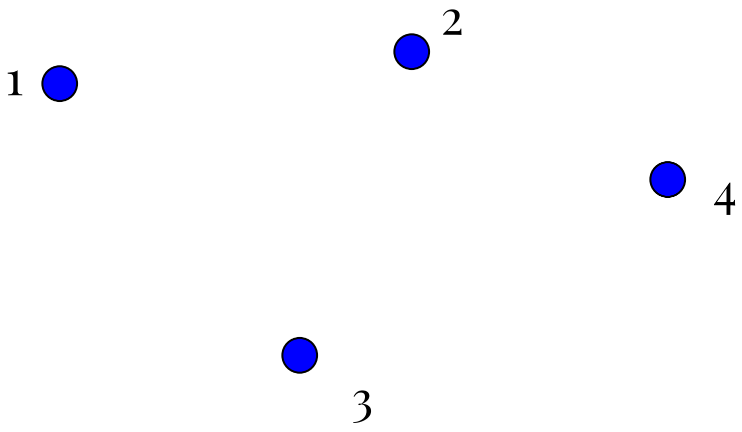

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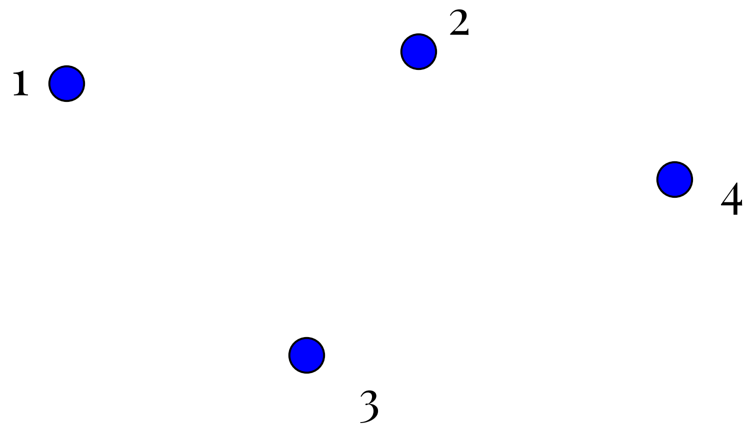

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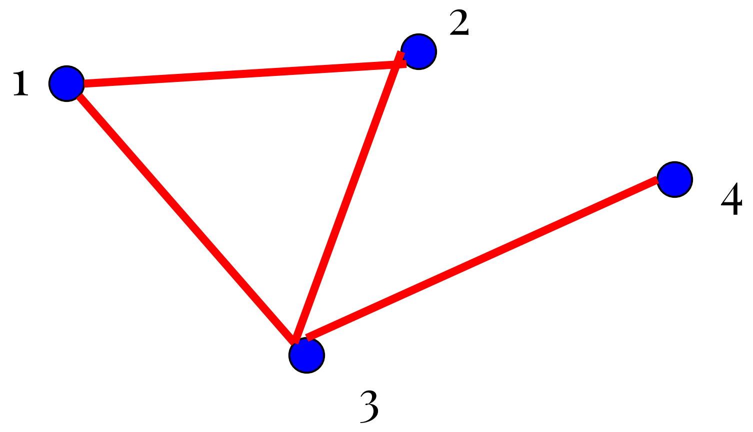

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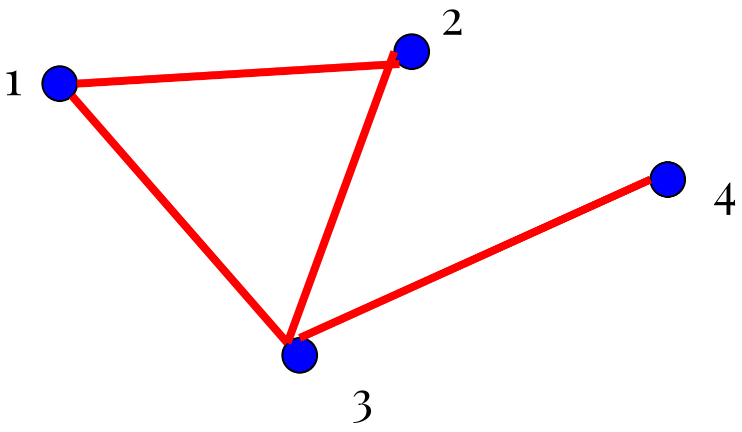

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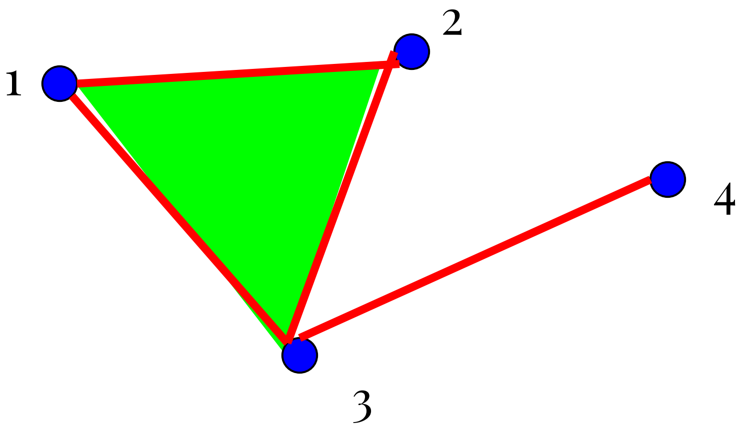

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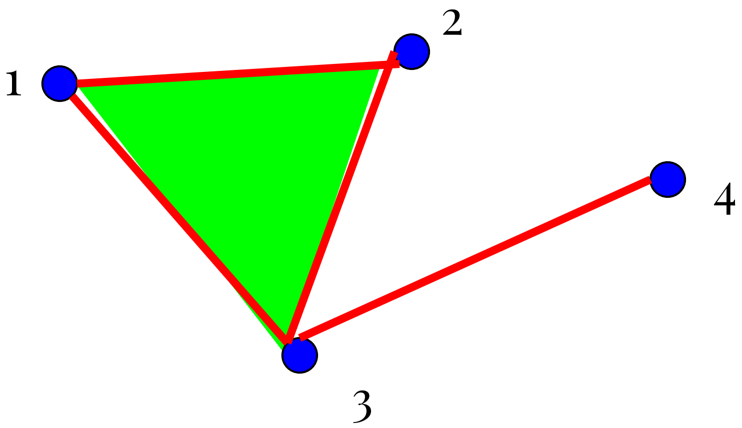


We can study the topology of a simplicial complex

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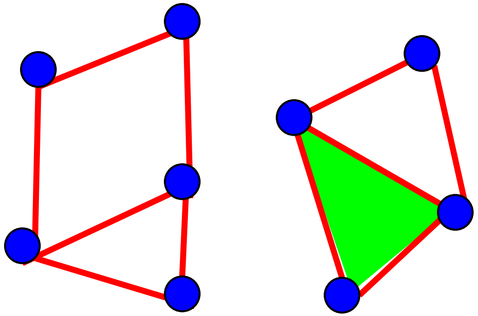
X= 

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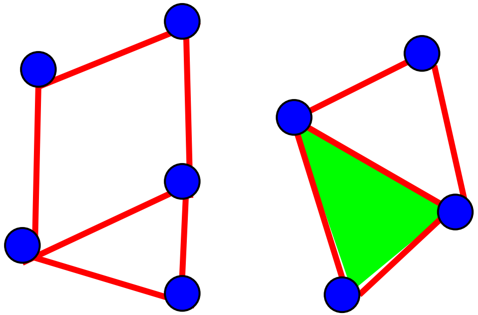

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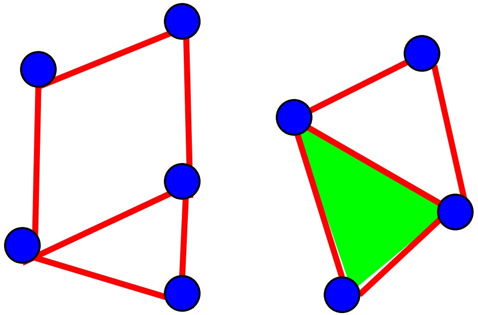
X= 

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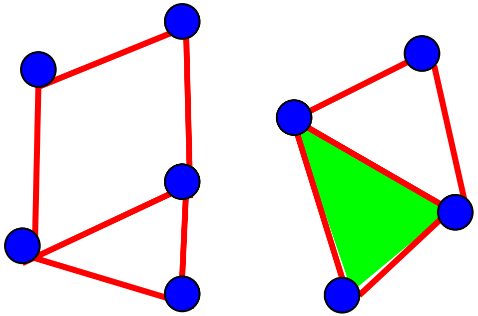

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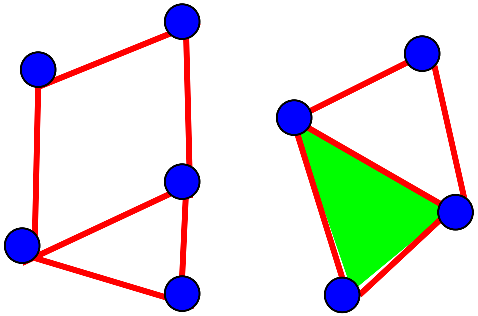
X= 

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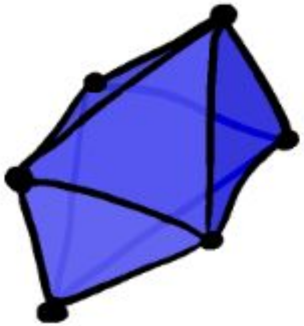
X= 

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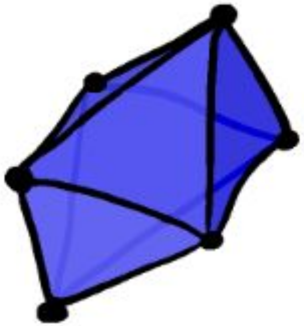
X= 

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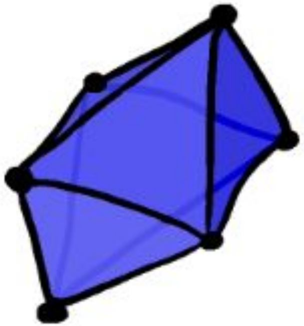
X= 

Triangulation of a sphere

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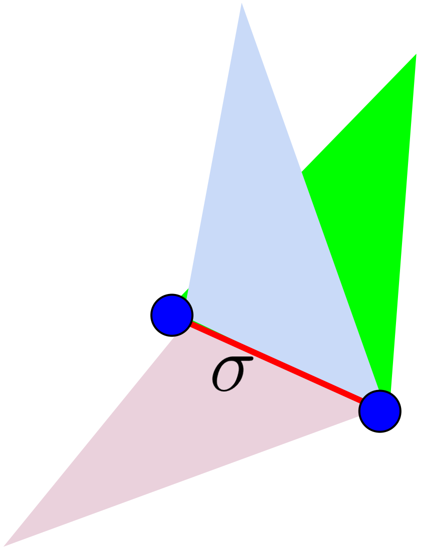

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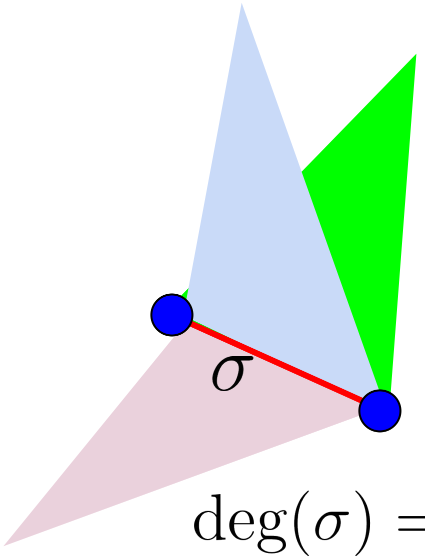


$\deg(\sigma) = 3$

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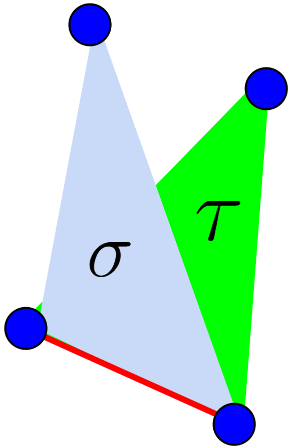

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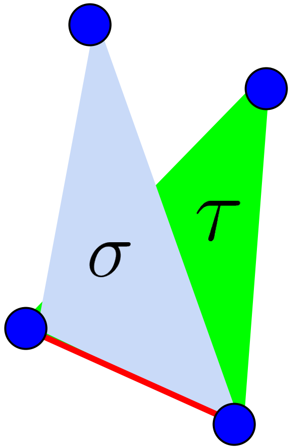

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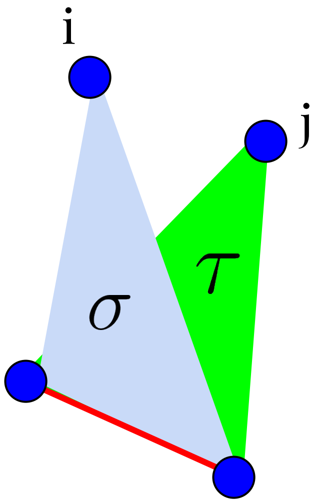

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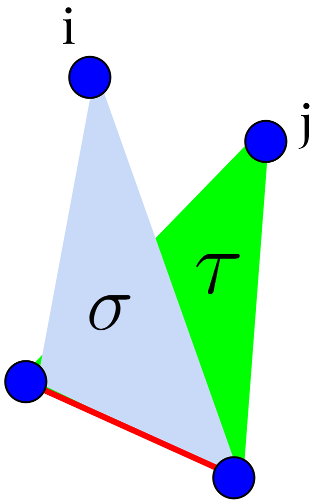

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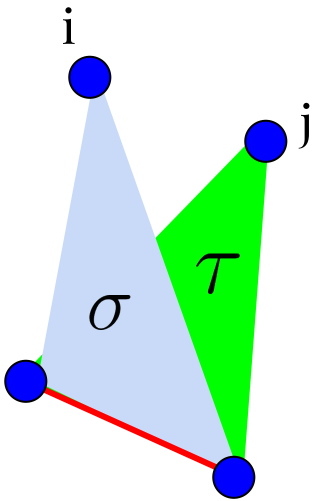


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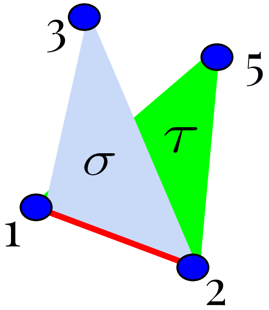

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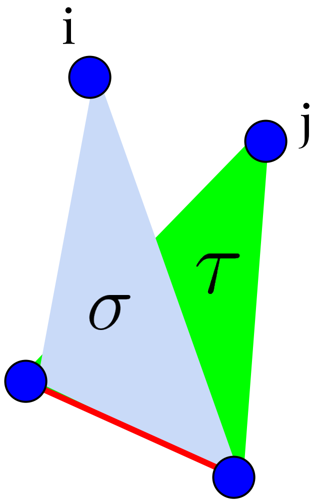


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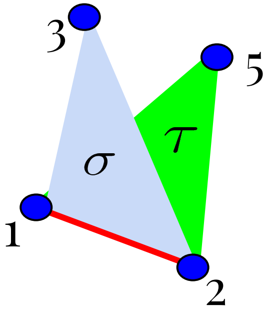

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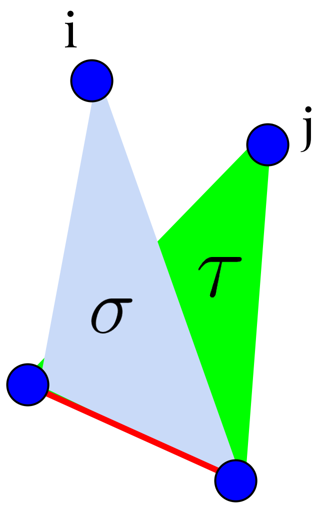


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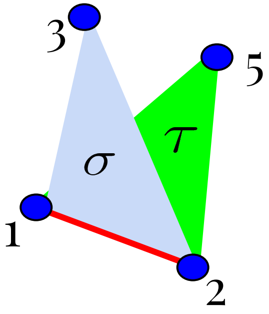


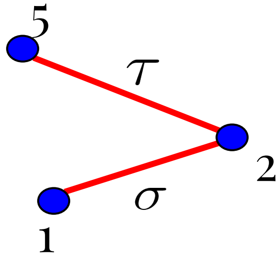

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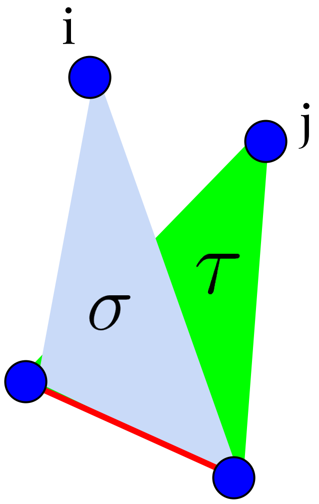


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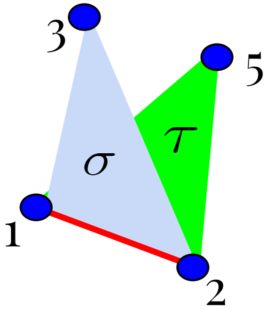


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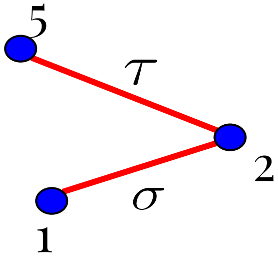

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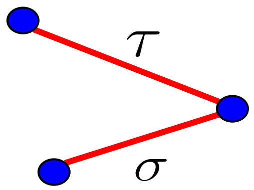


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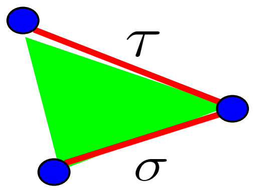

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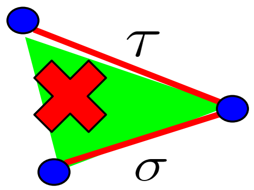

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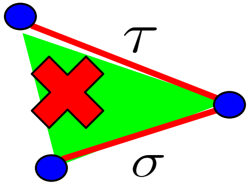

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Spectrum of $L_0(X)$: $\lambda_2, \cdots, \lambda_n, n$

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**Simplicial Hodge Theorem (Eckmann '44):** $\mathrm{Ker}(L_k(X)) \cong \tilde{H}_k(X)$

# Clique complexes / Independence complexes

Let G=(V,E) be a graph.

The clique complex of G: $X(G) = \{U \subset V : U \text{ is a clique in } G\}$

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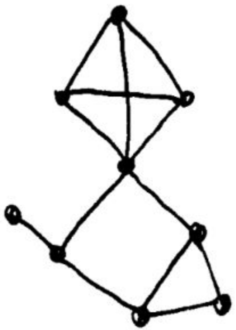
G = 

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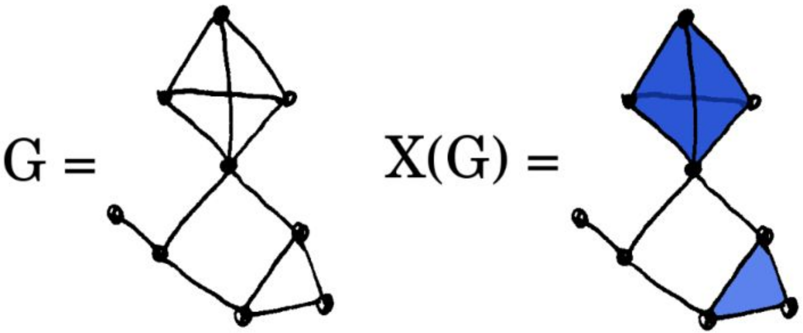

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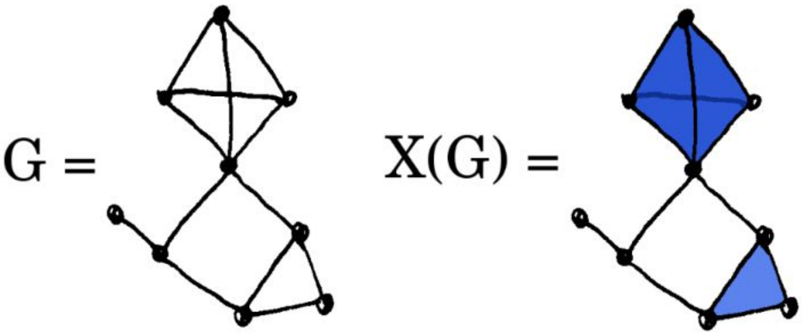


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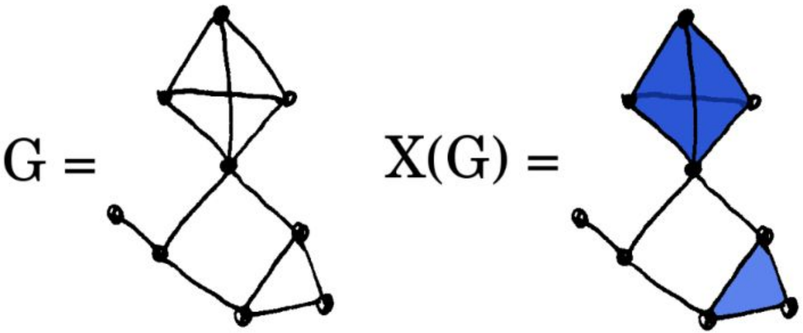


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**Theorem** (Aharoni, Berger, Meshulam '05): Let G=(V,E) be a graph on n vertices.

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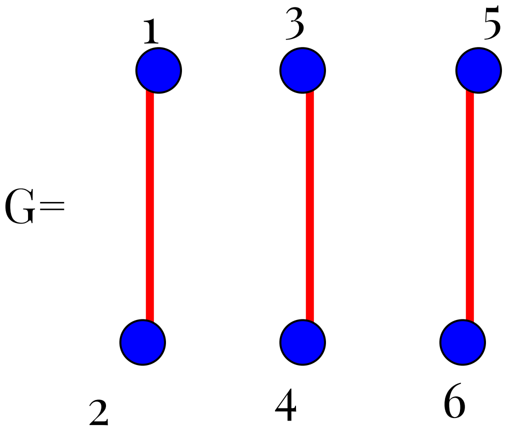

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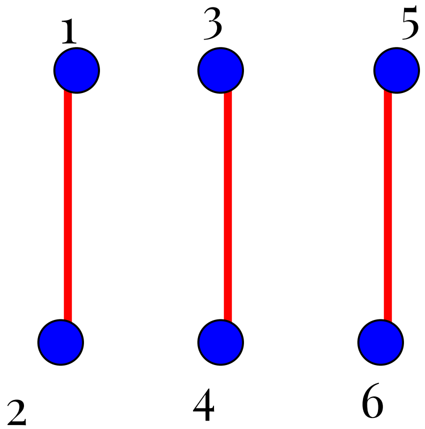


Eigenvalues of L(G): 0,0,0,2,2,2

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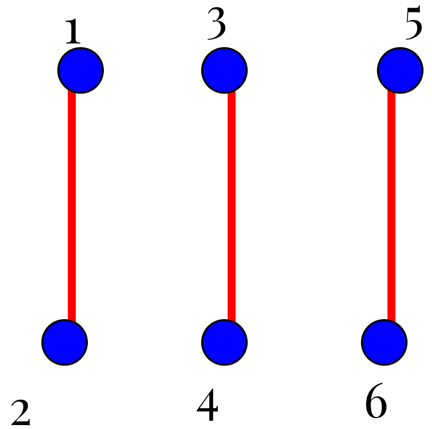


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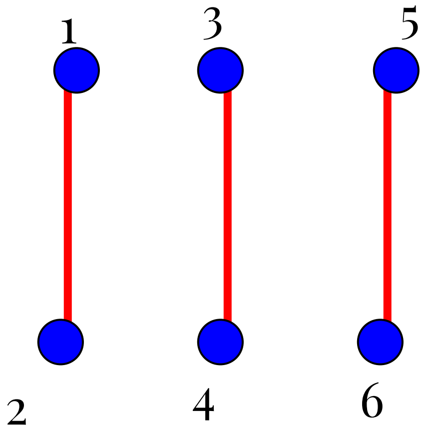


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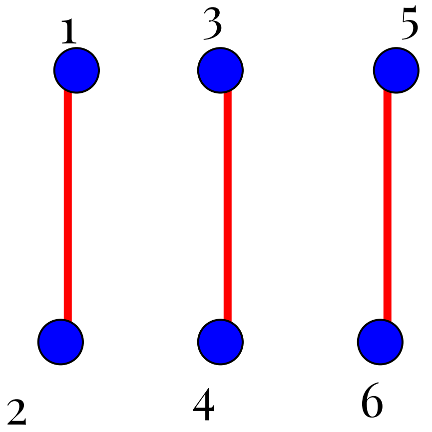


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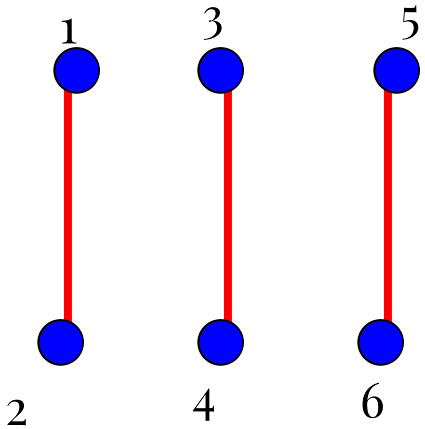


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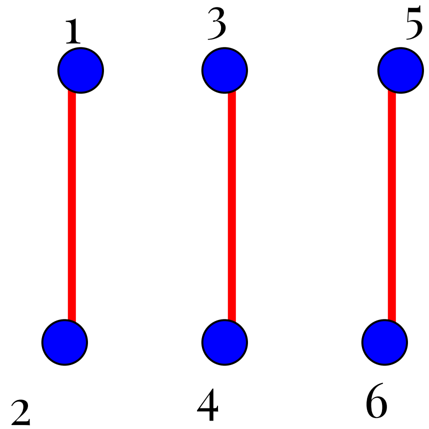


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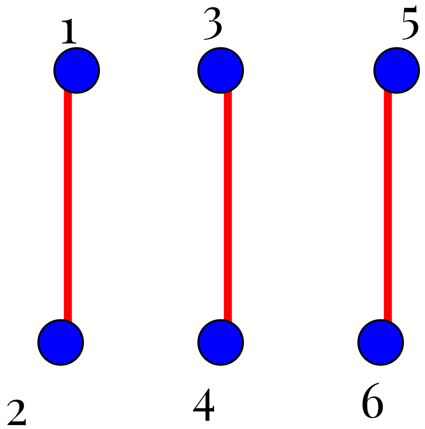
G= 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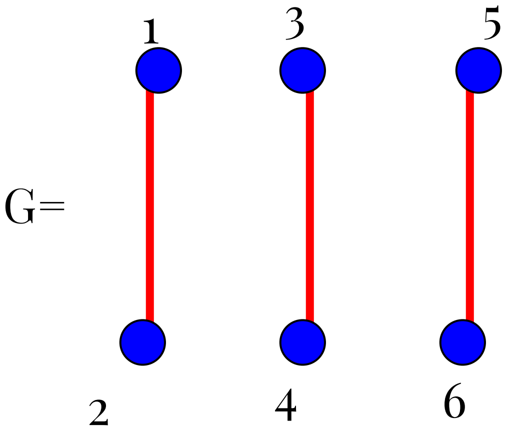


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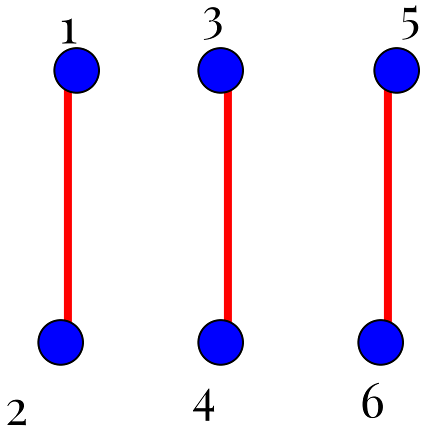


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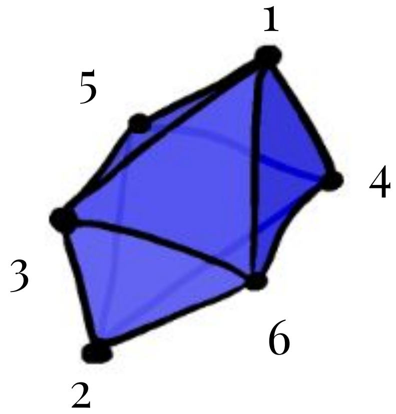


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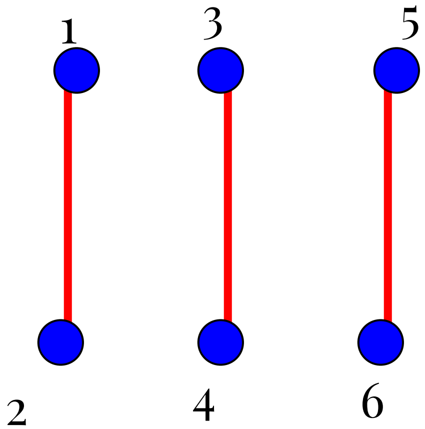


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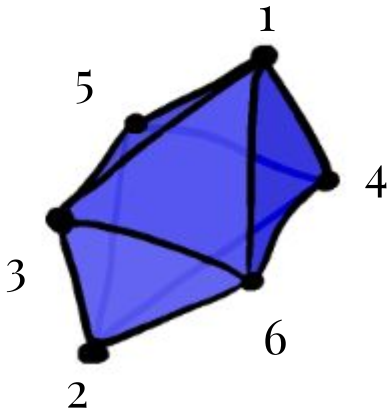


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Let V be a vector space.

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Exterior powers:

$\bigwedge^k V$ = the k-th exterior power of V

= vector space spanned by $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ for $v_1, \ldots, v_k \in V$.

## Additive compound matrices

If $e_1, \ldots, e_n$ is a basis of V, then

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

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Let $A : V \to V$ be a linear operator. Define $A^{[k]} : \bigwedge^k V \longrightarrow \bigwedge^k V$ by

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**k-th additive compound of A (Wielandt '67)**

# Additive compound matrices

Properties:

If A has eigenvalues $\lambda_1, \ldots, \lambda_n$, then the eigenvalues of $A^{[k]}$ are

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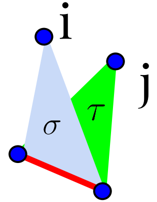


$$A^{[k]}_{\sigma,\tau} = \begin{cases} \sum_{i\in\sigma} A_{ii} & \text{if } \sigma = \tau \\ (-1)^{\epsilon(\sigma,\tau)} A_{ij} & \text{if } |\sigma \cap \tau| = k-1,\ \sigma \setminus \tau = \{i\},\ \tau \setminus \sigma = \{j\} \end{cases}$$

# Additive compound matrices

Properties:

If A has eigenvalues $\lambda_1, \ldots, \lambda_n$, then the eigenvalues of $A^{[k]}$ are

$$\mathcal{S}_k(A) = \{\lambda_{i_1} + \cdots + \lambda_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

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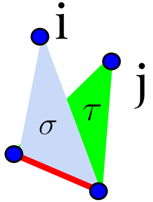


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$$L_0(X(G))_{u,v} = \begin{cases} \deg(u) + 1 & \text{if} \quad \text{u=v} \\ 1 & \text{if } \{u, v\} \notin E \\ 0 & \text{otherwise} \end{cases}$$

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# An additional application of additive compounds

**Proposition** (L 23+): Let G=(V,E) be a graph. Then

$$\sum_{i=1}^{k} \lambda_i^{\downarrow}(L(G)) \leq 2 \cdot \max_{\sigma \in \binom{V}{k}} |\{e \in E : e \cap \sigma \neq \emptyset\}|$$

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**Proof idea:** Apply Geršgorin's theorem on k-th additive compound of matrix.

# Some open problems

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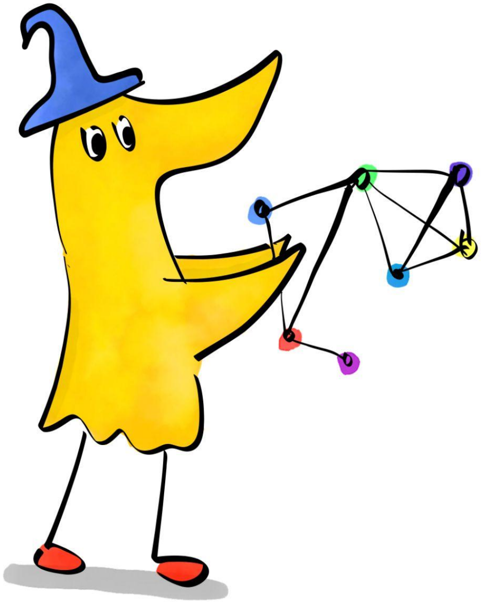
-The Garland-like argument of ABM can be extended to "generalized clique complexes" (L '18). Can we use additive compounds to obtain improved results in this setting?



Thank you for listening!