Combinatorial Nullstellensatz and the Erdős box problem

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Combinatorial Nullstellensatz

- $\mathbb{F}$: field; $f \in \mathbb{F}[x_1, \ldots, x_r]$; $[x_1^{d_1} \ldots x_r^{d_r}]f$: coeff. of $x_1^{d_1} \ldots x_r^{d_r}$ in f

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$[x_1^{d_1} \ldots x_r^{d_r}]f \neq 0$, $\deg f = d_1 + \cdots + d_r$ $\Rightarrow$ $\forall A_1, \ldots, A_r \subseteq \mathbb{F}$, $|A_i| \geq d_i + 1$ $\exists a_i \in A_i \;:\; f(a_1, \ldots, a_r) \neq 0$

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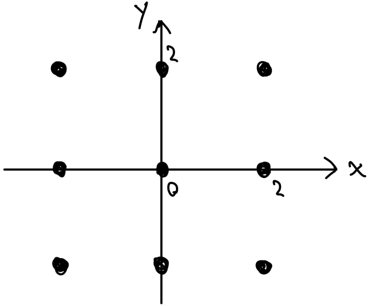
$$f(x, y) = \cdots + cx^2y^2$$

$$A_1 = A_2 = \{-2, 0, 2\}$$



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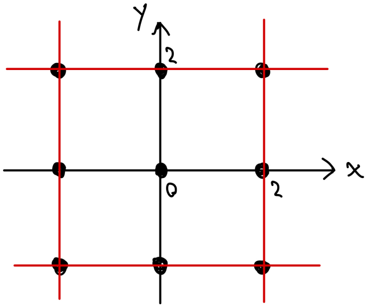

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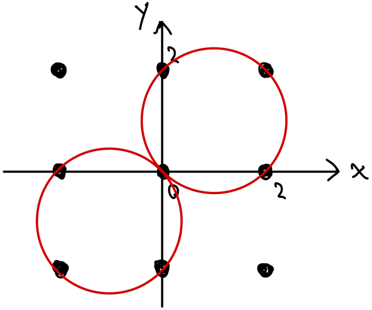

Example of a proof

Cauchy-Davenport theorem

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object of study $\xrightarrow{\text{interpret}}$ zero set of polynomial $\xrightarrow{\text{CN}}$ structure

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- *Example*: $f(x, y) = x^{100} + xy + y^{100}$
- **Schauz, 2008**: even more general theorem
- **In practically all known applications degree condition is sufficient!**

Turán numbers

- *Turán number* $\mathrm{ex}(n, G)$: max. # of edges in G-free graph on n vertices

Turán, 1941

$$\mathrm{ex}(n, K_{m+1}) = \left(1 - \frac{1}{m} + o(1)\right) \binom{n}{2}$$

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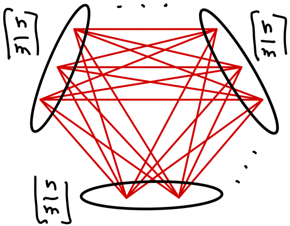

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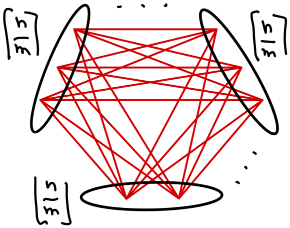

Erdős–Stone, 1946

$$\mathrm{ex}(n, G) = \left(1 - \frac{1}{\chi(G)-1} + o(1)\right)\binom{n}{2}$$

◇ determines $\mathrm{ex}(n, G)$ asymptotically when G is not bipartite ($\chi(G) > 2$)

r-partite r-graphs

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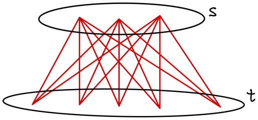

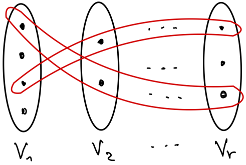

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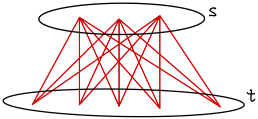
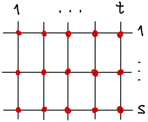
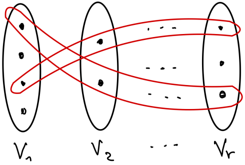
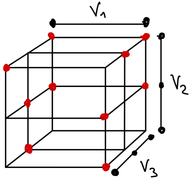
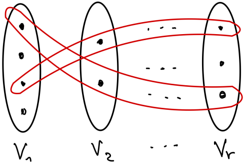


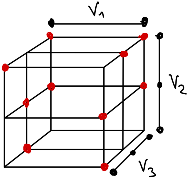






Turán number of $K^{(r)}_{s_1,\dots,s_r}$





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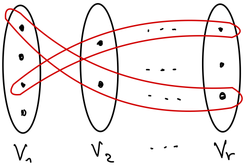

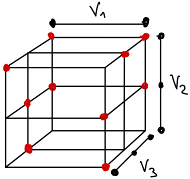

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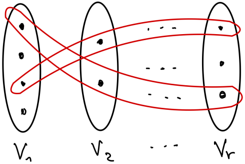

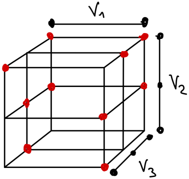

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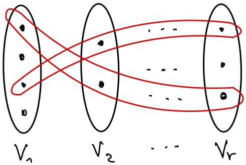

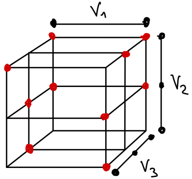

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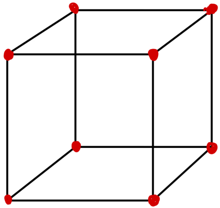
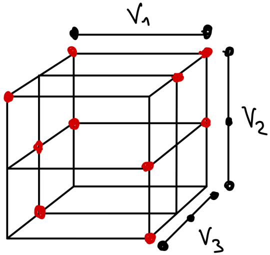
- • **True for** $K_{2,2}$ and $K_{3,3}$ — **E.Klein, 1934 and Brown, 1966**
- • **True for** $s_r \gg s_1, \dots, s_{r-1}$ — **Pohoata–Zakharov, 2021+**
  - ◇ norm hypergraphs (graphs: **Alon–Kollár–Rónyai–Szabó, 1990s**)
  - **Ma–Yuan–Zhang, 2018**
  - ◇ random algebraic method (**Blagojević–Bukh–Karasev, 2013**; **Bukh, 2015**)

Erdős box problem





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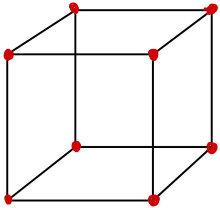

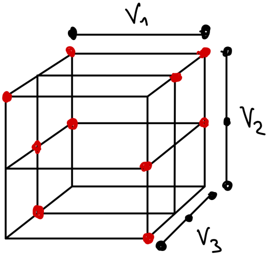

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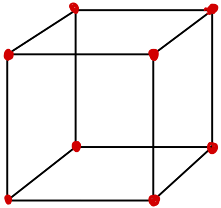

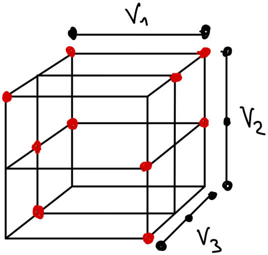

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- $\mathrm{ex}(n, K^{(3)}_{2,2,2}) = \Omega(n^{8/3})$ **Katz–Krop–Maggioni, 2002**

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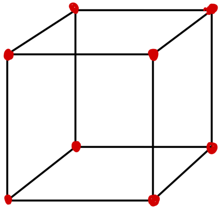

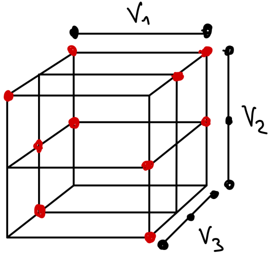

- $\mathrm{ex}(n, K^{(r)}_{2,\dots,2}) = O\left(n^{r-\frac{1}{2^{r-1}}}\right)$ for $r \geq 2$ **Erdős, 1964**
- $\mathrm{ex}(n, K_{2,2}) = \Theta(n^{3/2})$ **E.Klein, 1934**

◇ **no matching lower bound for $r > 2$ is known!**

- $\mathrm{ex}(n, K^{(3)}_{2,2,2}) = \Omega(n^{8/3})$ **Katz–Krop–Maggioni, 2002**
- $\mathrm{ex}(n, K^{(r)}_{2,\dots,2}) = \Omega\left(n^{r-\lceil\frac{2^r-1}{r}\rceil^{-1}}\right)$ **Conlon–Pohoata–Zakharov, 2021** (improving on **Gunderson–Rödl–Sidorenko, 1999**)

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$r - \frac{1}{2^{r-1}}$	1.5	2.75	3.875	4.9375	5.96875
$r - \lceil \frac{2^r-1}{r} \rceil^{-1}$	1.5	2.(6)	3.75	4.(857142)	5.(90)
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  - ◇ proof is much more complicated

The framework

◇ *The Generalized Combinatorial Lasoń–Alon–Zippel–Schwartz Nullstellensatz Lemma, arxiv:2305.10900* **Rote, 2023**

The framework

- $\mathbb{F}$: field; $\quad f \in \mathbb{F}[x_1, \ldots, x_r]$; $\quad B_1, \ldots, B_r \subseteq \mathbb{F}$; $\quad B = B_1 \times \cdots \times B_r$
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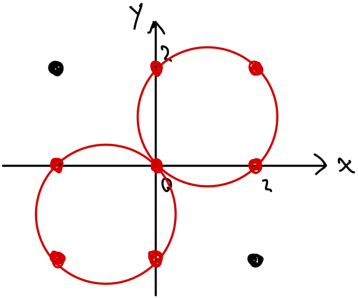

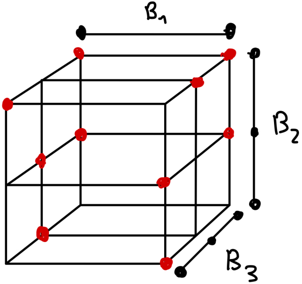

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Key Lemma (general)

$\forall \pi \in \mathcal{S}_r$: some maximal monomial of f divides $x_{\pi_1}^{d_1} \ldots x_{\pi_r}^{d_r}$ $\Rightarrow$ $H(f, B)$ is $K^{(r)}_{d_1+1,\ldots,d_r+1}$-free

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Corollary (retracing Erdős's proof)

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Construction

Lemma (G., 2024)

$$f(x_1, \ldots, x_r) = x_1 \ldots x_r + \sum_{i=1}^{r} \prod_{j=1}^{r-1} x_{i+j}^{p^r - p^j} \quad \text{where } x_{r+i} = x_i$$

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Theorem (G., 2024)

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Proof of Lemma

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