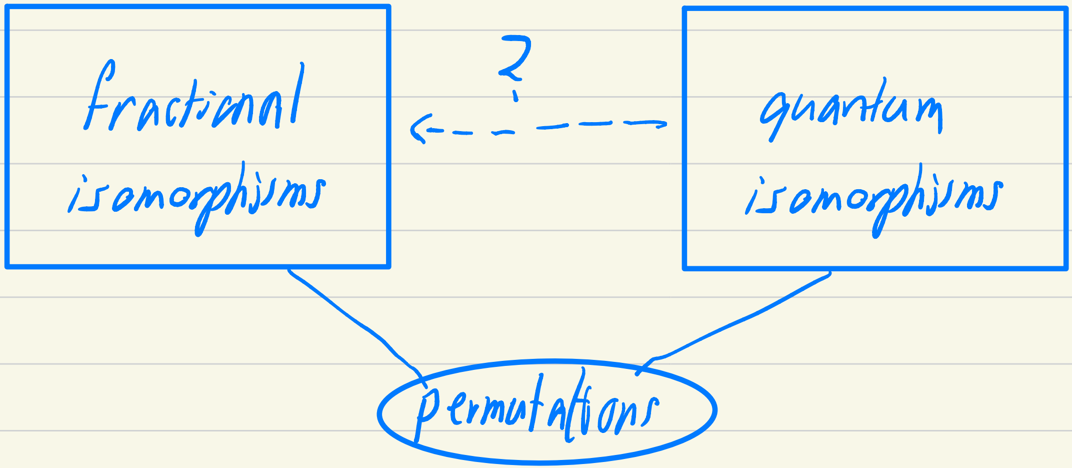
Fractional & Quantum Isomorphisms

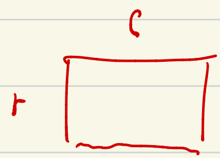

fractional isomorphisms
?
quantum isomorphisms
permutations

Doubly stochastic matrices

A matrix is doubly stochastic if its real, non-negative, and each row and column sums to 1.

e.g. permutation matrices

A doubly stochastic matrix is necessarily square.



The set of $n \times n$ doubly stochastic matrices is

(a) a convex polytope

(b) closed under transposes

monoid

(c) closed under multiplication

The permutation matrices are the vertices of the polytope. Birkhoff

$\langle M, N \rangle = \mathrm{tr}(M^* N)$

$\langle P, P \rangle = n$

$\langle P, S \rangle < n, \quad S \neq P$

$\Rightarrow P$ is a vertex

Fractional isomorphisms

Graphs X, Y; adjacency matrices A & B.

X & Y are fractionally isomorphic iff there is a doubly stochastic matrix S such that $AS = SB$.

- isomorphisms are fractional isomorphisms

- Any two k-regular graphs on n vertices are fractionally isomorphic.

$S = \frac{1}{n} J$

$AS = \frac{k}{n} J \quad SB = \frac{k}{n} J$

If $AS = SB$, then $BS^T = S^T A$.

So fractional isomorphism is an equivalence relation.

If $AS = SA$, then S is a fractional automorphism.

If $AS = SB$ & $BS^T = S^TA$, then

$$SS^TA = SBS^T = ASS^T$$

and so SS^T is a fractional automorphism of X.

We use $\mathcal{S}(X)$ to denote the set of all fractional automorphisms of X.

A graph is compact if all vertices of $\mathcal{S}(X)$ are permutation matrices. Tinhofer

Theorem If X & Y are fractionally isomorphic graphs on n vertices, there is a permutation matrix P such that

$$P\left[\underset{\sim}{1}\ A\underset{\sim}{1} \cdots A^{n-1}\underset{\sim}{1}\right] = \left[\underset{\sim}{1},\ B\underset{\sim}{1} \cdots B^{n-1}\underset{\sim}{1}\right].$$

rk = # main eigenvalues

Lemma If R & S are doubly stochastic and $y = Rx$, $x = Sy$, then $y = Px$ for some permutation matrix P.

Equitable partitions

Suppose π is a partition of $V(X)$ with normalized characteristic matrix N. (So N is $v \times |\pi|$ and $N^T N = I$.)

e.g. $N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{3} \\ 0 & 0 & 1/\sqrt{3} \\ 0 & 0 & 1/\sqrt{3} \end{pmatrix}$

NN^T is a projection

Theorem The following are equivalent:

(a) π is equitable.

(b) the column space of N is A-invariant.

functions on $V(X)$ constant on cells of π

(c) $AN = NC$ for some $|\pi| \times |\pi|$ matrix C.

(Orbit partitions are equitable.)

The projection onto the column space of N is NN^T, which is doubly stochastic. And $\mathrm{col}(N)$ is A-invariant if & only if A & NN^T commute.

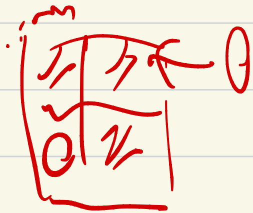
So equitable partitions give elements of $\mathcal{S}(X)$.

Fractional isomorphisms

give equitable partitions

We can view a doubly stochastic matrix S as a weighted adjacency matrix of a directed graph.

Because S is doubly stochastic, any weak component is strongly connected.



So we may assume S has the block-diagonal form

$$\begin{bmatrix} S_1 & & 0 \\ & \ddots & \\ 0 & & S_c \end{bmatrix},$$

where the diagonal blocks are doubly stochastic & irreducible.

If S is non-negative & irreducible, its largest eigenvalue is simple. So each S_i has spectral radius 1, with eigenvector $\underset{\sim}{1}$.

The strong components of S partition $V(X)$. If N is the normalized characteristic matrix of this partition, NN^T is projection on $\ker(S-I)$.

Claim NN^T is a polynomial in S, hence the partition given by S is equitable.

A graph X is controllable if $\langle A, J \rangle = Mat_{n\times n}(\mathbb{R})$

Almost all graphs are controllable O'Rourke & Touri

Lemma If X is controllable, $\mathcal{P}(X) = \{I\}$.

Quantum permutations

A quantum permutation is an $n \times n$ matrix P with entries from the ring $\mathrm{Mat}_{d\times d}(\mathbb{C})$ such that:

(a) $P_{i,j}$ is a projection for all i & j

(b) $\sum_r P_{i,r} = I_d = \sum_s P_{s,j}$

e.g. any permutation $(d=1)$

Remark If $Q_1, \dots, Q_k$ are $d \times d$ projections and $\sum_r Q_r = I$, then $Q_i Q_j = 0$ if $i \neq j$.

Proof. We have $I = I^2 = \sum_i Q_i + \sum_{i \neq j} Q_i Q_j$

Hence $0 = \sum_{i \neq j} Q_i Q_j$ and so

$$0 = tr\left(\sum_{i \neq j} Q_i Q_j\right) = \sum_{i \neq j} tr(Q_i Q_j)$$

Hence $tr(Q_i Q_j) = 0$ & so $Q_i Q_j = 0$.

1) A quantum permutation is unitary.

2) The product of two quantum permutations is not, in general, a quantum permutation.

3) If $L = (L_{i,j})$ is an $n \times n$ Latin square and $u_1, \dots, u_n$ is an orthonormal basis of $\mathbb{C}^d$, then

$$P = (u_{L_{i,j}} u^*_{L_{i,j}})$$

(if $L_{i,j} = k$, then $P_{i,j} = u_k u_k^*$)

is a quantum permutation.

Remark: If $u_1, \ldots, u_n$ is an orthonormal basis,

then $\sum_r u_r u_r^* = I$.

Converse?

We say a quantum permutation P of index d is a quantum automorphism of X if $A \otimes I_d$ and P commute.

$\tilde{A}$

Operations on quantum permutations:

coproduct: $(P \star Q)_{ij} := \sum_r P_{ir} \otimes Q_{rj}$

direct sum: $(P \oplus Q)_{ij} := P_{ij} \oplus Q_{ij}$ $\quad \begin{pmatrix} P_{ij} & 0 \\ 0 & Q_{ij} \end{pmatrix}$

Theorem If P, Q are quantum automorphisms of X, so are $P \star Q$ and $P \oplus Q$.

What make a quantum permutation quantum?

Suppose $P = (P_{i,j})$ is a quantum automorphism. If the entries of P commute, there is a change of basis that diagonalizes them — so we may assume $P_{i,j}$ is diagonal 01. It follows that P is the direct sum of permutations, which commute with X if P does.

So if we want something not classical, the algebra generated by the entries of P must not be commutative.

e.g. $pq \neq qp$,

$$\left(\begin{array}{cccc:c} p & 1-p & 0 & 0 & \\ 1-p & p & 0 & 0 & \\ 0 & 0 & q & 1-q & 0 \\ 0 & 0 & 1-q & q & \\ \hdashline & & 0 & & \end{array}\right)$$

Measuring quantum permutations.

A measurement is a sequence of projections $Q_1, \dots, Q_m$ such that $\sum_i Q_i = I$. A state is given by a density matrix D, i.e., a positive semidefinite matrix such that $\mathrm{tr}(D) = 1$.

The outcome of a measurement is an element of $\{1, \dots, m\}$. We observe i with probability $\langle Q_i, D \rangle = \mathrm{tr}(Q_i D)$.

Each row & each column of a quantum permutation is a measurement. If P is a quantum permutation of index d & D is a density matrix of order $d \times d$, we define the $n \times n$ matrix $\langle\langle P, D \rangle\rangle$ by

$$\langle\langle P, D \rangle\rangle_{i,j} = \langle P_{i,j}, D \rangle$$

Theorem $\langle\!\langle P, D \rangle\!\rangle$ is doubly stochastic.

If P is a quantum automorphism of X, then $\langle\!\langle P, D \rangle\!\rangle \in \mathcal{S}(X)$.

Theorem $\langle\!\langle P_1 * P_2 , D_1 \otimes D_2 \rangle\!\rangle = \langle\!\langle P_1, D_1 \rangle\!\rangle \langle\!\langle P_2, D_2 \rangle\!\rangle$

$$\langle\!\langle P_1 \oplus P_2 , aD_1 \oplus (1-a) D_2 \rangle\!\rangle = a \langle\!\langle P_1, D_1 \rangle\!\rangle + (1-a) \langle\!\langle P_2, D_2 \rangle\!\rangle$$

$0 \le a \le 1$

The matrices $\langle\langle P, D \rangle\rangle$, where P runs over the quantum automorphisms of X and D runs over density matrices form a convex subset of $\mathcal{S}(X)$. It is a monoid & is transpose-closed.

1) If $tr(P_{ij}) = 1$ for all i,j and $D = \frac{1}{d} I_d$,

then $\langle\langle P, D \rangle\rangle = \frac{1}{n} J$ and X is regular.

2) If X is controllable, its only quantum automorphism is the identity.

3) Most trees admit non-classical quantum automorphisms (Junk, Schmidt, Weber)

4) Quantum isomorphic trees are isomorphic.

The End(s)

thanks!

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