Entropy and the growth rate of universal covering trees

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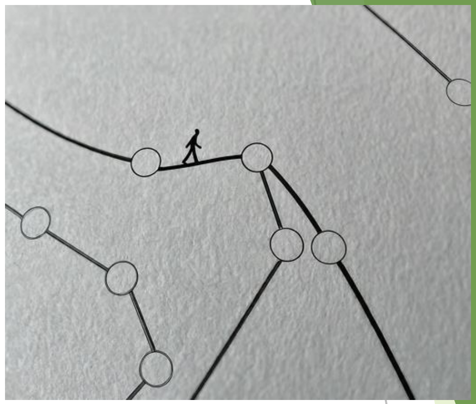
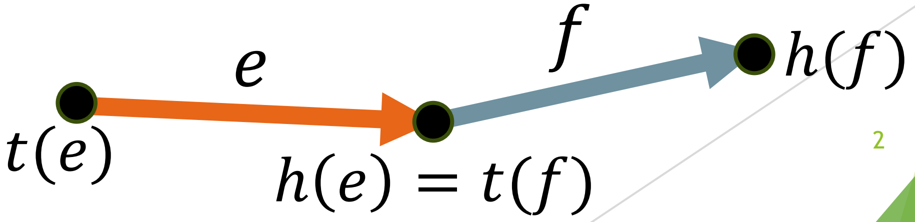
Non-backtracking walks on graphs



- Non-backtracking walk, Non-Backtracking Random Walk (NBRW)
- Directed edges $\vec{E}$ - each undirected edge is a pair of directed edges
- Edges connect $e \to f$ if $h(e) = t(f)$ and e is not inverse of f
- Non-backtracking adjacency operator B 0-1 $\vec{E} \times \vec{E}$ matrix, 1 if $e \to f$
- Transition matrix P where $P_{e,f} = \frac{1}{\mathrm{outdeg}(e)} = \frac{1}{\mathrm{indeg}(f)}$ if $e \to f$, zero otherwise
- Stationary distribution ν_s - uniform on $\vec{E}$
- Graph G – undirected, connected, no degree one vertices, not a cycle
- B and P are irreducible

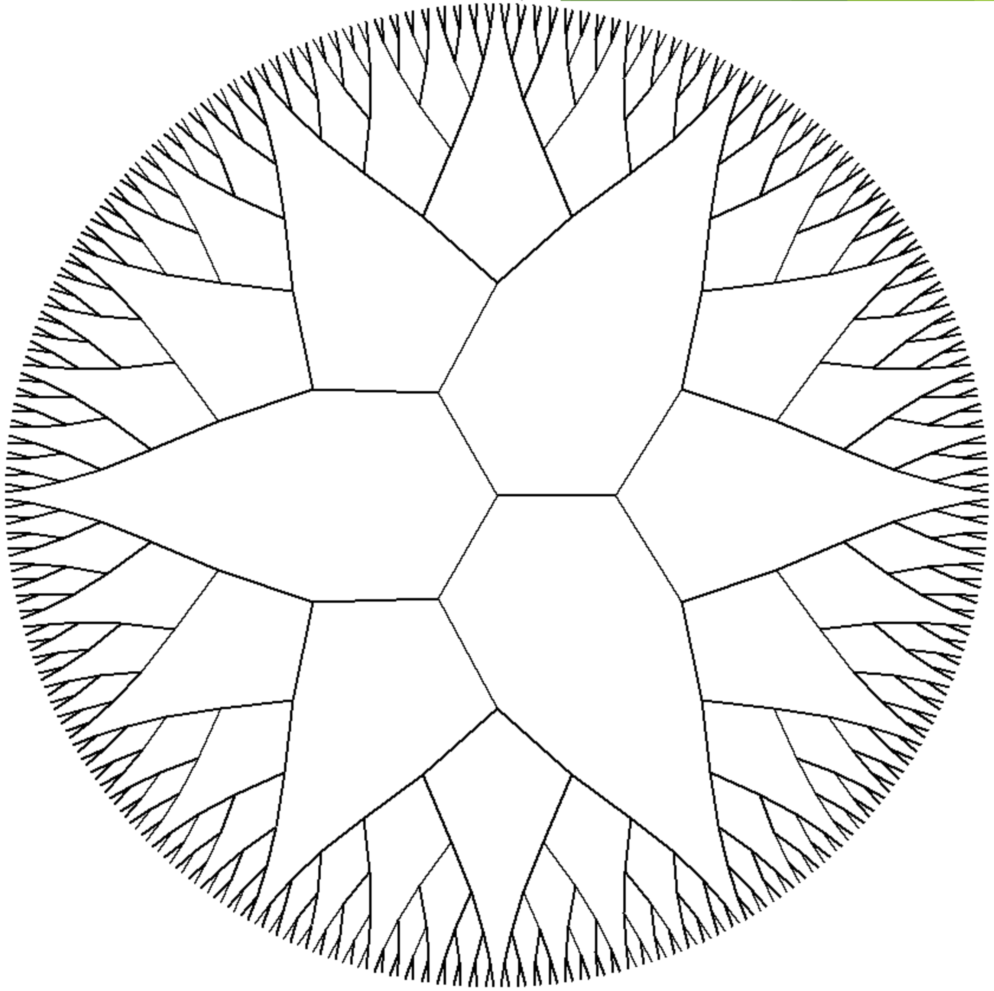


ρ - Growth Rate of the Universal Cover

- $\tilde{G}$ denotes the universal cover of G
- Infinite tree covering G:
  - Choose some vertex v_0 in G
  - $V(\tilde{G})$ – the set of all NB-walks from v_0
  - $E(\tilde{G})$ – NB-walks are adjacent if one extends the other by one step
- Rate of growth –

$$\rho(\tilde{G}) = \limsup_{r\to\infty} |B_r(v)|^{1/r}$$

- ρ = Perron eigenvalue of B



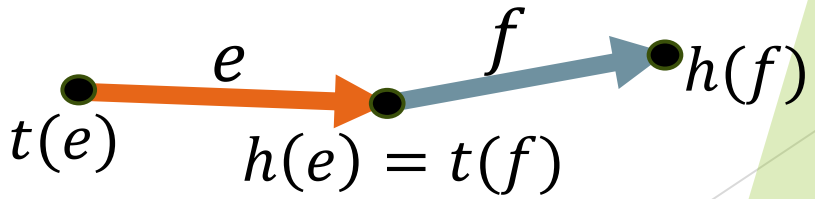
The universal cover of any 3-regular graph is the 3-regular tree

Λ - Rate of entropy growth

$$\Lambda = \prod_{v \in V} (\deg(v) - 1)^{\frac{\deg(v)}{|\vec{E}|}} = \prod_{e \in \vec{E}} \mathrm{outdeg}(e)^{\frac{1}{|\vec{E}|}}$$

$$E_{v_s}[\log_2 \mathrm{outdeg}(e)] = \log_2 \Lambda$$

Expected number of random bits per NBRW step.
Rate of entropy growth.



$\mathrm{outdeg}(e) = \deg(h(e) - 1) =$ the number of choices for f

ρ ≥ Λ

- Ω_l - set of length l nb-walks $\omega = (e_0, \ldots, e_l)$
- Probability space of length l NBRW walks from v_s

$$Pr[\omega] = \frac{1}{|\vec{E}|} \prod_{i=0}^{l-1} \frac{1}{\operatorname{outdeg}(e_i)}$$

- Number of randomness bits: $R(\omega) = \sum_{i=0}^{l-1} \log_2 \operatorname{outdeg}(e_i)$
- $E[R] = l \cdot E_{v_s}[\log_2 \operatorname{outdeg}(e)] = l \cdot \log_2 \Lambda$
- $v_s\, B^l\, \bar{1} = \frac{1}{|\vec{E}|} |\Omega_l| = \sum_{\omega=(e_0,\ldots,e_l)\in\Omega_l} Pr[\omega] \prod_{i=0}^{l-1} \operatorname{outdeg}(e_i)$

$$= \sum_{\omega} Pr[\omega]\, 2^{R(\omega)} = E_{\Omega_l}[2^R] \geq 2^{E_{\Omega_l}[R]} = \Lambda^l$$

- $\rho(\tilde{G}) = \rho(B) = \lim_{l\to\infty} \left(v_s\, B^l\, \bar{1}\right)^{\frac{1}{l}} \geq \Lambda$

Some known results about ρ, Λ

- Ben-Hamou and Salez 2017
  - Mixing time of NBRW on a random graph with given degrees is $\log_\Lambda n\,(1+o(1))$ whp
- Conchon-Kerjan 2022
  - Mixing time of NBRW on a random n-lift of G is $\log_\Lambda n\,(1+o(1))$ whp
  - Diameter of a random n-lift of G is $\log_\rho n\,(1+o(1))$ whp

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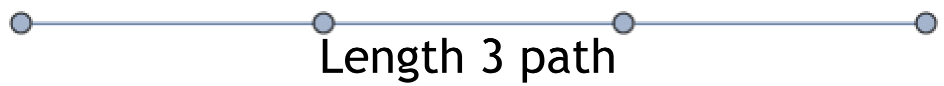
- The Moore bound – best-known upper bound on the girth g of a graph with n vertices
  - Alon, Hoory and Linial 2002 – Given the degree distribution, $g \le 2\log_\Lambda n\,(1+o(1))$
  - Hoory 2023 – For an n-lift of G, $g \le 2\log_\rho n\,(1+o(1))$

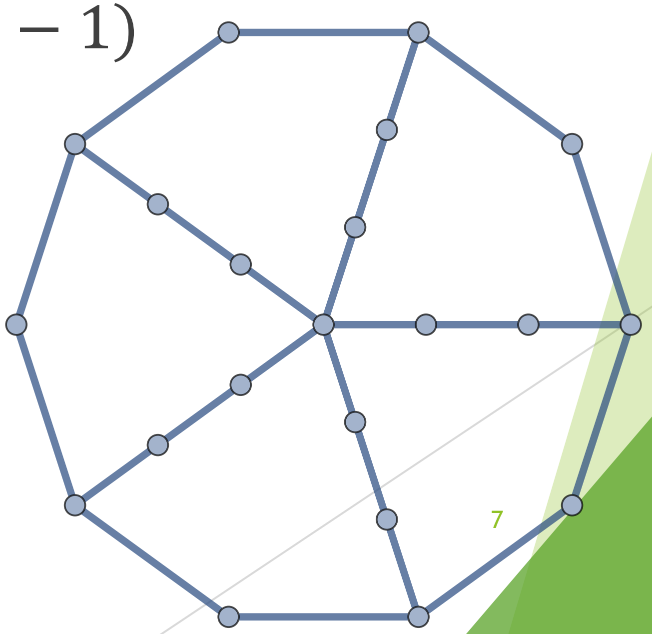
Graphs with ρ = Λ?

1. d-regular - $\rho = \Lambda = d - 1$
2. Bipartite d_L, d_R bi-regular - $\rho = \Lambda = \sqrt{(d_L - 1)(d_R - 1)}$
3. As above with edges replaced by length k paths
   $\rho = \Lambda = \sqrt[k]{d - 1}$ or $\rho = \Lambda = \sqrt[2k]{(d_L - 1)(d_R - 1)}$

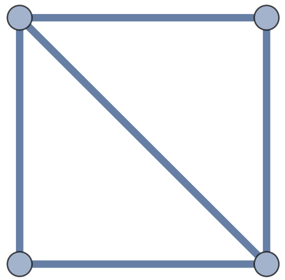


4. and infinitely more graphs…

such as this one with $\rho = \Lambda = \sqrt{2}$

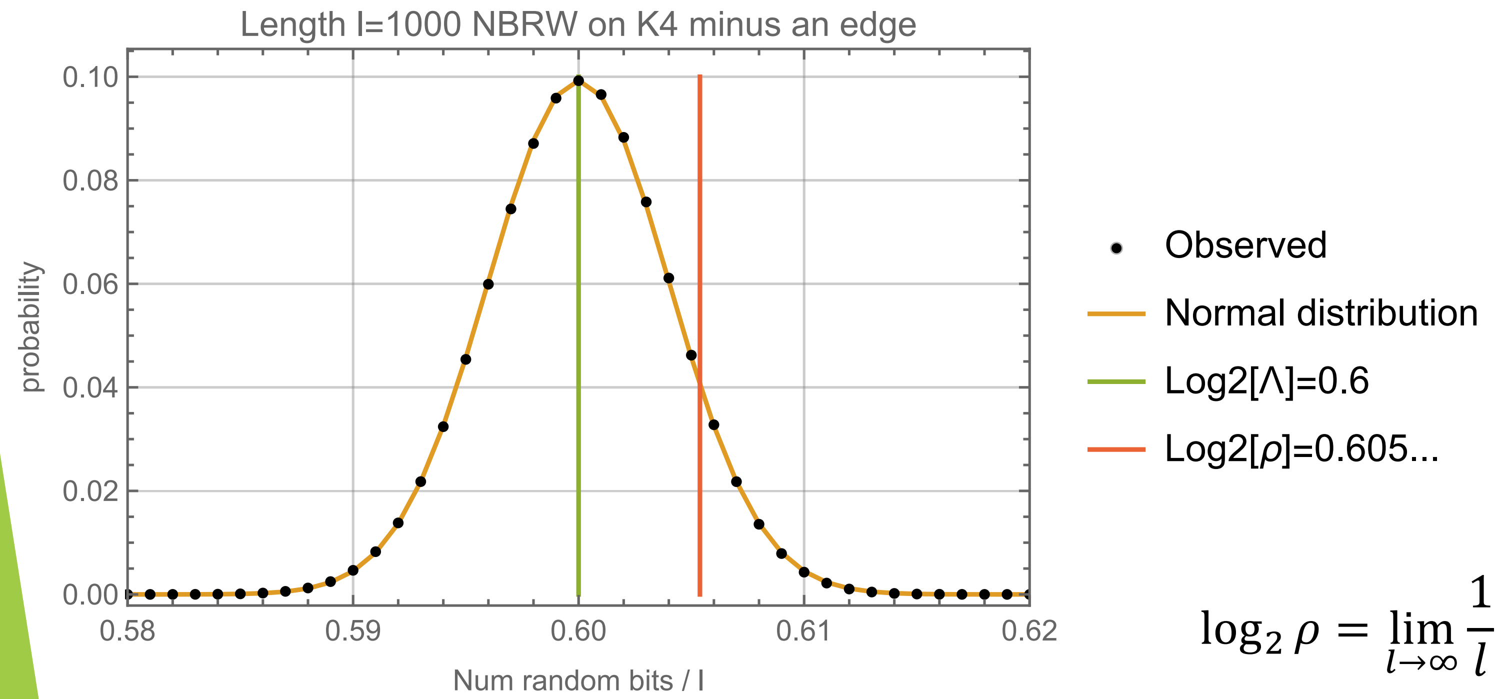


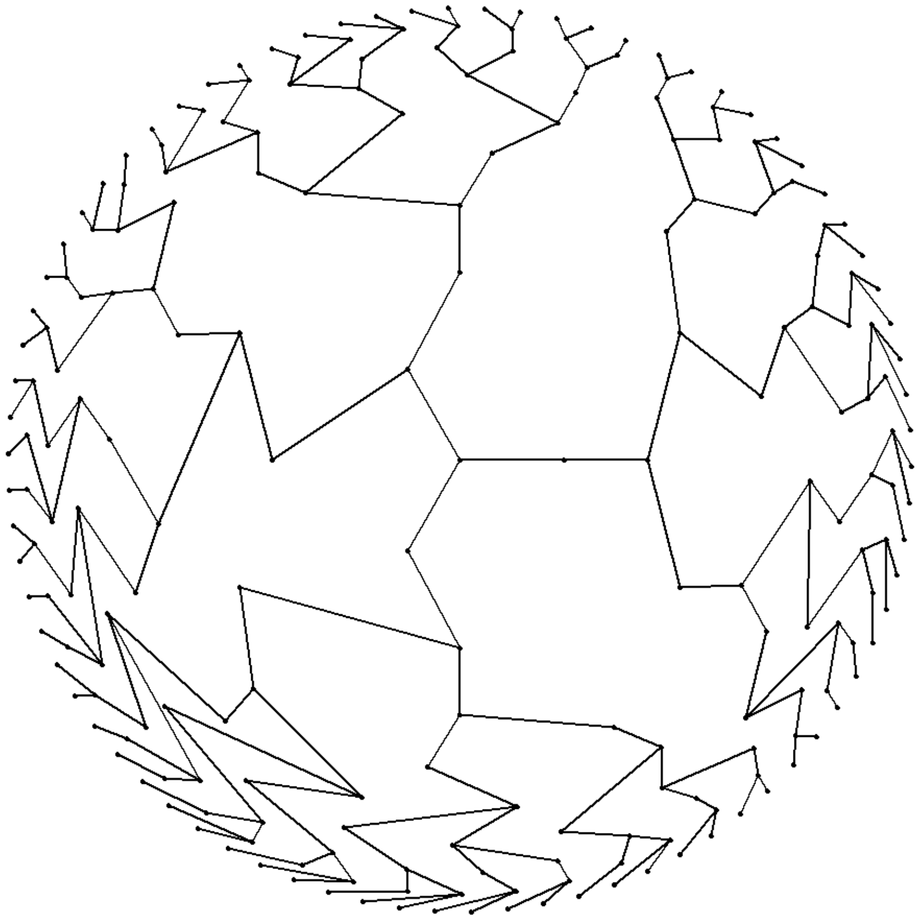
A Graph with ρ > Λ



K4 minus an edge

- $\Lambda = 2^{0.6} = 1.51\ldots$
- $\rho = 1.52\ldots$ is the Perron eigenvalue of $\begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$





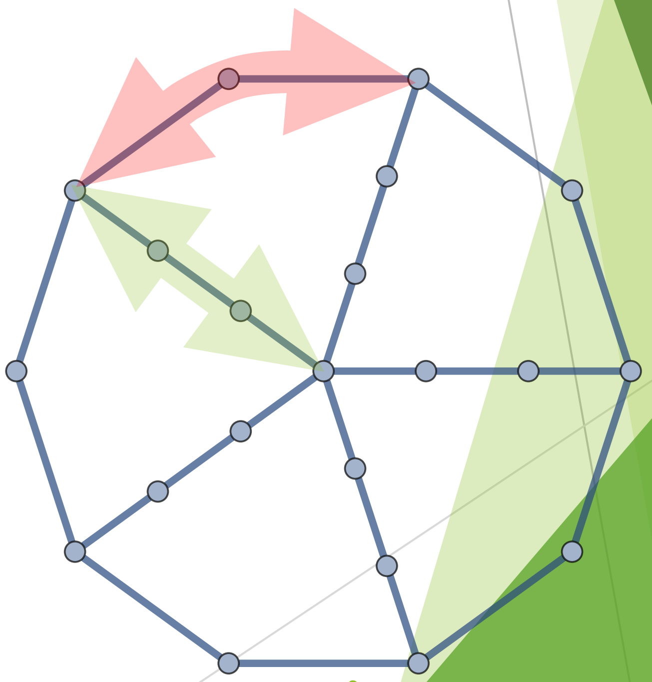
$$\log_2 \rho = \lim_{l\to\infty} \frac{1}{l} \log_2 \mathrm{E}[2^R]$$

Theorem 1

A suspended path is a nb-path where all internal vertices have degree 2 and the end vertices have larger degree.

The following conditions are equivalent:

1. $\rho = \Lambda$
2. For every nb-cycle C in G
$$\prod_{e \in C} \operatorname{outdeg}(e) = \Lambda^{|C|}$$
3. For every suspended path P in G
$$\operatorname{outdeg}(P)\operatorname{indeg}(P) = \Lambda^{2|P|}$$

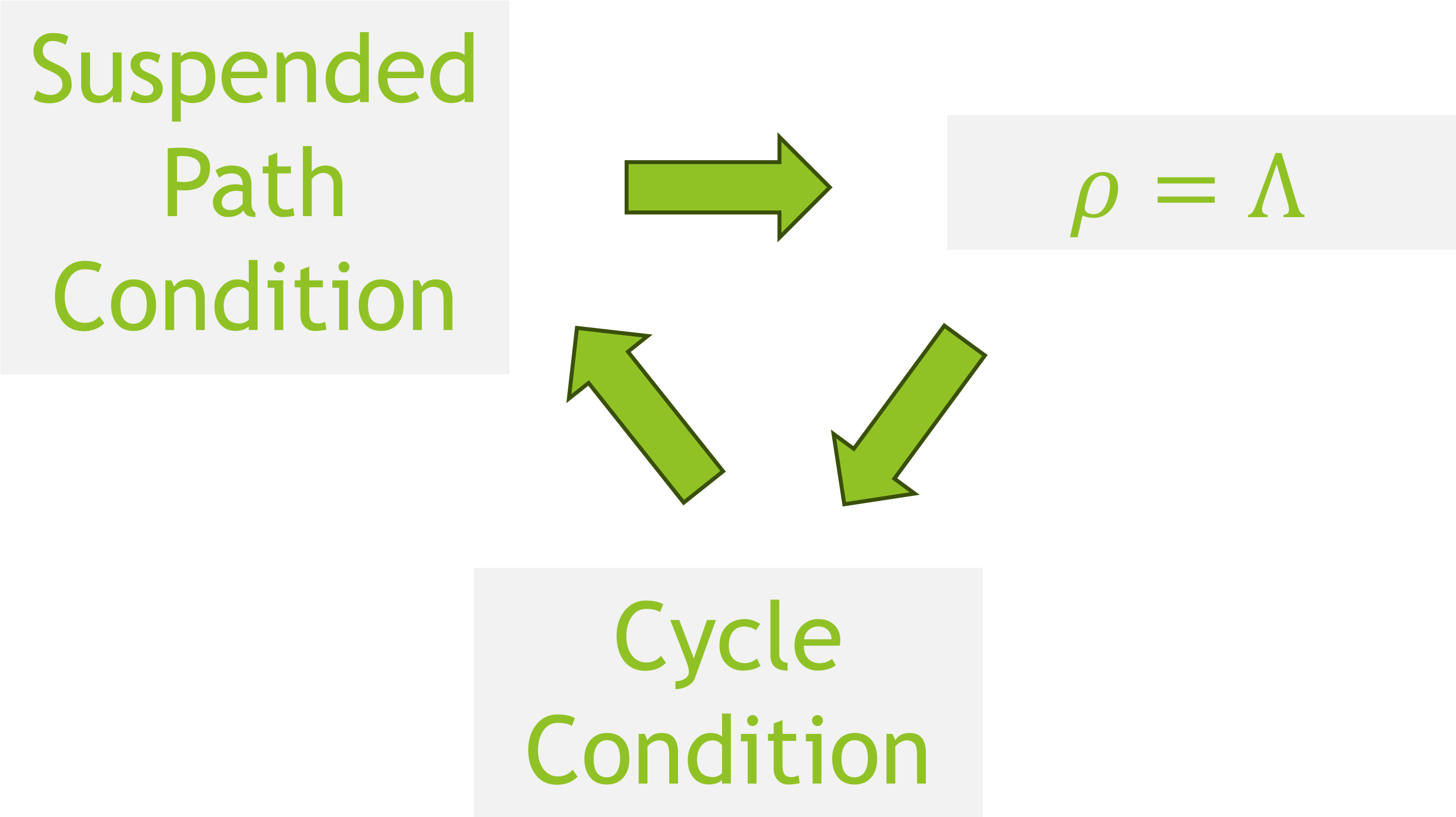


Theorem 2: The Variance Dichotomy

Random variable R_l denoting the number of random bits consumed by the length l NBRW from the stationary distribution.

$$\mathrm{Var}[R_l] = \begin{cases} O(1) & \text{if } \rho = \Lambda \\ \Theta(l) & \text{if } \rho > \Lambda \end{cases}$$

Proof of Theorem 1



Suspended path condition → ρ = Λ

- nb-walk $\omega = (e_0, \dots, e_l) = P_0 P_1 \cdots P_{s-1} P_s$ where P_i are suspended paths except possibly for P_0, P_s that may be partial suspended paths.
- $$f(\omega) = \prod_{i=0}^{l-1} \mathrm{outdeg}(e_i) = \prod_{i=0}^{s-1} \mathrm{outdeg}(P_i)$$
$$= \mathrm{outdeg}(P_0)^{\frac{1}{2}} \left[\prod_{i=1}^{s-1} \mathrm{indeg}(P_i)^{\frac{1}{2}} \mathrm{outdeg}(P_i)^{\frac{1}{2}} \right] \mathrm{indeg}(P_s)^{\frac{1}{2}}$$
$$= \Theta(1) \cdot \left[\prod_{i=1}^{s-1} \Lambda^{|P_i|} \right] = \Theta(1) \cdot \Lambda^l$$
- $$\rho = \rho(B) = \lim_{l\to\infty} \left(v_s \cdot B^l \cdot \bar{1}\right)^{\frac{1}{l}} = \lim_{l\to\infty} \left(\sum_{\omega \in \Omega_l} Pr[\omega] f(\omega) \right)^{\frac{1}{l}} = \Lambda$$ ■

ρ = Λ ⟶ cycle condition (1/5)

- M_t interpolation between P and B for $0 \leq t \leq 1$

$$(M_t)_{e,f} = (P)_{e,f}{}^{1-t} (B)_{e,f}{}^{t} = \begin{cases} \dfrac{1}{\mathrm{outdeg}(e)^{1-t}} & \text{if } e \to f \\ 0 & \text{otherwise} \end{cases}$$

Theorem (Kingman 1961):

$$\rho(M_t) \leq \rho(P)^{1-t} \rho(B)^t$$

ρ = Λ → cycle condition (2/5)

Lemma: If $\rho = \Lambda$ then $\rho(M_t) = \rho^t$ for all $0 \le t \le 1$

Proof:

Since $\rho(B) = \rho$ and $\rho(P) = 1$. By Kingman $\rho(M_t) \le \rho^t$.

$$\nu_s\,(M_t)^l\,\bar{1} = \sum_{\omega=(e_0,\dots,e_l)\in\Omega_l} Pr[\omega] \prod_{i=0}^{l-1} \mathrm{outdeg}(e_i)^t$$

$$= \sum_{\omega} Pr[\omega]\left(2^{R(\omega)}\right)^t = E_{\Omega_l}[2^{tR}] \ge 2^{t\,E_{\Omega_l}[R]} = \Lambda^{tl}$$

$$\rho(M_t) = \lim_{l\to\infty}\left(\nu_s\,(M_t)^l\,\bar{1}\right)^{\frac{1}{l}} \ge \Lambda^t = \rho^t$$

■

ρ = Λ → cycle condition (3/5)

Theorem (Nussbaum 1986):

If $\rho(M_t) = \rho(P)^{1-t}\rho(B)^t$ for some $0 < t < 1$ and P, B are non-negative irreducible matrices then:

1. Equality holds for all $0 \leq t \leq 1$

2. $P = kD^{-1}BD$
   For a scalar $k > 0$ and positive diagonal matrix D

ρ = Λ → cycle condition (4/5)

Theorem:

If $\rho = \Lambda$ then the cycle condition holds

Proof:

- By the Lemma, $\rho(M_t) = \rho^t$ for $0 < t < 1$
- By Nussbaum, $P = kD^{-1}BD$
- For all $e \to f$, $B_{e,f} = 1, P_{e,f} = \mathrm{outdeg}(e)^{-1}$
- There are $\lambda \in \mathbb{R}$, $\varphi: \vec{E} \to \mathbb{R}$ so that for every $e \to f$:

$$-\log \mathrm{outdeg}(e) = \lambda - \varphi(\mathrm{e}) + \varphi(f)$$

ρ = Λ ⟶ cycle condition (5/5)

- There are $\lambda \in \mathbb{R}$, $\varphi : \vec{E} \to \mathbb{R}$ so that for every $e \to f$:
$$-\log \operatorname{outdeg}(e) = \lambda - \varphi(\mathrm{e}) + \varphi(f) \qquad (*)$$
- Summing on all $e \to f$ with weights $\operatorname{outdeg}(e)^{-1}$
$$-\left|\vec{E}\right| \cdot \log \Lambda = \left|\vec{E}\right| \cdot \lambda$$
- Therefore, $\lambda = -\log \Lambda$.
- For any nb-cycle C:
$$\sum_{e \in C} \log \operatorname{outdeg}(e) = |C| \cdot \log \Lambda$$

■

$\neg$ suspended path condition $\rightarrow \neg$ cycle condition (1/3)

- $g(P) = [\text{outdeg}(P)\ \text{indeg}(P)]^{\frac{1}{2|P|}}$
- Suspended path condition: $g(P) = \Lambda$ for all P
- $\Lambda = \prod_{e \in \vec{E}} \text{outdeg}(e)^{\frac{1}{|\vec{E}|}} = \prod_{P} [\text{outdeg}(P)\ \text{indeg}(P)]^{\frac{1}{2|\vec{E}|}} = \prod_{P} g(P)^{\frac{|P|}{|\vec{E}|}}$
- Given a suspended path P_0 with $g(P_0) < \Lambda$
- Need to find cycle C with: $\prod_{e \in C} \text{outdeg}(e) > \Lambda^{|C|}$

$\neg$ suspended path condition $\rightarrow \neg$ cycle condition (2/3)

- $f(e) = \frac{1}{2}\log(\mathrm{outdeg}(e)\ \mathrm{indeg}(e))$
- $\mathrm{E}_{v_s}[f] = \log\Lambda$ and $\log g(P_0) = \mathrm{E}_{v_s}[f|P_0] < \log\Lambda$
- Denote $\mu_0 = v_s$ and μ_1 as μ_0 conditioned on $e \notin P_0$
- $\mathrm{E}_{\mu_1}[f] > \log\Lambda$

$\neg$ suspended path condition $\rightarrow \neg$ cycle condition (3/3)

- Define $\mu_2, \mu_3, \ldots$ each time reducing the support either by:
  - excluding a connected component U of $supp(\mu_i)$ with small $\mathrm{E}_{\mu_i}[f|U]$
  - excluding a suspended path P with small $\mathrm{E}_{\mu_i}[f|P]$
- Process ends when $supp(\mu_i)$ is a cycle C

$$\Lambda < \frac{1}{|C|}\sum_{e \in C} f(e) = \frac{1}{|C|}\sum_{e \in C} \log(\mathrm{outdeg}(e))$$

- Violating the cycle condition.

$\blacksquare$

Theorem 2: The Variance Dichotomy

$$\mathrm{Var}[R_l] = \begin{cases} O(1) & \text{if } \rho = \Lambda \\ \Theta(l) & \text{if } \rho > \Lambda \end{cases}$$

R_l is a random variable counting the number of bits consumed by a length l NBRW starting from the stationary distribution.

Proof of Theorem 2 - the $\rho = \Lambda$ case

- Suspended Path Condition holds
- NB-walk $\omega = (e_0, \dots, e_l)$ satisfies $\prod_{i=0}^{l-1} \text{outdeg}(e_i) = \Theta(1) \cdot \Lambda^l$
- $(R_l(\omega) - \text{E}[R_l])^2 = \left(\sum_{i=0}^{l-1} \log(\text{outdeg}(e)) - l \log_2 \Lambda\right)^2 \leq const$
- $\text{Var}[R_l] \leq const$

Proof of Theorem 2 – variance upper bound when $\rho > \Lambda$

- $\mathrm{Var}[R_l] \leq cl$ for some constant c is a consequence of CLT for Markov chains.
- One may directly compute $\lim_{l\to\infty} \frac{\mathrm{Var}[R_l]}{l}$ given the graph G.

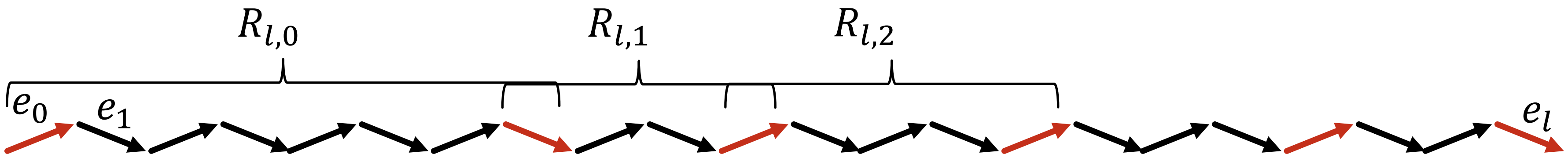
Proof of Theorem 2 – variance lower bound when $\rho > \Lambda$

- Cycle condition is violated
- Cycles C_1, C_2 with $|C_1| = |C_2| = l_0$, a common edge f and different geometric out-degree average
- Probability space $\Omega_{l_0,f,f}$ of length l_0 NBRW conditioned on start and end edges being f
- $R_{l_0,f,f}(\omega) = \sum_{i=0}^{l_0-1} \log_2 \text{outdeg}(e_i)$ where $\omega = (e_0, \cdots, e_{l_0}) \in \Omega_{l_0,f,f}$
- Then $\text{Var}\left[R_{l_0,f,f} | \Omega_{l_0,f,f}\right] > 0$

Proof of Theorem 2 – variance lower bound when $\rho > \Lambda$

- Given a length l NBRW $\omega = (e_0, \dots, e_l)$

  Perform a partial exposure χ of the edges and write $R_l = \sum_i R_{l,i}$

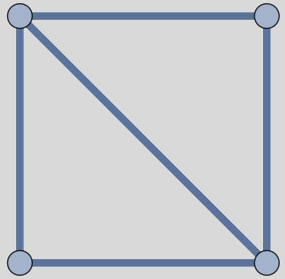


- $\mathrm{Var}[R_l|\chi] = \sum_i \mathrm{Var}\left[R_{l,i}|\chi\right]$

- W.h.p. for $c_1 l$ of the segments, $R_{l,i}|\chi$ are i.i.d. $R_{l_0,f,f}$ variables

- $\mathrm{Var}[R_l] \geq c_2 l$

$\Lambda = 2^{0.6} = 1.51\ldots,$
$\rho = 1.52\ldots.$



Questions and Thoughts

1. Is there a quantitative version of our result?

   G is δ-close to satisfying the suspended path condition iff $\rho < \Lambda + \varepsilon$.

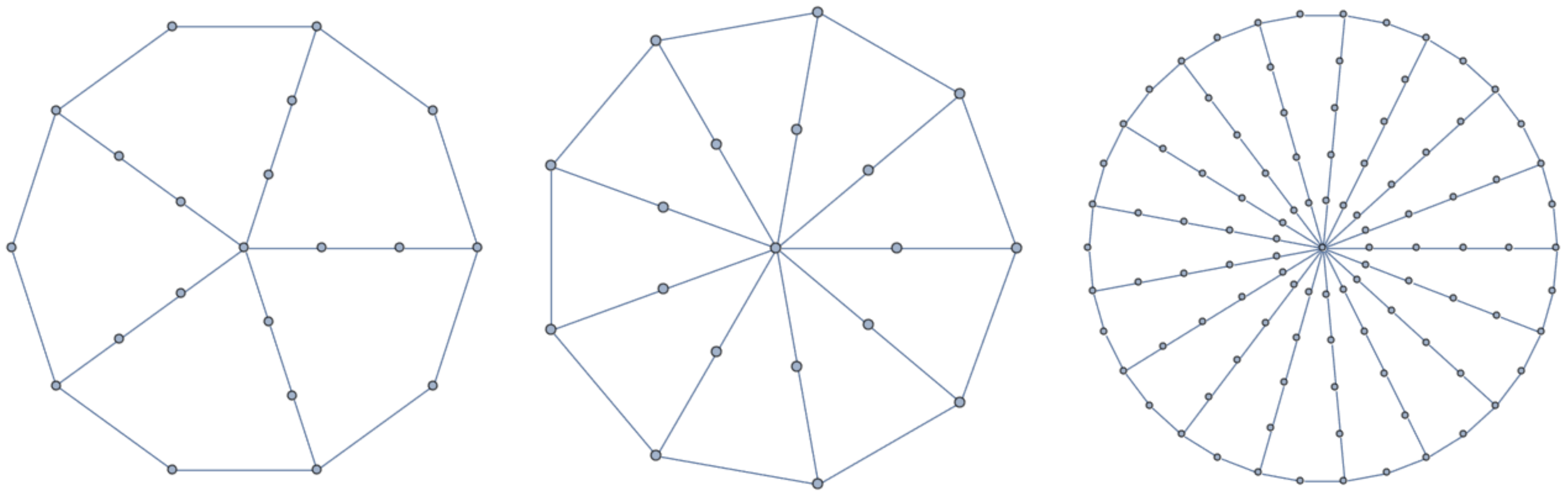
2. What is the Minimal ρ possible, given the degree distribution?

3. Prove that $\rho(K_4 - edge)$ has the minimal possible ρ for a graph with 50% deg 2 and 50% deg 3 vertices?

4. Prove that any girth g graph with 50% deg 2 and 50% deg 3 vertices,

   $g \leq 2 \log_\rho n \left(1 + o(1)\right)$ where $\rho = \rho(K_4 - edge)$.

More graphs with ρ = Λ



Questions?