On the eigenvalues of the graphs $D(5, q)$

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Motivation

The isoperimetric constant of a graph Γ measures how "well-connected" a graph is.

Definition (Isoperimetric constant)

$$h(\Gamma) := \min\left\{\frac{|E(S,S^c)|}{|S|} : S \subset V(\Gamma), 0 < |S| \leq \frac{|V(\Gamma)|}{2}\right\}$$

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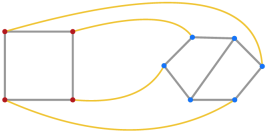

$|S| = 4$, $|E(S,S^c)| = 4 \implies \frac{|E(S,S^c)|}{|S|} = 1.$

Expanders

Definition (Expanders)

An infinite family of k-regular finite graphs $\{\Gamma_n\}$ is called *expander family*

- if there exists $C := C(k) > 0$ such that $h(\Gamma_n) \geq C$ for all n,
- and $|V(\Gamma_n)| \to \infty$ as $n \to \infty$.

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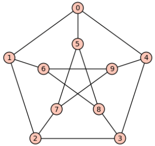
Expander graphs and their Applications - Hoory, Linial and Wigderson 2006 Coding Theory, Probability theory, Computer Science.

Spectral graph theory

- The **adjacency matrix** A of a graph G with n vertices is a $n \times n$ matrix with $A(x,y) =$ the number of edges between x and y.



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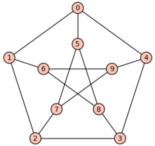
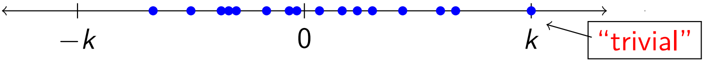


3^1, 1^5, -2^4

- **Fact**: For a connected, k-regular graph G on n vertices

$$k = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq -k.$$



Spectral Expanders

Theorem (Alon-Milman 1985, Dodziuk 1984, and Mohar 1989)

Let Γ be a k-regular connected graph with second largest eigenvalue λ_2. Then

$$\frac{k-\lambda_2}{2} \leq h(\Gamma) \leq \sqrt{(k-\lambda_2)(k+\lambda_2)}.$$

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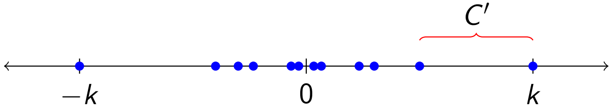

Alon-Boppana Bound and Ramanujan Graphs

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Let $\{\Gamma_n\}$ be a family of k-regular connected graphs, with $|V_n| \to \infty$ as $n \to \infty$. Then

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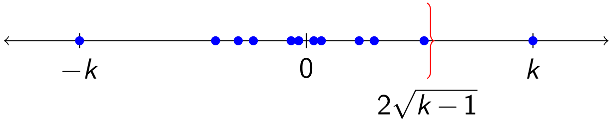

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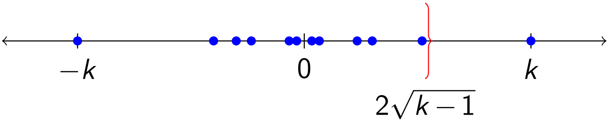

Ramanujan graphs $\longleftrightarrow$ best possible expanders.

Known constructions of Ramanujan graphs

- **Lubotzky, Phillips, & Sarnak 1988 and independently Margulis 1988** constructed an infinite families of $(p+1)$-regular graphs where p is a prime.

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- **Morgenstern 1994** extended their constructions to prime powers.
- **Marcus, Spielman, & Srivastava 2015** showed there exist infinite families of bipartite Ramanujan graphs for any degree greater than 2.

Graphs $D(k,q)$ - Lazebnik and Ustimenko 1995

Let q be a prime power, $k \geq 2$, $P = L = \mathbb{F}_q^k$, and $V = P \cup L$.

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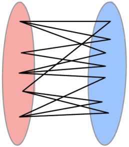


$$(p) = (p_1, p_2, p_3, \ldots, p_j, \ldots p_k) \sim [\ell] = [\ell_1, \ell_2, \ell_3, \ldots, \ell_j, \ldots, \ell_k]$$

$$\Updownarrow$$

$$p_2 + \ell_2 = p_1 \ell_1$$

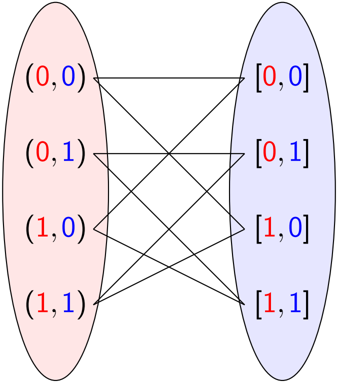
$$p_3 + \ell_3 = p_1 \ell_2$$

$$p_j + \ell_j = \begin{cases} p_{j-2}\ell_1 & \text{if } j = 0,1 \pmod 4 \\ p_1 \ell_{j-2} & \text{if } j = 2,3 \pmod 4. \end{cases}$$

An Example

Let $q = 2$ and $k = 2$, $P = L = \mathbb{F}_2^2$, $V = P \cup L$

$$(p_1, p_2) \sim [l_1, l_2] \quad \longleftrightarrow \quad p_2 + \ell_2 = p_1 \ell_1$$



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  A component is denoted $CD(k,q)$
- **Lazebnik and Ustimenko 1995** The girth of $D(k,q)$ is at least $k+4$ when k is even, and $k+5$ when k is odd.

  Thus gives asymptotically the best known general lower bound (except for $k=5$) for $ex(n, \{C_{2k+1}, \ldots, C_4, C_3\})$, i.e.,

$$ex(n, \{C_{2k+1}, \ldots, C_4, C_3\}) \geq n^{1+\frac{2}{3k-3+\varepsilon}}$$

  where ε is 1 if k is even, and 0 if k is odd.

Main Problem

Ustimenko's Conjecture

$$\lambda_2(CD(k,q)) \leq 2\sqrt{q}$$

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Progress so far!

- For $k = 2, 3$ and all q (Li, Lu, and Wang 2009, Cioabă, Lazebnik, and Li 2014).
- For $k = 4$ and all q (Moorhouse, Sun, and Williford 2017).
- For $k = 5, 6$ and all odd q (G. and Taranchuk 2023+).

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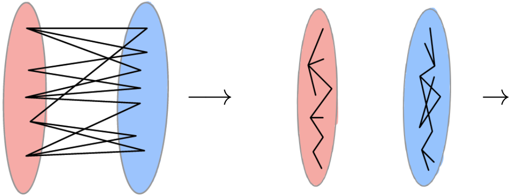

The Point collinearity graph is a Cayley graph. Make use of the Representation theory.

Cayley Graphs

Let G be a group, and let S be a nonempty, finite subset of G such that $S = S^{-1}$ and $1 \notin S$.

Definition (Cayley graph)

The Cayley graph $\mathscr{G}(G, S)$ is the graph with vertex set $V = G$ and edge set $E = \{\{x, y\} | \exists\ s \in S : y = xs\}$.

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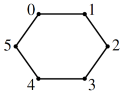
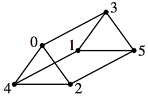
Examples:



$G = \mathbb{Z}/6\mathbb{Z},\ S = \{1,-1\}$



$G = \mathbb{Z}/6\mathbb{Z},\ S = \{2,-2,3\}$

Representation theory of finite groups

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- Two representations $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ are said to be **equivalent** if there exists an isomorphism $T : V \to W$ such that $\sigma(g) = T\rho(g)T^{-1}\ \forall g \in G$.

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- A representation of degree n is a group homomorphism $\rho : G \to GL_n(\mathbb{C})$.

Examples

Let G be a finite group.

- **Trivial Representation** $\rho : G \to \mathbb{C}^*$, $g \mapsto 1 \; \forall g \in G$.

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- **Regular Representation** $\lambda : G \to GL_{|G|}(\mathbb{C})$, $g \mapsto R_g \ \forall g \in G$, where $R_g = [r_{x,y}^{(g)}]_{x,y \in G}$ and

$$r_{x,y}^{(g)} = \begin{cases} 1, & \text{if } y = xg \\ 0, & \text{otherwise.} \end{cases}$$

Irreducible representations

Let $\rho : G \to GL(V)$ be a representation of G.

- A subspace W of V is G-**invariant** if $\forall g \in G$ and $w \in W$, one has $\rho(g)(w) \in W$.

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- A representation ρ of a group G is **irreducible** if the only G-invariant subspaces are trivial, i.e., $\{0\}$ and V.
- If (ρ_1, W_1) and (ρ_2, W_2) are representations of a group G and $V = W_1 \oplus W_2$, then $\rho = \rho_1 \oplus \rho_2$ is their **direct sum**, that is

$$\rho(g)v = \rho_1(g)w_1 + \rho_2(g)w_2 \text{ where } v = w_1 + w_2.$$

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Theorem

Let G be a finite group, $\lambda : G \to GL_{|G|}(\mathbb{C})$, $g \mapsto R_g$ the regular representation, and $\widehat{G} = \{\rho_1, \rho_2 \ldots, \rho_t\}$. Then

- $\lambda \simeq d_{\rho_1}\rho_1 \oplus d_{\rho_2}\rho_2 \oplus \cdots \oplus d_{\rho_t}\rho_t$
- $|G| = \sum_{i=1}^{t} d_{\rho_i}^2$.

Spectra of Cayley Graphs

Definition (Recall)

The Cayley graph $\mathscr{G}(G,S)$ is the graph with vertex set $V = G$ and edge set $E = \{\{x,y\}|\ \exists\ s \in S : y = xs\}$.

Lemma

Let A be the adjacency matrix of a Cayley graph $\mathscr{G}(G,S)$. Then

$$A = \sum_{s \in S} R_s$$

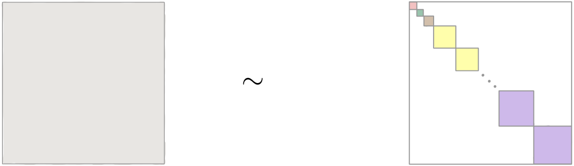
Spectra of Cayley Graphs

Let $\widehat{G} = \{\rho_1, \rho_2, \ldots, \rho_t\}$. For each i, let $\rho_i(S) = \sum_{s \in S} \rho_i(s)$.

Theorem

Let A be the adjacency matrix of a Cayley graph $\mathscr{G}(G, S)$. The characteristic polynomial of A is given by

$$\Phi(x) = \det(xI_{|G|} - A) = \prod_{i=1}^{t} \det[xI_{d_{\rho_i}} - \rho_i(S)]^{d_{\rho_i}}$$



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G. and Taranchuk 2023+

Let $q = p^e$, where p is an odd prime and e is a positive integer. Then

$$\lambda_2(D(5,q)) \leq 2\sqrt{q}$$

First Lemma

Lemma 1

Let q be an odd prime power, $P = L = \mathbb{F}_q^5$, and $V = P \cup L$. Then

$$D(5,q) \cong \Gamma(5,q)$$

For $D(5,q)$

$$(p_1, p_2, p_3, p_4, p_5) \sim [l_1, l_2, l_3, l_4, l_5]$$
$$\Updownarrow$$
$$p_2 + l_2 = p_1 l_1$$
$$p_3 + l_3 = p_1 l_2$$
$$p_4 + l_4 = p_2 l_1$$
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For $\Gamma(5,q)$

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$$p_2 + l_2 = p_1 l_1$$
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$$p_4 + l_4 = p_1^2 l_1$$
$$p_5 + l_5 = p_1^2 l_1^2$$

Second Lemma

Let $G = (\mathbb{F}_q^5, \cdot)$, be a group, where

$$X \cdot Y = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3,\ x_4 + y_4 + 2x_1y_2,\ x_5 + y_5 + 2x_1y_3).$$

Let $S = \{(x, xa, xa^2, x^2a, x^2a^2) : a, x \in \mathbb{F}_q, x \neq 0\}$ be a subset of G.

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Lemma 2

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$$(p_1, p_2, p_3, p_4, p_5) \sim (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$$
$$(r_1, r_2, r_3, r_4, r_5) \sim (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$$

$$\begin{aligned} p_2 - r_2 &= (p_1 - r_1)\ell_1, \\ p_3 - r_3 &= (p_1 - r_1)\ell_1^2, \\ p_4 - r_4 &= (p_1^2 - r_1^2)\ell_1, \\ p_5 - r_5 &= (p_1^2 - r_1^2)\ell_1^2. \end{aligned}$$

Third Lemma

Lemma 3

Every eigenvalue λ of the point graph with multiplicity m corresponds to a pair of eigenvalues $\pm\sqrt{\lambda + q}$ of the graph $\Gamma(5, q)$ each with multiplicity m.

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Recall Ustimenko's Conjecture is that the second largest eigenvalue of graph $CD(k, q)$ is at most $2\sqrt{q}$.

Aim To show second largest eigenvalue of the point graph (which is Cayley) is at most $3q$.

Irreducible representations of the group G

- Group $G = (\mathbb{F}_q^5, \cdot)$ where

$$X \cdot Y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 + 2x_1y_2, x_5 + y_5 + 2x_1y_3).$$

$\zeta = e^{\frac{2\pi i}{p}}$ $\quad tr : \mathbb{F}_q \to \mathbb{F}_p \quad tr(a) = a + a^p + \cdots + a^{p^{e-1}}$

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- Irreducible representations of G are:
  - **Type 1** For $\alpha, \beta, \gamma \in \mathbb{F}_q$, $\chi_{\alpha,\beta,\gamma} : G \to GL_1(\mathbb{C})$

$$\chi_{\alpha,\beta,\gamma}(X) := \zeta^{tr(\alpha x_1 + \beta x_2 + \gamma x_3)}.$$

  - **Type 2** For $\alpha, \beta, \gamma \in \mathbb{F}_q$ with $\alpha \neq 0$, $M_{\alpha,\beta,\gamma} : G \to GL_q(\mathbb{C})$

$$M_{\alpha,\beta,\gamma}(X) := [\zeta^{tr\left(\left(x_2 + \frac{\beta}{\alpha} x_3\right)j + \alpha x_4 + \beta x_5 + \gamma x_3\right)} \delta_{2x_1\alpha + j,k}]_{j,k \in \mathbb{F}_q}.$$

  - **Type 3** For $\tau, \mu \in \mathbb{F}_q$ with $\tau \neq 0$, $N_{\tau,\mu} : G \to GL_q(\mathbb{C})$

$$N_{\tau,\mu}(X) := [\zeta^{tr(x_3 j + \tau x_5 + \mu x_2)} \delta_{2x_1\tau + j,k}]_{j,k \in \mathbb{F}_q}.$$

$\zeta = e^{\frac{2\pi i}{p}}$ $\quad tr : \mathbb{F}_q \to \mathbb{F}_p \quad tr(a) = a + a^p + \cdots + a^{p^{e-1}}$

Eigenvalues from the irreducible representations

Generating set $S = \{(x, xa, xa^2, x^2a, x^2a^2) : a, x \in \mathbb{F}_q, x \neq 0\}$.

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Eigenvalues coming from Type 2 and Type 3 are sums involving multiplicative and additive character of $\mathbb{F}_q$. We bound its absolute value by using Weil's bound for character sums.

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On the eigenvalues of the graphs D(5,q),
arXiv:2207.04629 - July 2022,
G. and Taranchuk.