L-systems and the Lovász number

William Linz

University of South Carolina

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Hoffman bound

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$$\alpha(G) \leq \frac{-\lambda_n}{\lambda_1 - \lambda_n} n,$$

where $\alpha(G)$ is the maximum size of an independent set in G.

Sketch of proof of Hoffman bound

- Define the matrix

$$M = A - \lambda_n I - \frac{\lambda_1 - \lambda_n}{n} J,$$

where J is the $n \times n$ all 1s matrix.

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  where J is the $n \times n$ all 1s matrix.
- Fact: M is positive semidefinite.
- Since M is positive semidefinite, for any vector x,

$$0 \leq x^T M x = x^T A x - \lambda_n x^T x - \frac{\lambda_1 - \lambda_n}{n} x^T J x.$$

Sketch of proof continued

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- Let s be the characteristic vector of the independent set S. Since S is an independent set, $s^T A s = \sum_{u,v} a_{uv} s_u s_v = 0$, so setting $x = s$ in (1),

$$0 \leq -\lambda_n |S| - \frac{\lambda_1 - \lambda_n}{n} |S|^2.$$

Sketch of proof continued

$$0 \leq x^T Mx = x^T Ax - \lambda_n x^T x - \frac{\lambda_1 - \lambda_n}{n} x^T Jx. \qquad (1)$$

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$$0 \leq -\lambda_n |S| - \frac{\lambda_1 - \lambda_n}{n} |S|^2.$$

- Rearranging this inequality gives Hoffman's bound.

Pseudoadjacency matrices

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We call any matrix which satisfies these two properties a *pseudoadjacency matrix* for the graph G.

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Let G be an n-vertex regular graph, and let A be a pseudoadjacency matrix for G with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then,

$$\alpha(G) \leq \frac{-\lambda_n}{\lambda_1 - \lambda_n} n.$$

How do we find the best possible bound for the independence number from pseudoadjacency matrices?

Lovász number definition

Lovász number

Let G be an n-vertex graph. For a matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, denote the largest eigenvalue of A by $\text{lev}(A)$. The Lovász number $\vartheta(G)$ is defined to be

$$\vartheta(G) = \min\{\text{lev}(A) : A \text{ is symmetric}, \;\; a_{ij} = 1 \text{ if } i = j \text{ or } i \not\sim j.\}$$

Sandwich property of the Lovász number

Recall the Shannon capacity of a graph G is defined to be

$$\Theta(G) = \sup(\alpha(G^k))^{\frac{1}{k}} = \lim_{k\to\infty}(\alpha(G^k))^{\frac{1}{k}}.$$

Sandwich Theorem (Lovász)

$$\alpha(G) \leq \Theta(G) \leq \vartheta(G).$$

Alternate Lovász number characterization

The Lovász number can be written as a semidefinite program, so it can be computed in polynomial time up to arbitrary precision.

Theorem (Lovász)

Let G be an n-vertex graph, and let $B = (b_{ij})_{1 \le i,j \le n}$ range over all positive semidefinite matrices with $b_{ij} = 0$ whenever $i \sim j$ and $\mathrm{Tr}(B) = 1$. Then,

$$\vartheta(G) = \max_{B} \mathrm{Tr}(BJ).$$

Connection to pseudoadjacency matrices

The Lovász number is, in some sense, the best bound for the independence number that could be obtained by using pseudoadjacency matrices.

Theorem (Lovász)

Let G be a regular graph with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then,

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- $J - xA$ satisfies the conditions for the definition of the Lovász number, so

$$\vartheta(G) \leq \text{lev}(J - xA).$$

- The eigenvalues of $J - xA$ are $n - x\lambda_1, -x\lambda_2, \dots, -x\lambda_n$, so

$$\text{lev}(J - xA) = \max\{n - x\lambda_1, -x\lambda_n\}.$$

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- Choosing x so that $n - x\lambda_1 = -x\lambda_n$, *i.e.* $x = n/(\lambda_1 - \lambda_n)$ gives

$$\vartheta(G) \leq \frac{-\lambda_n}{\lambda_1 - \lambda_n} n.$$

Erdos-Ko-Rado theorem

A family of sets $\mathcal{F}$ is *intersecting* if any two sets in the family have a nonempty intersection.

Erdős-Ko-Rado

Let $n \geq 2k$. Then, if $\mathcal{F} \subset \binom{[n]}{k}$ is an intersecting family, we have

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One of the many proofs of EKR uses the Hoffman bound.

Sketch of proof of EKR with Kneser graphs

- Recall the *Kneser graph* $G(n, k)$, which has $V(G(n, k)) = \binom{[n]}{k}$, and $A \sim B$ if and only if $A \cap B = \emptyset$.

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- Independent sets in the Kneser graph correspond to intersecting k-uniform families.
- The eigenvalues of the Kneser graph $G(n,k)$ are $(-1)^i \binom{n-k-i}{k-i}$, $i = 0, \ldots, k$, so

$$\binom{n-1}{k-1} \leq \alpha(G) \leq \vartheta(G) \leq \frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k} + \binom{n-k-1}{k-1}} \binom{n}{k} = \binom{n-1}{k-1}.$$

Erdos-Ko-Rado for t-intersecting families

A family of sets $\mathcal{F}$ is *t-intersecting* if every two sets in the family intersect in at least t elements.

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Theorem (Wilson)

If $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting, then for $n \geq (t+1)(k-t+1)$,

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

t-Kneser graphs

- Define the *t-Kneser graph* $G(n, k, t)$ as the graph with $V(G(n, k, t)) = \binom{[n]}{k}$, and $A \sim B$ if and only if $|A \cap B| < t$.

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- Unfortunately, the Hoffman bound for the adjacency matrix does not give a tight bound for the independence number of $G(n,k,t)$.
- Wilson constructed a suitable pseudoadjacency matrix for $G(n,k,t)$ to show

$$\binom{n-t}{k-t} \leq \alpha(G(n,k,t)) \leq \vartheta(G(n,k,t)) \leq \binom{n-t}{k-t}.$$

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- Theorem (Ahlswede-Khachatrian): If $\mathcal{F} \subset \binom{[n]}{k}$ is a t-intersecting family, then

$$|\mathcal{F}| \leq \max_i \{F \in \binom{[n]}{k} | |F \cap [t+2i]| \geq t+i\}.$$

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- Is there a proof of the Ahlswede-Khachatrian complete intersection theorem using pseudoadjacency matrices?

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- But $\vartheta(G(11,5,2)) = 105$, so no suitable pseudoadjacency matrix can exist.

L-systems

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- For positive integers n and k, and a set of integers $L \subseteq [0, k-1]$, an L-system is a collection of sets $\mathcal{F} \subset \binom{[n]}{k}$ such that for any two distinct sets $A, B \in \mathcal{F}$, $|A \cap B| \in L$.

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- Intersecting k-uniform families correspond to L-systems with $L = \{1, 2, \ldots, k-1\}$; t-intersecting families correspond to L-systems with $L = \{t, t+1, \ldots, k-1\}$.

L-systems and generalized Johnson graphs

Definition (Generalized Johnson graphs)

Let n and k be positive integers with $n > k$ and $L \subset [0, k-1]$. The *generalized Johnson graph* $G = G(n, k, L)$ is the graph with $V(G) = \binom{[n]}{k}$, and $AB \in E(G) \iff |A \cap B| \notin L$.

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L-systems correspond to independent sets in generalized Johnson graphs.

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Theorem (L.)

For any $\epsilon > 0$, there is an explicit construction of a graph on n vertices which has $\vartheta(G)/\alpha(G) = \Omega(n^{\frac{1}{2}-\epsilon})$.

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Note that for random graphs $G(n, \frac{1}{2})$, $\vartheta(G(n, \frac{1}{2})) = \Theta(\sqrt{n})$ with high probability, while $\alpha(G(n, \frac{1}{2})) = \log_2(n)$ with high probability.

Construction

- Let $G = G(n, 2\ell + 1, \ell)$ and set $|V(G)| = N = \binom{n}{2\ell+1}$. Then, we have $\vartheta(G) = \Theta(N^{\frac{2\ell}{2\ell+1}})$, while $\alpha(G) = \Theta(N^{\frac{\ell}{2\ell+1}})$.

Construction

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- This provides the promised explicit construction - indeed, choose ℓ sufficiently large so that $\frac{1}{2} - \frac{\ell}{2\ell+1} = \frac{1}{4\ell+2} < \epsilon$.

The Lovász number

- The graphs $G(n, k, \ell)$ are regular and edge-transitive. Hence, by a result of Lovász,

$$\vartheta(G(n,k,\ell)) = \frac{-\lambda_{\binom{n}{k}}}{\lambda_1 - \lambda_{\binom{n}{k}}}\binom{n}{k}.$$

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- The eigenvalues of $G(n,k,\ell)$ are

$$p_{k-\ell}(j) = \sum_{r=k-\ell}^{k}(-1)^{r-k+\ell+j}\binom{r}{k-\ell}\binom{n-2r}{k-r}\binom{n-r-j}{r-j}$$
$$= \sum_{r=0}^{k-\ell}(-1)^{r}\binom{j}{r}\binom{k-j}{k-\ell-r}\binom{n-k-j}{k-\ell-r}$$

for $j = 0,\ldots,k$.

The Lovász number (continued)

- The largest eigenvalue is

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- Therefore,

$$\vartheta(G(n,k,\ell)) = \Theta(n^{k-1}).$$

Independence number

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- So, if $k = 2\ell + 1$, then $\vartheta(G(n, 2\ell+1, \ell)) = \Theta(n^{2\ell})$, while $\alpha(G(n, 2\ell+1, \ell)) = \Theta(n^{\ell})$.

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Let n and k be positive integers, and let $L \subset [0, k-1]$. Then, if $\mathcal{F}$ is an L-system with k and L fixed and $n \to \infty$,

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- This is true if $L = [0, k-1] \setminus \{\ell\}$.
- This is the same order of magnitude as the two general bounds for the maximum size of an L-system due to Deza-Erdős-Frankl and Ray-Chaudhuri–Wilson.

- For which generalized Johnson graphs $G(n, k, L)$ is it the case that

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- This equality holds by the (t-intersecting) Erdős-Ko-Rado theorem for $n \geq (t+1)(k-t+1)$ and $L = \{t, t+1, \ldots, k-1\}$.

The Lovász number of unions of classes of other association schemes

- More generally, the Lovász number of graphs whose edge-sets are unions of classes of other association schemes could be studied.
- For example, the analogues of L-systems for vector spaces over a finite field correspond to independent sets in unions of classes of graphs from the Grassmann scheme.