Neumaier graphs

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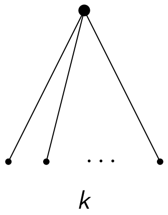
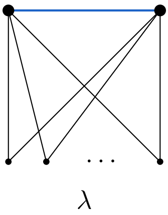
Neumaier graphs

2 Regularity in graphs

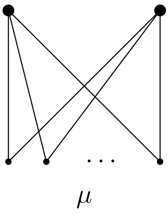
k-regular



edge-regular



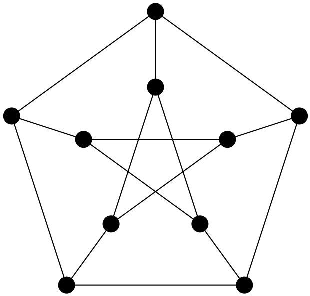
co-edge-regular



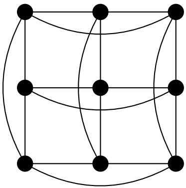
3 Strongly regular graphs

Definition

A regular graph is ***strongly regular*** if it is edge-regular and co-edge-regular.



The Petersen graph
srg$(10, 3, 0, 1)$



The 3×3 rook's graph
srg$(9, 4, 1, 2)$

4 Regularity of subsets

Definition

A vertex subset S is *e-regular* if for every vertex $x \notin S$ we have $|N(x) \cap S| = e$.

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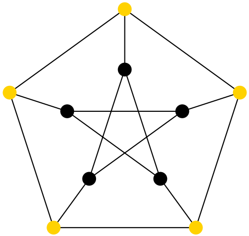
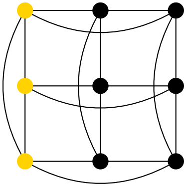


1-regular subset
No regular cliques



A 1-regular clique

5 Neumaier's question

Theorem (Neumaier, 1981)

A vertex-transitive and edge-transitive graph with a regular clique is strongly regular.

Problem (Neumaier)

Is a regular, edge-regular graph with a regular clique necessarily strongly regular?

5 Neumaier's question

Theorem (Neumaier, 1981)

A vertex-transitive and edge-transitive graph with a regular clique is strongly regular.

Problem (Neumaier)

Is a regular, edge-regular graph with a regular clique necessarily strongly regular?

Definition

A ***Neumaier graph*** is a regular, edge-regular graph with a regular clique. It is a ***strictly Neumaier graph*** if it is not strongly regular.
A Neumaier graph has parameters $(v, k, \lambda; e, s)$ if it is an edge-regular graph with parameters (v, k, λ), admitting an e-regular clique of size s.

6 The main questions

Remark

There are 'many' strongly regular (i.e. non-strictly) Neumaier graphs.

Problem

Do strictly Neumaier graphs exist?

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Problem

Do strictly Neumaier graphs exist?

Problem

For which parameter sets $(v, k, \lambda; e, s)$ do strictly Neumaier graphs exist?

Feasibility conditions

7 Counting

Theorem (folklore; Neumaier, 1981; Evans-Goryainov-Panasenko, 2019)

If there is a Neumaier graph with parameters $(v, k, \lambda; e, s)$, then

(i) $v > k > \lambda$ *and* $v - 2k + \lambda \geq 0$*;*

(ii) $vk \equiv 0 \pmod 2$, $k\lambda \equiv 0 \pmod 2$ *and* $vk\lambda \equiv 0 \pmod 6$*;*

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(ii) $vk \equiv 0 \pmod 2$, $k\lambda \equiv 0 \pmod 2$ *and* $vk\lambda \equiv 0 \pmod 6$;

(iii) $s(k - s + 1) = (v - s)e$;

(iv) $s(s - 1)(\lambda - s + 2) = (v - s)e(e - 1)$;

(v) $k - s + e - \lambda - 1 \geq 0$.

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(iv) $s(s - 1)(\lambda - s + 2) = (v - s)e(e - 1)$;

(v) $k - s + e - \lambda - 1 \geq 0$.

If there is a strictly Neumaier graph with parameters $(v, k, \lambda; e, s)$, *then moreover*

(i*) $v - 1 > k$ *and* $v - 2k + \lambda \geq 2$;

(v*) $k - s + e - \lambda - 1 \geq 1$;

(vi) $\lambda + 3 > s \geq 4$;

(vii) $1 \leq e < s - 1$.

8 And more counting

Theorem (Abiad-Castryck-DB-Koolen-Zeijlemaker, 2021)

If there is a Neumaier graph with parameters $(v, k, \lambda; e, s)$*, then*
$(v-k-1)(v-k-2) - k(v-2k+\lambda) \geq 0.$
If there is a strictly Neumaier graph with parameters $(v, k, \lambda; e, s)$*, then*
$(v-k-1)(v-k-2) - k(v-2k+\lambda) > 0.$
(This result is independent of e and s, true for all edge-regular graphs.)

9 Table of admissible parameters (strictly)

v	k	λ	e	s
16	9	4	2	4
22	12	5	2	4*
24	8	2	1	4
25	12	5	2	5
	16	9	3	5
26	15	8	3	6
28	9	2	1	4
	15	6	2	4
		8	3	7
	18	11	4	7
33	24	17	6	9

* Non-existence by computer search Evans-Goryainov-Panasenko and Abiad-De Boeck-Zeijlemaker.

v	k	λ	e	s
34	18	7	2	4
35	10	3	1	5
	16	6	2	5
	18	9	3	7
	22	12	3	5
36	11	2	1	4
	15	6	2	6
	20	10	3	6
	21	12	4	8
	25	16	4	6
40	12	2	1	4
	21	8	2	4
		12	4	10
	27	18	6	10
	30	22	7	10

10 Non-existence by ILP

We can model a (strictly) Neumaier graph with given parameters by an ILP.

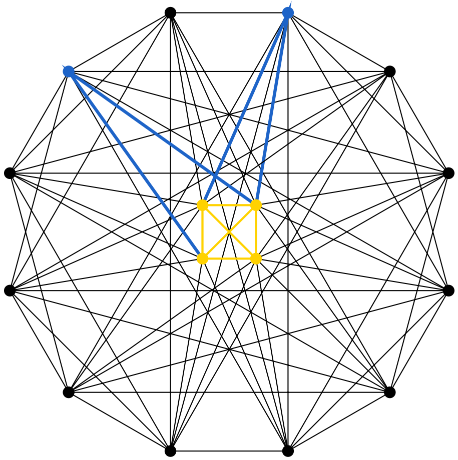
- For each pair of vertices $\{u, v\}$ a variable x_{uv} that is 1 or 0 (edge or not).
- For each pair $\{u, \{v, w\}\}$ a variable y_{uvw} that is 1 or 0 (u adjacent to both v and w, or not).
  - $x_{uv} \geq y_{uvw}$, $x_{uw} \geq y_{uvw}$
  - $x_{uv} + x_{uw} - 1 \leq y_{uvw}$
- Linear equations/inequalities to describe (edge-)regularity.
- Clique $E \rightarrow$ fix $x_{uv} = 1$ with $u, v \in E$.
- Linear equation (or fixed edges) for clique regularity
- Fixed edge and inequalities to break co-edge-regularity (if necessary).

Corollary (Abiad-DB-Zeijlemaker, 2023)

For strictly Neumaier graphs $(25, 16, 9; 3, 5)$, $(28, 18, 11, 4, 7)$, $(33, 24, 17; 6, 9)$, $(35, 22, 12; 3, 5)$ *and* $(55, 30, 18; 3, 5)$ *are not admissible as parameter sets.*

Existence

Strictly Neumaier graphs do exist



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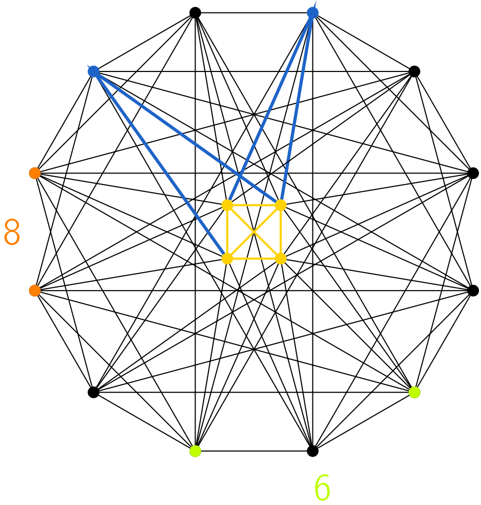

12 How many strictly Neumaier graphs?

Theorem (Greaves-Koolen, 2018)

There are (infinitely many) strictly Neumaier graphs (with $e = 1$).

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Theorem (Evans-Goryainov-Panasenko, 2019)

For every $n \geq 2$, there is a strictly Neumaier graph with parameters $(2^{2n}, (2^{n-1}+1)(2^n-1), 2(2^{n-2}+1)(2^{n-1}-1); 2^{n-1}, 2^n)$

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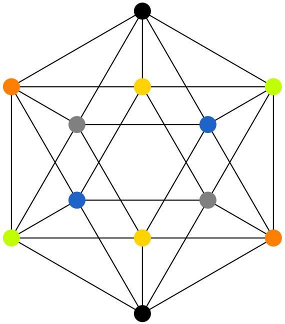
Theorem (Evans-Goryainov-Panasenko, 2019)

The Neumaier graph with parameters $(16, 9, 4; 2, 4)$ *is unique up to isomorphism.*

Evans-Goryainov-Panasenko (2019): computer-assisted proof
Abiad-De Boeck-Zeijlemaker (2023): computer-free proof

13 A strictly Neumaier graph on 24 vertices



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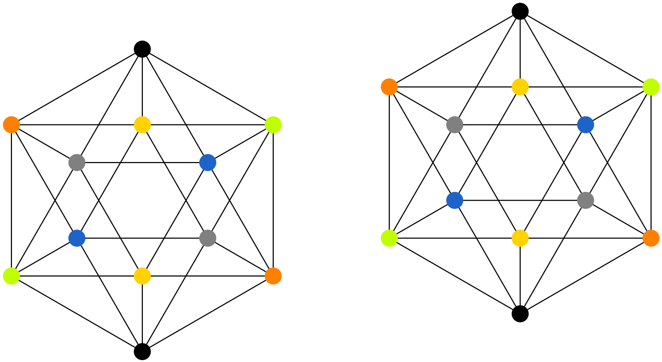

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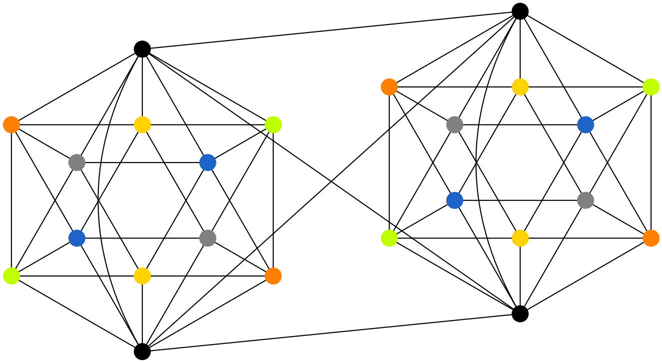

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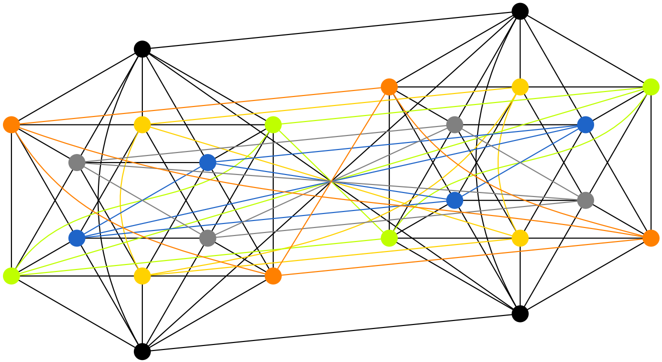

14 Evans-Goryainov technique

Inspired by the Greaves-Koolen construction.

Theorem (Evans, 2020 ; Evans-Goryainov-Konstantinova-Mednykh, 2021)

Let $\Gamma_1 = (V_1, E_1), \ldots, \Gamma_t = (V_t, E_t)$ be t edge-regular graphs with parameters (v, k, λ) such that each Γ_i admits a partition in 1-regular cocliques, $C_{i,1}, \ldots, C_{i,k+1}$. The graph $F(\Gamma_1, \ldots, \Gamma_t)$ is the graph

- *with as vertex set $V_1 \cup \cdots \cup V_t$,*
- *and where two vertices $x \in C_{i,k}$ and $y \in C_{j,l}$ are adjacent if and only if $i = j$ and $x \sim y$ in Γ_i, or if $k = l$.*

If $t = \frac{(\lambda+2)(k+1)}{v} \in \mathbb{N}$, then $F(\Gamma_1, \ldots, \Gamma_t)$ is a Neumaier graph with parameters $(vt, k + \lambda + 1, \lambda; 1, \lambda + 2)$; it admits a spread of 1-regular cliques.

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If $t \geq 2$, then $F(\Gamma_1, \dots, \Gamma_t)$ is a strictly Neumaier graph.

ERG's with a regular coclique partition?

Theorem (Greaves-Koolen, 2019)

Take $V_1, \dots, V_t$ distance-regular a-antipodal graphs of diameter 3.

Example

- Taylor graphs
- Thas-Somma graphs, edge-regular graphs with parameters $(q^{2n+1}, q^{2n}-1, q^{2n-1}-2)$ for a prime power q. You need to take q^{2n-2} copies, $n \geq 2$. You get a strictly Neumaier graph with parameters $(q^{4n-1}, q^{2n-1}(q+1)-2, q^{2n-1}-2; 1, q^{2n-1})$.

15 ERG's with a regular coclique partition?

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Take $V_1, \ldots, V_t$ distance-regular a-antipodal graphs of diameter 3.

Theorem (Greaves-Koolen, 2018)

Take $V_1, \ldots, V_t$ a (specificly described) Cayley graph on $(\mathbb{Z}/2\mathbb{Z})^m \times (\mathbb{F}_q, +)$, with $m \in \{2, 3\}$ and q a prime power with $q \equiv 1 \pmod{2^{m+1} - 2}$.

$m = 2$: $q \in \{7, 13, 19, 37, 49, \ldots\}$, $m = 3$: $q \in \{29, 43, 71, 127, \ldots\}$

A new look at the table (strictly)

v	k	λ	e	s
16	9	4	2	4
24	8	2	1	4^*
25	12	5	2	5
	16	9	3	5
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		8	3	7
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$*$: 4 vertex-transitive, ≥ 2 non-vertex transitive (Evans, EGP)

$\circ$: 2 vertex-transitive, ≥ 2 non-vertex transitive (Evans, EGP)

v	k	λ	e	s
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35	10	3	1	5
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A new construction

18 Example

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Example

- $p = 13$, $q = 5$, $a = 2$
- $S_{65} = \{1, 2, 4, 8, 16, 32, 64 = -1, 63, 61, 57, 49, 33\}$
- $\Gamma_{65}(2)$ is edge-regular with parameters $(65, 12, 3)$, and has a spread of 1-regular cocliques: cosets of $\{0, 13, 26, 39, 52\}$ in $\mathbb{Z}/65\mathbb{Z}, +$

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- $t = \frac{(\lambda+2)(k+1)}{v} = \frac{(3+2)(12+1)}{65} = 1$
- $F(\Gamma_{65}(2))$ is a strictly Neumaier graph.

Theoretically

Definition

Let a be such that $a^i \equiv -1 \pmod{n}$, where $2i$ is the order of a in $(\mathbb{Z}/n\mathbb{Z})^*, \cdot$.
Then $S_n(a) = \{a^j \in \mathbb{Z}/n\mathbb{Z} \mid 0 \leq j < 2i\}$.
$\Gamma_n(a)$ is the Cayley graph on $\mathbb{Z}/n\mathbb{Z}, +$ with $S_n(a)$ as generating set.

Theorem (Abiad-Castryck-DB-Koolen-Zeijlemaker, 2021)

Let $p > 2$ be a prime, $q \in \mathbb{N}$ odd. Let $a \in \mathbb{Z}$ be such that it has order $p-1$ in $(\mathbb{Z}/p\mathbb{Z})^*, \cdot$ and such that $a^{\frac{p-1}{2}} \equiv -1 \pmod{pq}$.
Then, the Cayley graph $\Gamma_{pq}(a)$ is an edge-regular graph with parameters $(pq, p-1, \lambda)$, with $\lambda = |S_{pq}(a) \cap (S_{pq}(a) + 1)|$, that has a spread of 1-regular cocliques.

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Remark

In general we need that $\frac{(\lambda+2)(k+1)}{v} = \frac{|S_{pq}(a) \cap (S_{pq}(a)+1)|+2}{q}$ is an integer. In other words, $|S_{pq}(a) \cap (S_{pq}(a)+1)| \equiv -2 \pmod{q}$.

Overview of new examples

q	p	a	t	v	k	λ	s
5	13	2	1	65	16	3	5
	37	2	1	185	40	3	5
	61	17	4	1220	79	18	20
	149	13	4	2980	167	18	20
		2	7	5215	182	33	35
7	79	54	1	553	84	5	7
	103	45	1	721	108	5	7
	127	12	2	1778	139	12	14
	139	26	4	3892	165	26	28
11	131	2	1	1441	140	9	11
13	61	2	1	793	72	11	13
	397	6	2	10322	421	24	26
		20	2	10322	421	24	26

Overview of new examples

q	p	a	t	v	k	λ	s
25	1021	77	2	51050	1069	48	50
		122	2	51050	1069	48	50
	1181	42	2	59050	1229	48	50
	1301	3	2	65050	1349	48	50
		73	2	65050	1349	48	50
	1381	42	2	69050	1429	48	50
		123	2	69050	1429	48	50
	1621	88	2	81050	1669	48	50
		113	2	81050	1669	48	50
	1741	197	2	87050	1789	48	50
	2141	58	2	107050	2189	48	50
		112	2	107050	2189	48	50

The admissible q's: some number theory

21 Main questions about construction

Problem

For which q can we find primes p and a corresponding integer a such that the construction produces a strictly Neumaier graph?

Main questions about construction

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For which q can we find primes p and a corresponding integer a such that the construction produces a strictly Neumaier graph?

- Does this construction produce an infinite number of examples?
- Are there q's for which it produces an infinite number of examples?
- Are there an infinite number of q's for which it produces an infinite number of examples?

We need to look at $|S_{pq}(a) \cap (S_{pq}(a)+1)| \pmod{q}$. Is it -2?

An explicit formula

Theorem (Abiad-Castryck-DB-Koolen-Zeijlemaker, 2021)

$$|S_{pq}(a) \cap (S_{pq}(a)+1)| = \frac{1}{n^2}\left((p+1)\,|B| + \sum_{1 \leq i \leq j < n-i} 2(2-\delta_{i,j})\Re(c_{i,j}J(\chi^i,\chi^j))\right)$$

where $c_{i,j} = \sum_{b \in B} \psi(b)^{-i}\psi(1-b)^{-j}$ *and* $\delta_{i,j}$ *is the Kronecker symbol.*

An explicit formula

Notation

- $\alpha = a \pmod p$, $\beta = a \pmod q$, n is the order of β in $(\mathbb{Z}/q\mathbb{Z})^*$
- $\xi : \mathbb{F}_p^* \to \langle\beta\rangle : \alpha^j \mapsto \beta^j$ and $\psi : \langle\beta\rangle \to \mu_n : \beta^j \mapsto e^{2\pi \mathbf{i} j/n}$ and $\chi = \psi \circ \xi$
- $B = \{b \in \langle\beta\rangle | b - 1 \in \langle\beta\rangle\}$
- J is the Jacobi sum of two characters: $J(\chi, \lambda) = \sum_{c \in \mathbb{F}_p} \chi(c)\lambda(1-c)$

Theorem (Abiad-Castryck-DB-Koolen-Zeijlemaker, 2021)

$$|S_{pq}(a) \cap (S_{pq}(a)+1)| = \frac{1}{n^2}\left((p+1)\,|B| + \sum_{1 \le i \le j < n-i} 2(2-\delta_{i,j})\Re(c_{i,j} J(\chi^i, \chi^j))\right)$$

where $c_{i,j} = \sum_{b \in B} \psi(b)^{-i}\psi(1-b)^{-j}$ *and* $\delta_{i,j}$ *is the Kronecker symbol.*

To give you an idea

Example ($q = 5$)

If $\beta = -1$, then $B = \emptyset$, so $|S \cap (S+1)| = 0$.
For $\beta = 2$, we have $\psi(\beta) = \mathbf{i}$ and must have $p \equiv 5 \pmod 8$. We find that

$$|\,S \cap (S+1)\,| = \frac{1}{16}\left(3p + 3 + 2\Re((-1+2\mathbf{i})J(\chi,\chi)) + 4\Re((1-2\mathbf{i})J(\chi,\chi^2))\right).$$

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There are x, y such that

$$p = x^2 + y^2, \qquad x \equiv 1 \pmod 4, \qquad y \equiv x\alpha^{\frac{p-1}{4}} \pmod p.$$

We can express the Jacobi sums in terms x and y and find that

$$|\, S \cap (S+1)\,| = \frac{3}{16}(p + 1 + 2x + 4y).$$

A sledgehammer from number theory

Theorem (Dirichlet, Neukirch)

Let $R = \mathbb{Z}[\mathbf{i}]$ *or* $R = \mathbb{Z}[\zeta_6]$ *and consider* $m \in R \setminus \{0\}$*. Let* $a \in R$ *be coprime with* m*. Then there exist infinitely many prime elements* $\pi \in R$ *such that* $m \mid \pi - a$*.*

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Example ($q = 5$, continued)

We want

$$
\begin{aligned}
& |\, S \cap (S+1)\,| = \frac{3}{16}(x^2 + y^2 + 1 + 2x + 4y) \equiv 3 \pmod 5 \\
\iff \quad & x^2 + 2x + y^2 + 4y \equiv 0 \pmod 5 .
\end{aligned}
$$

There are infinitely many prime elements $\pi \in \mathbb{Z}[i]$ such that $20 \mid \pi - (5 + 6\mathbf{i})$. Any $p = \pi\overline{\pi}$ satisfies the conditions.

25 Infinite infiniteness

Theorem

If $q = 5$ *or* $q = \ell_1^{e_1} \cdots \ell_k^{e_k} \geq 7$ *such that all primes* ℓ_i *satisfy* $\ell_i \equiv 1 \bmod 6$, *then there is an infinite number of primes* p *and integers* a *such that* $|S_{pq}(a) \cap (S_{pq}(a) + 1)| \equiv -2 \pmod{q}$.

Consequently, for an infinite number of q*'s the construction produces an infinite number of strictly Neumaier graphs!*

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Consequently, for an infinite number of q's the construction produces an infinite number of strictly Neumaier graphs!

Theorem

For $q = 5$, the density of the primes p for which we can find an integer a such that $|S_{pq}(a) \cap (S_{pq}(a) + 1)| \equiv -2 \pmod{q}$, equals $\frac{7}{64}$. For $q = 7$ this density equals $\frac{1}{12}$.

26 Non-admissible q's

Theorem (Abiad-Castryck-DB-Koolen-Zeijlemaker)

This construction produces no new examples of (strictly) Neumaier graphs if $3|q$.

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Definition

A Fermat prime is a prime of the form $2^{2^n}+1$ for some integer n. The known Fermat primes are 3, 5, 17, 257 and 65537. It is conjectured there are no others.

Theorem (Abiad-Castryck-DB-Koolen-Zeijlemaker)

If q *is divisible by both a Fermat prime* $p' \geq 5$ *and prime* $p'' \equiv 3 \pmod 4$, *then* $|S_{pq}(a) \cap (S_{pq}(a)+1)| = 0$ *for any* p *and* a *satisfying the conditions.*

Example

No examples for $q = 35, 55, 95, 119, \ldots$.

Bonus track

A Latin square graph

Example

Given the Latin square

a	b	c	d	e
b	a	d	e	c
c	e	a	b	d
d	c	e	a	b
e	d	b	c	a

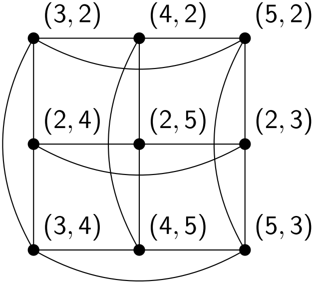
we define the Latin square graph Γ with

- Vertices $\{1, \ldots, 5\}^2$
- $(i, j) \sim (i', j')$ iff
  - $i = i'$,
  - $j = j'$, or
  - same entry on (i, j) and (i', j').

Γ is an strongly-regular Neumaier graph with parameters $(25, 12, 5; 2, 5)$.

A switching

A subgraph of Γ



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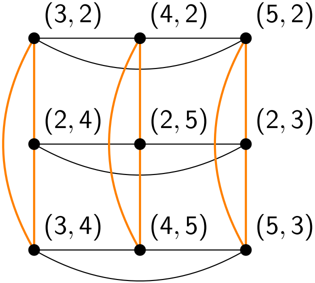
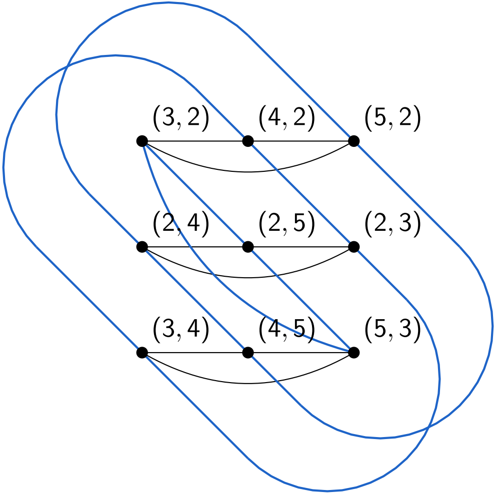


A switching

A switched subgraph of Γ



A new strictly Neumaier graph

Example (Abiad-DB-Zeijlemaker)

The graph Γ that results from switching the subgraph is a strictly Neumaier graph with parameters $(25, 12, 5; 2, 5)$.

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Remark

This was the first known strictly Neumaier graph with $e \notin \left\{1, \frac{s}{2}\right\}$. Among those, it is still the only one known which is not vertex-transitive.

Open questions

31 General questions

Problem

Which sets are admissible as parameter sets of strictly Neumaier graphs? Which for vertex-transitive strictly Neumaier graphs?

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Can a strictly Neumaier graph have five eigenvalues?
UPDATE (Sept. 19, 2023) YES - (Goryainov-Koolen)
An example with parameters $(48, 14, 2; 1, 4)$.

Q&A

Thank you for your attention

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Questions?