Cayley incidence graphs

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Introduction

Evra, Feigon, Maurischat, and Parzanchevski defined a class of biregular graphs which they called Cayley bigraphs. These graphs are bipartite variants of Cayley graphs. They constructed bipartite expanders using such graphs.

In joint work with Árnadóttir, Gordeev, Lato, and Randrianarisoa, we gave a definition that is equivalent to the one given by Evra, Feigon, Maurischat, and Parzanchevski. We explored some basic properties of such graphs and the relations that they have to other combinatorial objects. We opted for a different name to avoid confusion with bi-Cayley graphs and Cayley digraphs.

Definition

Let G be a group (with identity e) and let $\pi = \{C_1, \cdots C_\ell\}$ be a collection of subsets of G containing e. We say that π satisfies the T-axiom if for all C_i and $g \in C_i$, it holds that

$$g^{-1}C_i \in \pi$$

The T stands for translate - translating any C_i by g^{-1} with $g \in C_i$ gives some $C_j \in \pi$. We will call the sets C_i cells.

Example

Let $G = \mathbb{Z}/7\mathbb{Z}$. The sets

$$C_1 = \{0, 1, 2\}$$
$$C_2 = \{0, 6, 5\}$$
$$C_3 = \{0, 1, 6\}$$

satisfy the T-axiom.

Definition

Let G be a group (with identity e) and let $\pi = \{C_1, \cdots C_\ell\}$ be a collection of subsets of G containing e. We say that π satisfies the T-axiom if for all C_i and $g \in C_i$, it holds that

$$g^{-1}C_i \in \pi$$

Example

If H is a subgroup of H, $g^{-1}H = H$ for any $g \in H$, and therefore $\{H\}$ satisfies the T-axiom.

More generally a collection of subgroups $\{H_1, \cdots, H_\ell\}$ satisfies the T-axiom.

Cayley incidence graphs

Definition

Let G be a group and let $\pi = \{C_1, \dots C_\ell\}$ such that each cell has size k, any two distinct cells intersect in the identity

$$C_i \cap C_j = \{e\},$$

and such that π satisfies the T-axiom. We then define the Cayley incidence graph $\mathrm{Cin}(G, \pi)$ to be the bipartite graph having biparts γ and β, where

$$\gamma = G$$
$$\beta = \{gC_i \;:\; C_i \in \pi\}.$$

The edges are given by $g * gC_i$.

Note: the elements $gC_i \in \beta$ are not uniquely represented, as one may have $gC_i = hC_j$.

Defining edges to only lie between g and gC_i is equivalent to saying that we have an edge between g and hC_j if and only if $g \in hC_j$.The proof of this is straightforward:

'only if:' It is clear that $g \in gC_i$, since $e \in C_i$.
'if:' If $g \in hC_j$ for some j, then

$$g = hg_a, \text{ where } g_a \in C_j.$$

By the T-axiom, $g_a^{-1} C_j = C_m$ for some m, and we see that

$$gC_m = g(g_a^{-1} C_j) = hg_a(g_a^{-1} C_j) = hC_j,$$

and thus one has $g * hC_j$.

One naturally has a group action of G on $\mathrm{Cin}(G, \pi)$. It is given by

$$g \cdot (h) = gh \text{ for } h \in \gamma$$
$$g \cdot (hC_i) = ghC_i \text{ for } hC_i \in \beta.$$

The action is regular on γ, but it is not necessarily regular on β. In fact, the action on β can both fail to be free (semiregular) and transitive. We will later consider when it is regular.

Examples

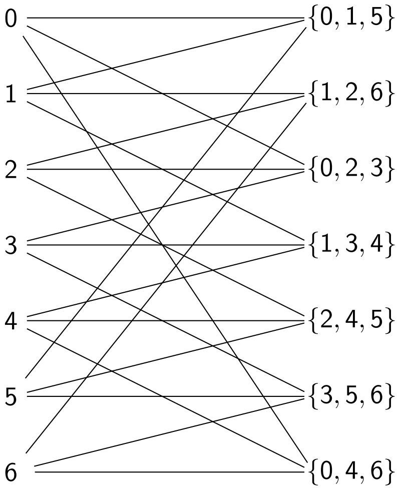
Let $G = \mathbb{Z}/7\mathbb{Z}$, and let

$$C_1 = \{0, 1, 5\},$$
$$C_2 = \{0, 4, 6\},$$
$$C_3 = \{0, 2, 3\}.$$

The translates of the cells are given by

$$\begin{array}{llll} C_1 & = \{0,1,5\} & 1 + C_1 & = \{1,2,6\} \\ 2 + C_1 & = \{0,2,3\} & 3 + C_1 & = \{1,3,4\} \\ 4 + C_1 & = \{2,4,5\} & 5 + C_1 & = \{3,5,6\} \\ 6 + C_1 & = \{0,4,6\}. & & \end{array}$$

Note that the cells containing the identity are precisely those that appear in π.



This Cayley incidence graph is the incidence graph of the Fano plane.

The previous example fits into a general family of examples related to $(n, k, 1)$ difference sets.

Definition

Let G be a group. We say that $D \subseteq G$ is called an (n, k, λ) difference set if $|D| = k$, and every non-identity $g \in G$ can be written as a product $d_1 d_2^{-1}$ with $d_1, d_2 \in D$ in λ ways.

If D is an $(n, k, 1)$ difference set, one can define $\pi_D = \{d^{-1}D : d \in D\}$. We need to check that this collection satisfies the three properties we need to define $\mathrm{Cin}(G, \pi_D)$.

- T-axiom (if $g \in C_i$, $g^{-1}C_i \in \pi$). If $d^{-1}d' \in d^{-1}D$, then $(d^{-1}d')^{-1}d^{-1}D = (d')^{-1}D$, so the T-axiom holds.
- All cells have size k. This is clear by construction.
- $C_i \cap C_j = \{e\}$ if $i \neq j$. To show this one needs to show that $d_i^{-1}d_a = d_j^{-1}d_b$ if and only if $a = i$ and $b = j$. The equation $d_i^{-1}d_a = d_j^{-1}d_b$ is equivalent to $d_a d_b^{-1} = d_i d_j^{-1}$. Since D is a $(n, k, 1)$ difference set, this is true if only if $a = i$ and $b = j$ or if $a = b$ and $i = j$. But $i \neq j$, so we have $a = i$ and $b = j$.

The resulting partition π_D is a partition such that each non-identity $g \in G$ appears in exactly one cell.

Difference sets lead to $(n, k, 1)$-designs with a regular group action on the points of the design. More generally, Cayley incidence graphs $\mathrm{Cin}(G, \pi)$ such that each non-identity $g \in G$ appears in exactly one cell $C_i \in \pi$ correspond to $(n, k, 1)$-designs with a regular group action on the points.

Using difference sets, one can construct incidence graphs of affine planes $\mathbb{F}_q^n$ and Desarguesian projective planes. The Cayley incidence graphs $\mathrm{Cin}(G, \pi_D)$ will be the incidence graphs of these planes.

Another combinatorial structure which leads to Cayley incidence graphs is that of a coset geometry. A coset geometry is defined using subgroups $H_1, \cdots, H_\ell \leq G$. We can define the incidence graph Γ of a coset geometry as having vertices

$$G/H_1 \cup G/H_2 \cup \cdots \cup G/H_\ell.$$

and one puts an edge if $i \neq j$ and $gH_i \cap g'H_j \neq \emptyset$.

If $H_i \cap H_j = \{e\}$ for all $i \neq j$, and if the subgroups have the same size k, then one can define a Cayley incidence graph by setting $\pi = \{H_1, \ldots, H_\ell\}$.

The graph $\mathrm{Cin}(G, \pi)$ has biparts $\gamma = G$ and $\beta = G/H_1 \cup G/H_2 \cup \cdots \cup G/H_\ell$. One puts an edge $g * gH$ if $g \in gH_i$. The graph $\mathrm{Cin}(G, \pi)$ is related to Γ because the halved graph H_β with vertex set β and edge set

$$\{gH_i * g'H_j : \text{ there exists a } h \in \gamma \text{ with } h * gH_i \text{ and } h * g'H_j\}$$

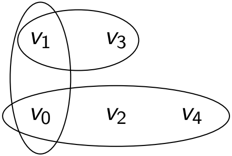
is isomorphic to Γ.

Example

Let $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, and let $H_1 = \mathbb{Z}/3\mathbb{Z} \times \{0\}$ and $H_2 = \{0\} \times \mathbb{Z}/3\mathbb{Z}$. The resulting graph $\mathrm{Cin}(G, \{H_1, H_2\})$ is the barycentric subdivision graph of $K_{3,3}$, with the vertices of subdivided edges being elements of γ, and the original vertices of $K_{3,3}$ given by β. The incidence graph of the coset geometry is given by a complete bipartite graph $K_{3,3}$.

Using hypergraphs

We will now provide a definition of Cayley incidence graphs as incidence graphs of hypergraphs.
A hypergraph $\mathcal{H} = (V, \mathcal{E})$ consists of a set V of vertices and a set $\mathcal{E} = \{c_1, \ldots, c_m\}$ of subsets of V (called hyperedges).



Example

$V = \{v_0, v_1, v_2, v_3, v_4\}$.
$\mathcal{E} = \{\{v_0, v_1\}, \{v_0, v_2 v_4\}, \{v_1, v_3\}\}$.

An automorphism of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a map $\varphi : V \to V$ such that $\varphi(c)$ is a hyperedge if and only if c is a hyperedge.

The automorphisms of a hypergraph form a group $\mathrm{Aut}(\mathcal{H})$. A group action on a hypergraph is a group homomorphism $G \to \mathrm{Aut}(\mathcal{H})$.

Definition

The incidence graph of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a graph with vertex set $V \cup \mathcal{E}$ and with edge set

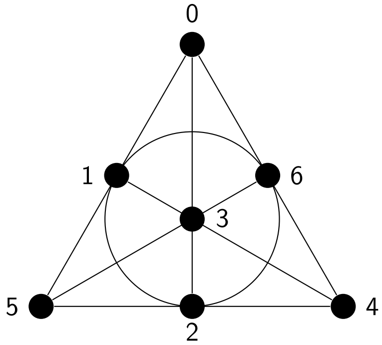
$$\{v * c_i \;:\; v \in c_i\}$$

The incidence graph is a bipartite graph with the two biparts being V and $\mathcal{E}$. One can also recover the hypergraph if one is given an incidence graph and the partition into two parts.

Cayley incidence graphs are incidence graphs of hypergraphs. More precisely $\mathrm{Cin}(G, \pi)$ is the incidence graph of the hypergraph $(V, \mathcal{E})$ having vertex set $V = G$ and hyperedge set

$$\mathcal{E} = \{gC_i \ : \ C_i \in \pi, g \in G\}.$$

We call these hypergraphs group hypergraphs associated to $\mathrm{Cin}(G, \pi)$, and we use the notation $CH(G, \pi)$ to denote them. One has a group action $G \to \mathrm{Aut}(CH(G, \pi))$. Each g induces a hypergraph automorphism by $g \mapsto \mathcal{L}_g$, where $\mathcal{L}_g : G \to G : h \mapsto gh$.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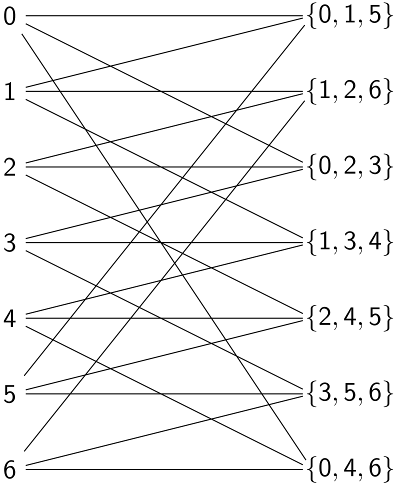
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{3, 5, 6}
{0, 4, 6}

Definition

A hypergraph is

- uniform if every hyperedge has the same size k,
- regular if every vertex is contained in ℓ hyperedges.
- linear if every two hyperedges intersect in at most 1 vertex.

Translated into a statement about the incidence graphs (with biparts $\gamma = V$ and $\beta = \mathcal{E}$) these are equivalent to the following

- $\mathcal{H}$ is uniform $\iff$ every vertex $v \in \beta$ has degree k
- $\mathcal{H}$ is regular $\iff$ every vertex $v \in \gamma$ has degree ℓ
- $\mathcal{H}$ is linear $\iff$ the girth (length shortest cycle) is at least 6.

Definition

A hypergraph is

- uniform if every hyperedge has the same size k,
- regular if every vertex is contained in ℓ hyperedges.
- linear if every two hyperedges intersect in at most 1 vertex.

Cayley incidence graphs $\mathrm{Cin}(G, \pi)$ are incidence graphs of hypergraphs $\mathcal{H} = (V, \mathcal{E})$, where $V = G$, and

$$\mathcal{E} = \{gC_i : C_i \in \pi\}.$$

These hypergraphs are

- uniform, since each hyperedge has size $|gC_i| = |C_i| = k$,
- regular, since each vertex g is contained in the hyperedges $gC_1, \dots, gC_\ell$,
- linear, which follows (after a short argument) from the property that $C_i \cap C_j = \{e\}$ for distinct $C_i, C_j \in \pi$

There have been several suggestions to generalize the definition of a Cayley graph to hypergraphs throughout the years (Teh and Shee (1976), Shee (1990), Buratti (1994), Lee and Kwon (2013), Jajcay and Jajcayová (2024)).
One perspective is that a variant of Sabidussi's theorem should hold:

Theorem

(Sabidussi) A graph is a Cayley graph if and only if there is a group that acts regularly on the vertex set.

Jajcay (2002) and Lee and Kwon (2013) proved variants of this theorem for hypergraphs - stating that any hypergraph for which a group acts regularly on the vertex set must be given by certain constructions. Based on these theorems, we gave the following characterisation of Cayley incidence graphs:

Theorem

A graph Γ *is isomorphic to a Cayley incidence graph* $\mathrm{Cin}(G, \pi)$ *if and only if there is a uniform, regular, and linear hypergraph* $\mathcal{H}$ *upon which the group* G *acts regularly, such that* Γ *is the incidence graph of* $\mathcal{H}$*.*

Theorem

A graph Γ is isomorphic to a Cayley incidence graph $\mathrm{Cin}(G, \pi)$ if and only if there is a uniform, regular, and linear hypergraph $\mathcal{H}$ upon which the group G acts regularly, such that Γ is the incidence graph of $\mathcal{H}$.

proof:
'only if': ✓
'if' : (sketch) Let $\mathcal{H}$ be a hypergraph as in the statement of the theorem. We need to check that its incidence graph is a Cayley incidence graph.
Pick $v \in V(\mathcal{H})$. Let $c_1, \cdots c_\ell$ be the hyperedges containing v. Each c_i is given by

$$\{v, v_1^i, \cdots v_{k-1}^i.\}$$

We can then define $g_{i,j}$ to be the group element such that

$$g_{i,j} v = v_j^i.$$

We can then define $C_i = \{g_{i,0}, \ldots, g_{i,k-1}\}$, where $g_{i,0} = e$ for each i. It can then be checked that $\mathrm{Cin}(G, \{C_1, \ldots, C_\ell\})$ is isomorphic to the incidence graph of $\mathcal{H}$.

Automorphisms

Similarly to the case for Cayley graphs (Godsil 1981), and Bi-Cayley graphs (Zhou and Feng 2016), one can consider when group automorphisms of the group G induce graph automorphisms of the Cayley incidence graph $\mathrm{Cin}(G,\pi)$. These group automorphisms turn out to determine the normalizer

$$N_{\mathrm{Aut}(\mathrm{Cin}(G,\pi))}(G) = \{\varphi \in \mathrm{Aut}(\mathrm{Cin}(G,\pi)) \ : \ \forall g \in G \ \exists h \in G \ \varphi \circ \mathcal{L}_g \circ \varphi^{-1} = \mathcal{L}_h\},$$

where $\mathcal{L}_g$ denotes the action of g on $V(\mathrm{Cin}(G,\pi))$.

For a bipartite graph $X = (\gamma \cup \beta, E)$, an automorphism $\varphi : X \to X$ either satisfies

$$\varphi(\gamma) = \gamma, \qquad \varphi(\beta) = \beta,$$

or

$$\varphi(\gamma) = \beta \qquad \varphi(\beta) = \gamma.$$

This defines a group homomorphism:

$$\sigma : \mathrm{Aut}(X) \to \mathbb{Z}/2\mathbb{Z} : \begin{cases} \varphi \mapsto 0 \text{ if } \varphi(\gamma) = \gamma \\ \varphi \mapsto 1 \text{ if } \varphi(\gamma) = \beta \end{cases}.$$

We will first consider automorphisms in $\ker(\sigma) \cap N_{\mathrm{Aut}(\mathrm{Cin}(G,\pi))}(G)$.

An automorphism $\varphi : \mathrm{Cin}(G,\pi) \to \mathrm{Cin}(G,\pi)$ with $\sigma(\varphi) = 0$ can be viewed as a hypergraph automorphism $\varphi' : CH(G,\pi) \to CH(g,\pi)$, since for any $gC_i \in \beta$, the image $\varphi(gC_i)$ is determined by the images of the neighbours of gC_i.

Such an isomorphism is then given by a (set) map $\varphi : G \to G$. To compute $N_{\mathrm{Aut}(\mathrm{Cin}(G,\pi))}(G)$ it suffices to consider the case where $\varphi(e) = e$, since one can multiply by elements of G to make this the case. An automorphism then normalizes G if for every g there exists an h such that

$$\varphi \circ \mathcal{L}_g \circ \varphi^{-1} = \mathcal{L}_h.$$

By applying to the identity e, one finds

$$\varphi(g) = \varphi \circ \mathcal{L}_g \circ \varphi^{-1}(e) = \mathcal{L}_h(e) = h,$$

and hence $\varphi \circ \mathcal{L}_g \circ \varphi^{-1} = \mathcal{L}_{\varphi(g)}$. One then sees that

$$\begin{aligned}\mathcal{L}_{\varphi(gh)} &= \varphi\mathcal{L}_{gh}\varphi^{-1} \\ &= \varphi\mathcal{L}_g\varphi^{-1}\varphi\mathcal{L}_h\varphi^{-1} \\ &= \mathcal{L}_{\varphi(g)}\mathcal{L}_{\varphi(h)}\end{aligned}$$

We thus see that any $\varphi \in N_{\mathrm{Aut}(\mathrm{Cin}(G,\pi))}$ satisfies $\varphi(gh) = \varphi(g)\varphi(h)$ (under the assumption that $\varphi(e) = e$). Thus φ comes from a group automorphism. Moreover, if we consider the fact that the image of an edge containing the identity must be an edge containing the identity, we see that for every $i \in \{1, \dots \ell\}$ there must be some j such that $\varphi(C_i) = C_j$. Hence hypergraph automorphisms such that $\varphi \circ \mathcal{L}_g \circ \varphi^{-1} = \mathcal{L}_h$ must come from group automorphisms permuting π.

Any group automorphism $\varphi : G \to G$ which induces a permutation of π (meaning that for every i there is some j such that $\varphi(C_i) = C_j$) also defines a hypergraph automorphism. To show this, we note that $\varphi(gC_i) = \varphi(g)\varphi(C_i) = \varphi(g)C_j$. Since φ is a group automorphism $\varphi \circ \mathcal{L}_g \circ \varphi = \mathcal{L}_{\varphi(g)}$, thus $\varphi \in N_{\mathrm{Aut}(\mathrm{Cin}(G,\pi))}(G)$. We have now determined $N_{\mathrm{Aut}(\mathrm{Cin}(G,\pi))}(G) \cap \ker(\sigma)$.

Theorem

$$N_{\mathrm{Aut}(\mathrm{Cin}(G,\pi))}(G) \cap \ker(\sigma) \cong \mathrm{Aut}(G,\pi) \ltimes G,$$

where $\mathrm{Aut}(G,\pi)$ *is the group of group automorphisms that permute* π*.*

Dual hypergraphs

To understand the graph automorphisms which are not in the kernel, we must make a digression on dual hypergraphs, and the action of G on the hyperedges of $CH(G, \pi)$.

For any hypergraph $\mathcal{H} = (V, \mathcal{E})$, one can define the dual hypergraph. $\mathcal{H}^*$ is the hypergraph having vertex set $\mathcal{E}$ and edge set $\mathcal{E}^*$

$$\begin{aligned} \mathcal{E}^* &= \{c_v \subseteq \mathcal{E}\}, \text{ where} \\ c_v &= \{c_i \in \mathcal{E} \ : \ v \in c_i\}. \end{aligned}$$

In terms of the bipartite graphs, this means that one swaps the roles of γ and β. We now investigate when the dual of a hypergraph $CH(G, \pi)$ associated to a Cayley incidence graph gives another $CH(G, \pi')$ associated to a Cayley incidence graph. For this to be possible, we need the group G to act regularly on the set of hyperedges $\mathcal{E}$ of $CH(G, \pi)$.

We now want to understand when the action of G on the edges of $CH(G, \pi)$ is regular. To this end we first consider what a stabilizer looks like. First we look at when the action is free (semiregular).

We consider the cells C_i. If $gC_i = C_i$ then $g \cdot e = g \in C_i$. Since $g^{-1}C_i = C_i$, we also have $g^{-1} \in C_i$. Conversely, if both $g, g^{-1} \in C_i$, then we see that

$$\begin{aligned} gC_i &= C_j && \text{for some } j \text{, by the } T\text{-axiom} \\ g &\in C_j, g \in C_i && \text{by definition, and since } e \in C_i \end{aligned}$$

Since $C_i \cap C_j = \{e\}$ if $C_i \neq C_j$ we see that j must be equal to i, and hence $gC_i = C_i$. Hence, we see that the stabilizer of C_i is given by $\{g \in G \mid g \in C_i, g^{-1} \in C_i\}$.

For other cells of the form gC_i, the stabilizer $\mathrm{Stab}(gC_i) = g\mathrm{Stab}(C_i)g^{-1}$, by generalities on group actions. Thus, we see that the action is free if and only if at most one of g, g^{-1} is an element of C_i for all cells C_i.

We now want to understand when the action of G on the hyperedges is regular. We now consider when the action is transitive.

It suffices that for each C_i with $i \in \{1, \dots, \ell\}$, there exists a g_i such that $g_i C_1 = C_i$. This forces the cells $C_2, \cdots, C_\ell$ to be given by $g_2^{-1} C_1, \dots, g_\ell^{-1} C_1$ for some $g_2, \dots g_\ell \in C_1$.

If the action is regular, it follows that the cells $C_1, \dots C_\ell$ are given by:

$$
\begin{aligned}
C_1 &= \{e, g_2, \dots, g_k\} \\
C_2 &= g_2^{-1} C_1 = \{g_2^{-1}, e, \dots, g_2^{-1} g_k\} \\
&\vdots \\
C_\ell &= g_k^{-1} C_1 = \{g_k^{-1}, g_k^{-1} g_2, \dots, g_k^{-1} g_{k-1}, e\}
\end{aligned}
$$

In particular, one must have $k = \ell$

If the action of G on the edges of $CH(G, \pi)$ is regular, combining these facts gives that $k = \ell$, and the cells $C_1, \ldots C_\ell$ are given by:

$$\begin{aligned} C_1 &= \{e, g_2, \ldots, g_k\} \\ C_2 &= \{g_2^{-1}, e, \ldots, g_2^{-1} g_k\} \\ &\vdots \\ C_\ell &= \{g_k^{-1}, g_k^{-1} g_2, \ldots, g_k^{-1} g_{k-1}, e\} \end{aligned}$$

For such collections π, we can understand the hypergraph dual, which is isomorphic to a group hypergraph associated to a Cayley incidence graph.

Hyperedges in the dual graph $\mathcal{H}^*$ are given by collections $\{hC_1, \ldots, hC_\ell\}$, since these are all edges containing h. In particular $\{C_1, \ldots, C_\ell\}$ is such a hyperedge in $\mathcal{H}^*$. Treating C_1 as the identity in the new Cayley incidence graph, we see that since $C_i = g_i^{-1} C_1$, that taking one cell should be given by

$$e, g_2^{-1}, \ldots, g_k^{-1}.$$

It turns out that defining π^* using the cell $\{e, g_2^{-1}, \ldots, g_k^{-1}\}$ leads to a group hypergraph isomorphic to $\mathcal{H}^*$.

Conclusion: If $CH(G, \pi)$ is a group hypergraph associated to a Cayley incidence graph, and if the action of G is regular on the edges of $CH(G, \pi)$, then the cells of π are given by

$$
\begin{aligned}
C_1 &= \{e, g_2, \ldots, g_k\} \\
C_2 &= \{g_2^{-1}, e, \ldots, g_2^{-1} g_k\} \\
&\vdots \\
C_\ell &= \{g_k^{-1}, g_k^{-1} g_2, \ldots, g_k^{-1} g_{k-1}, e\},
\end{aligned}
$$

and furthermore $CH(G, \pi)^* = CH(G, \pi^*)$, where the cells of π^* are given by:

$$
\begin{aligned}
C_1^* &= \{e, g_2^{-1}, \ldots, g_k^{-1}\} \\
C_2^* &= \{g_2, e, \ldots, g_2 g_k^{-1}\} \\
&\vdots \\
C_\ell^* &= \{g_k, g_k g_2^{-1}, \ldots, g_k g_{k-1}^{-1}, e\}
\end{aligned}
$$

We can now return to our study of graph automorphisms!

Since the map

$$\sigma : \mathrm{Aut}(X) \to \mathbb{Z}/2\mathbb{Z} : \begin{cases} \varphi \mapsto 0 \text{ if } \varphi(\gamma) = \gamma \\ \varphi \mapsto 1 \text{ if } \varphi(\gamma) = \beta \end{cases}$$

is a group homomorphism, we have that $\mathrm{Aut}(X) = \ker(\sigma)$ or $\mathrm{Aut}(X) = \ker(\sigma) \cup \varphi^* \ker(\sigma)$, where φ^* is any graph automorphism with $\sigma(\varphi^*) = 1$. It thus suffices to consider if there exists any graph automorphism which swaps the sides (i.e. $\sigma(\varphi^*) = 1$).

A graph automorphism with $\sigma(\varphi) = 1$ corresponds to a hypergraph isomorphism between $\mathcal{H}$ and $\mathcal{H}^*$. Hence it seems likely that a graph isomorphism with $\sigma(\varphi^*) = 1$ exists in the normalizer of G if there is a group automorphism $\varphi : G \to G$ with $\varphi(C_1) = C_i^*$. This turns out to be the case.
We will only show one direction.

We suppose that $\varphi(C_1) = C_i^*$ for some i. We will use this to define a graph automorphism of $\mathrm{Cin}(G, \pi)$.
Considering C_1 and C_1^* as sets of group elements, we know there is some g_φ such that $\varphi(C_1) = g_\varphi C_1^*$. We can thus define $f_\varphi : \gamma \cup \beta \to \gamma \cup \beta$ by

$$\begin{cases} g \mapsto \varphi(g) C_1 & \text{for } g \in \gamma \\ gC_1 \mapsto \varphi(g) g_\varphi & \text{for } gC_1 \in \beta \end{cases}$$

We have

$$\begin{aligned} h \in gC_1 &\iff \varphi(g^{-1}h) \in \varphi(C_1) \\ &\iff \varphi(g^{-1}h) \in g_\varphi C_1^* \\ &\iff g_\varphi^{-1} \varphi(g^{-1}h) \in C_1^* \\ &\iff \varphi(h^{-1}g) g_\varphi \in C_1 \text{ (taking inverses)} \\ &\iff \varphi(g) g_\varphi \in \varphi(h) C_1 \end{aligned}$$

so f_φ is a graph automorphism. Moreover, it holds that

$$f_\varphi \circ \mathcal{L}_g = \mathcal{L}_{\varphi(g)} \circ f_\varphi,$$

so we have $f_\varphi \in N_{\mathrm{Aut}(\mathrm{Cin}(G,\pi))}(G)$.

Theorem

Let $\mathrm{Cin}(G,\pi)$ be a Cayley incidence graph with regular action on β.
If there is a group automorphism $\varphi : G \to G$ such that $\varphi(C_1) = C_j^*$ for some j, then:

$$N_{\mathrm{Aut}(\mathrm{Cin}(G,\pi))}(G) \cong \langle \mathrm{Aut}(G,\pi) \ltimes G, \varphi \rangle.$$

Otherwise

$$N_{\mathrm{Aut}(\mathrm{Cin}(G,\pi))}(G) \cong \mathrm{Aut}(G,\pi) \ltimes G.$$

Example

For any $\mathrm{Cin}(G,\pi)$ defined using an abelian group G with regular action on β, the automorphism

$$G \mapsto G : g \mapsto g^{-1}$$

defines an automorphism which swaps the sides.

Example

We consider the quaternion group Q_8, with $\pi = \{\{1,i,-j\},\{1,-i,k\},\{1,j,-k\}\}$, then π^* is given by $\{\{1,-i,j\},\{1,i,-k\},\{1,-j,k\}\}$ and we let $\varphi : Q_8 \mapsto Q_8$ be given by $i \mapsto j$,$j \mapsto i$, $k \mapsto -k$. This induces an automorphism of $\mathrm{Cin}(Q_8,\pi)$ which swaps the two sides

Thank you for listening!

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