Quantum Automorphism Groups of Trees

Joint work with Josse van Dobben de Bruyn, David Roberson, Simon Schmidt and Peter Zeman.

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- It is known that almost all graphs do not have quantum symmetry(Lupini et al. [2020]), while almost all trees do(Junk et al. [2020]). This makes the study of quantum automorphism groups of trees particularly interesting.
- In what follows, we shall see a quantum analogue of Jordan's theorem. The main idea of the proof is similar to the classical proof.

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$$\begin{aligned} u_{ij} &= u_{ij}^2 = u_{ij}^* \\ \sum_j u_{ij} &= \sum_i u_{ij} = 1 \\ A_G u &= u A_G \end{aligned} \tag{1}$$

  together with the comultiplication map $\Delta : C(\mathrm{Qut}(G)) \to C(\mathrm{Qut}(G)) \otimes C(\mathrm{Qut}(G))$ defined by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$,

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- A matrix $[u_{ij}]_{ij}$ satisfying the first two conditions of 1 is known as a *magic unitary.*

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- A (vertex-coloured) graph is said to have *quantum symmetry* if the C^*-algebra $\mathrm{Qut}(G)$ is non-commutative.

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- Another closely related topic to quantum automorphism groups is that of *quantum isomorphism* introduced in Atserias et al. [2019]. Two (possibly coloured) graphs G, H are said to be *quantum isomorphic* if there is a non-zero unital C^*-algebra A, and a magic unitary $[u_{x,y}]_{x \in V(G), y \in V(H)}$ such that $A_G u = u A_H$, and $u_{xy} = 0$ if $c(x) \neq c(y)$.

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- It was also shown that these relations are equivalence relations.
- The partitions of $V(G)$ induced by $\sim_1$ are called the *orbits* of $\mathrm{Qut}(G)$, and the partitions of $V(G) \times V(G)$ induced by $\sim_2$ are known as the *orbitals* of $\mathrm{Qut}(G)$.

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- The colour is the iteratively updated to the multiset of colours of the neighbours of each vertex.
- This refinement eventually stabilises. This stable partition is the output of the algorithm.

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Lemma (van Dobben de Bruyn, K., Roberson, Schmidt, Zeman)

Let X be a (possibly coloured) graph and suppose that $P_1, \dots, P_r \subseteq V(X)$ is the partition of $V(X)$ found by colour-refinement. If $i \in P_l$ and $j \in P_k$ for $l \neq k$, then $u_{ij} = 0$, i.e., i and j are in different orbits of $\mathrm{Qut}_c(X)$.

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In other words, the output of colour refinement is a coarse graining of the orbits of $\mathrm{Qut}(G)$.

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- If the resulting partition has a part that contains a different number of vertices of G and H, then we can conclude that G and H are not isomorphic.
- It is known that all of the resulting partitions have the same number of vertices from G and H if and only if G and H are fractionally isomorphic.

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- Additionally, it is known from Immerman and Lander [1990] that any two trees are fractionally isomorphic if and only if they are isomorphic. Therefore we have the following:

Lemma (van Dobben de Bruyn, K., Roberson, Schmidt, Zeman)

If F and F' are trees, then they are quantum isomorphic if and only if they are isomorphic.

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1. If $S \subseteq V(X)$ is an independent set that is a union of colour classes, then adding $\binom{|S|}{2}$ edges, one for each distinct pair of vertices in S does not change the automorphism group of X.

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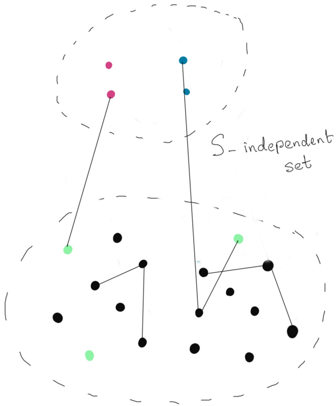

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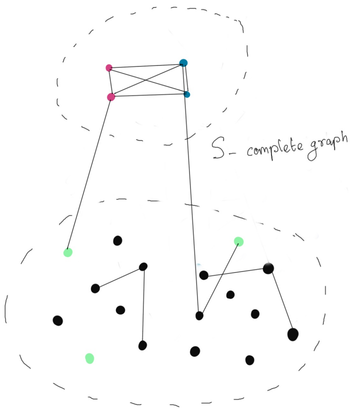

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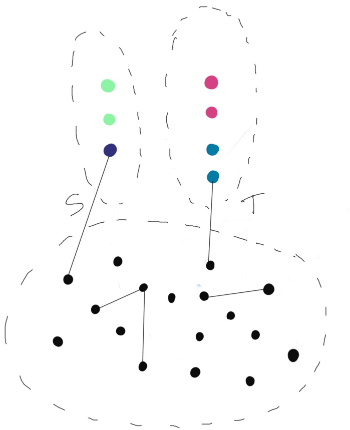

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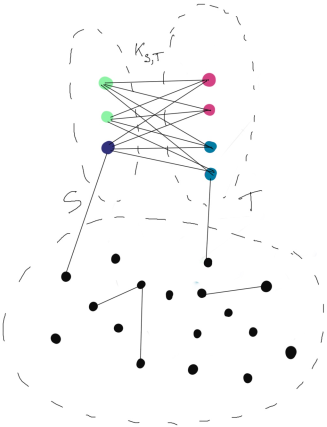

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3. Let $S \subseteq V(X)$ be a monochromatic vertex set that is a union of quantum orbits of X. Then changing the color of S to a new color (that does not occur elsewhere) does not change the quantum automorphism group.

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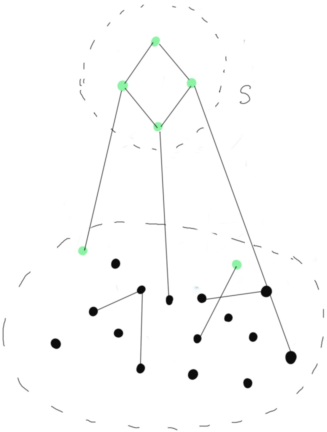

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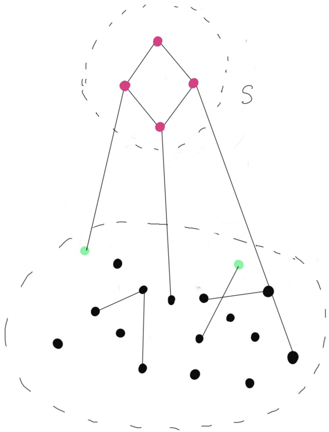

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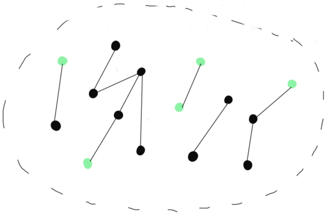

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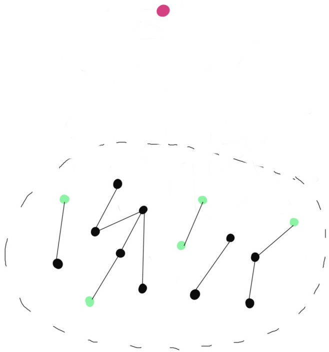

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- Indeed, the Jordan center can be found by iteratively removing the leaves of a tree.
- Since the leaves of a tree are a union of quantum orbits, we see that after each iteration, we are left with a union of quantum orbits.
- Hence, the Jordan center is a union of quantum orbits. We now describe the rootification process.

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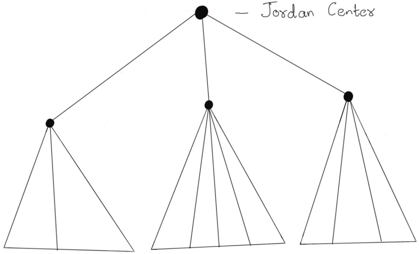

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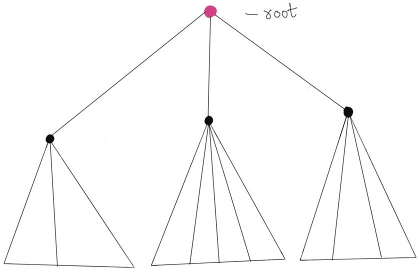

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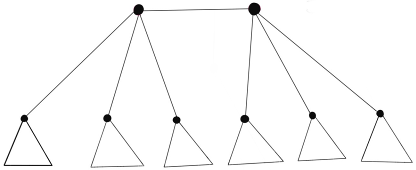

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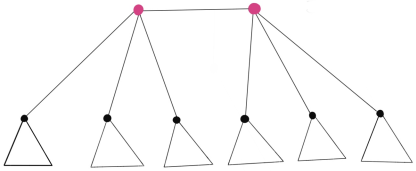

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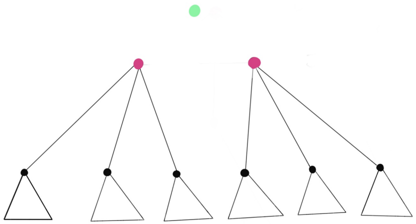

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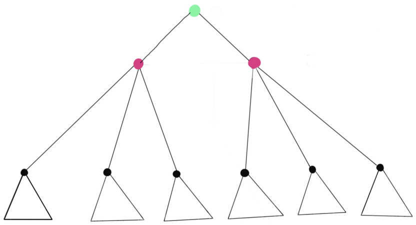

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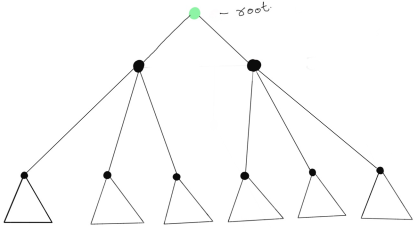

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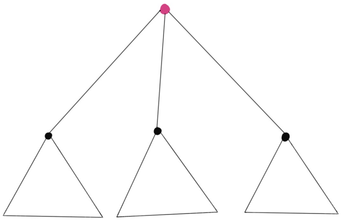

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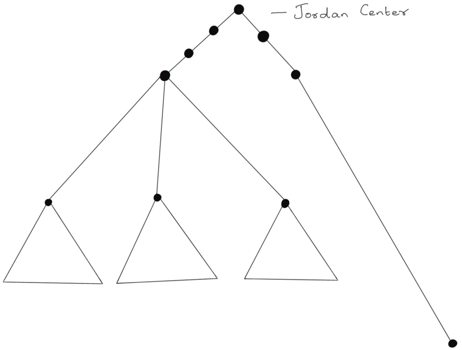

Free Product

- Let $(C(\mathrm{Qut}(G)), u)$ and $(C(\mathrm{Qut}(H)), v)$ be the quantum automorphism groups of two graphs. Their *direct product* $\mathrm{Qut}(G) * \mathrm{Qut}(H)$ is defined as the quantum permutation group $(C(\mathrm{Qut}(G)) * C(\mathrm{Qut}(H)), u \oplus v)$.

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- It was shown in Schmidt [2020a] that if G, H are two graphs that are not quantum isomorphic, then $\mathrm{Qut}(G \sqcup H) = \mathrm{Qut}(G) * \mathrm{Qut}(H)$. This result can be strengthened to the following:

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Lemma (van Dobben de Bruyn, K., Roberson, Schmidt, Zeman)

Let $X_1, \ldots, X_n$ be vertex colored graphs such that for any $i \neq j$, no connected component of X_i is quantum isomorphic to a connected component of X_j. Then,

$$\mathrm{Qut}_c\left(\bigsqcup_{i=1}^{n} X_i\right) = *_{i=1}^{n}\, \mathrm{Qut}_c(X_i) \qquad (3)$$

where $\bigsqcup_{i=1}^{n} X_i$ denotes the disjoint union of $X_1, \ldots, X_n$.

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Theorem (van Dobben de Bruyn, K., Roberson, Schmidt, Zeman)

Let G *be a connected vertex colored graph and* $n \in \mathbb{N}$*. Let* $\bigsqcup_{i=1}^{n} X$ *denote the disjoint union of* n *copies of* X*, all with the same coloring. Then,* $\mathrm{Qut}_c(\bigsqcup_{i=1}^{n} G) = \mathrm{Qut}_c(G) \wr_* \mathbb{S}_n^+$*, where* $\wr_*$ *denotes the free wreath product and* $\mathbb{S}_n^+$ *denotes* $\mathrm{Qut}(K_n)$*.*

From Rooted Trees to Forests

Lemma (van Dobben de Bruyn, K., Roberson, Schmidt, Zeman)

Let F be a forest of rooted trees, and let $\widetilde{F}$ be the rooted tree obtained by connecting the roots of the individual trees of F to a single new root. Then, $\mathrm{Qut}_c(F) \cong \mathrm{Qut}_r(\widetilde{F})$.

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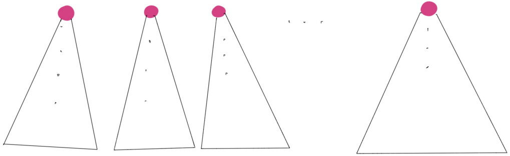

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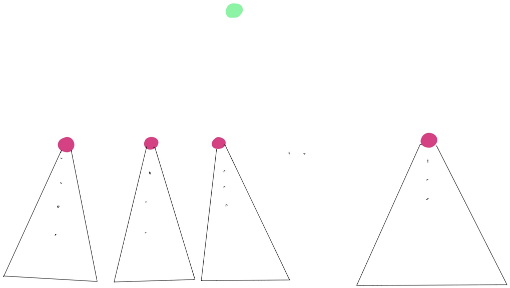

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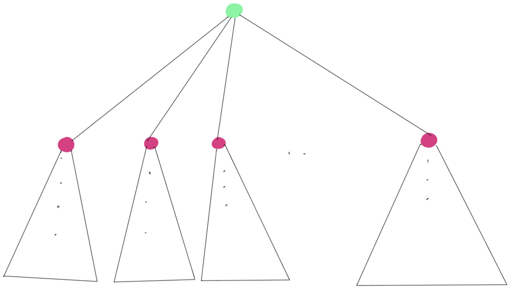

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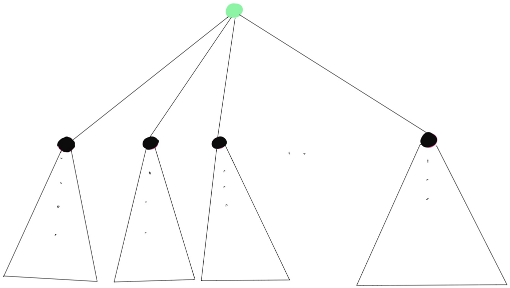

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Theorem (van Dobben de Bruyn, K., Roberson, Schmidt, Zeman)

The class $\mathcal{T}$ of all quantum automorphism groups of trees can be constructed inductively as follows:

1. $\mathbf{1} \in \mathcal{T}$.
2. *If* $\mathbb{G}, \mathbb{H} \in \mathcal{T}$, *then* $\mathbb{G} * \mathbb{H} \in \mathcal{T}$.
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Here, $\mathbf{1}$ can be though of as the quantum automorphism group of the tree with a single vertex and no edges.

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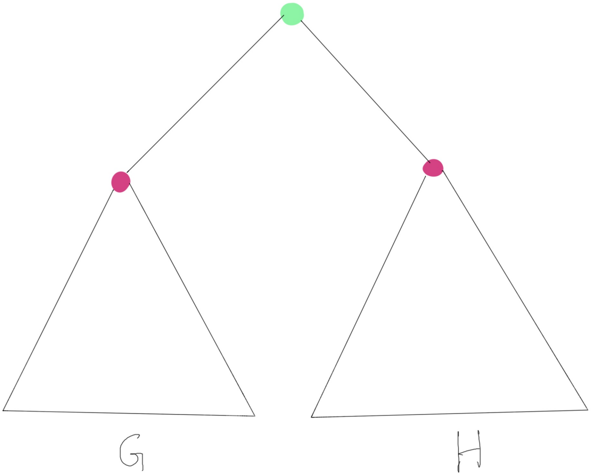

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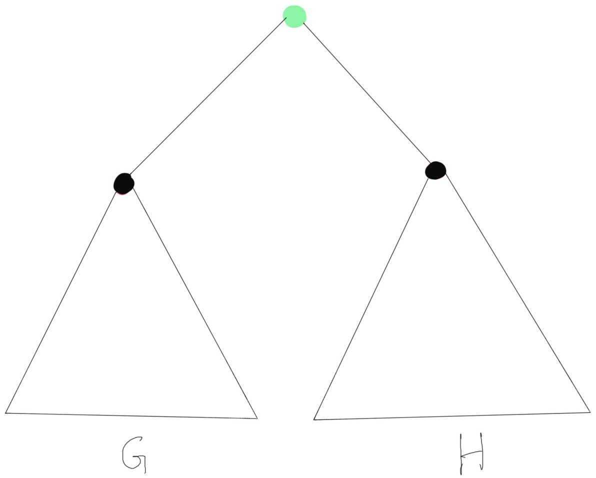

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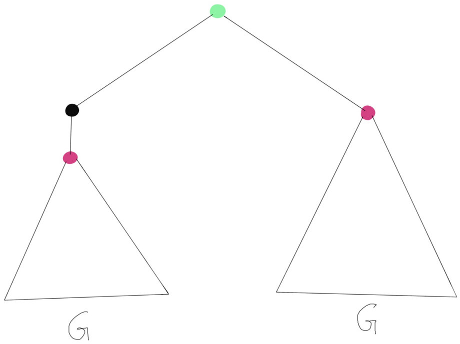

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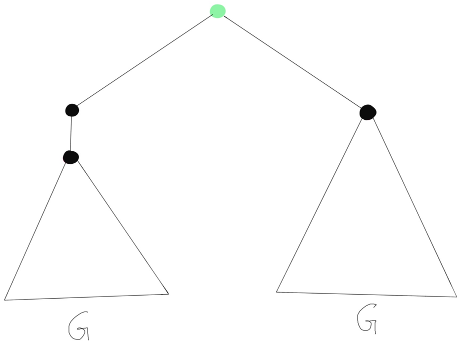

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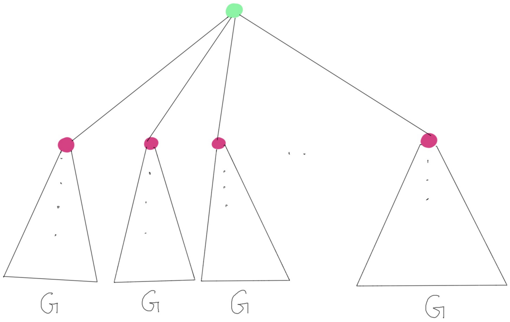

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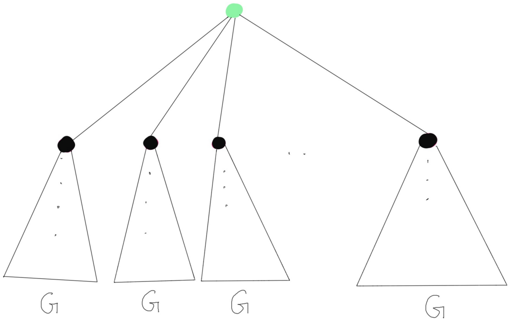

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- Hence, $\mathrm{Qut}_r(T)$ can be constructed iteratively using $1-3$.

Thank You!

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