Distance-regular graphs that support a uniform structure

(In collaboration with B. Fernández, Š.Miklavič, and G. Monzillo)

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Combinatorics Seminar-University of Waterloo
13/Nov/2023

Aim of the talk

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Classify non-bipartite distance-regular graphs with classical parameters which support a uniform structure.

We analyze three cases:

- Non-bipartite distance regular graphs of negative type.
- Non-bipartite distance regular graphs with classical parameters with $q = 1$.
- Non-bipartite distance regular graphs with classical parameters with $q \geq 2$.

Part 1

Preliminaries

Preliminaries

- $\Gamma = (X, \mathcal{R})$: simple, finite, and connected graph,
- $\partial(x, y)$:= distance between x and y, where $x, y \in X$,
- $\varepsilon(x) = max\{\partial(x, y) \,|\, y \in X\}$ (eccentricity of x),
- $D = max\{\varepsilon(x) \,|\, x \in X\}$ (diameter of Γ)
- $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$ (In particular, $\Gamma(x) = \Gamma_1(x)$).
- For an integer $k \geq 0$, we say that Γ is *regular* with valency k whenever $|\Gamma(x)| = k$ for all $x \in X$.

Preliminaries

- Adjacency matrix of Γ defined by

$$(A)_{xy} = \begin{cases} 1 & \partial(x,y) = 1, \\ 0 & \partial(x,y) \neq 1 \end{cases}$$

- V: vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X.
  $M_{|X|}(\mathbb{C})$: $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$.
  V: the *standard module*

Preliminaries

Definition 3.1

Fix $x \in X$ *and let* $\varepsilon = \varepsilon(x)$*. For* $0 \leq i \leq \varepsilon$ *let* $E_i^* = E_i^*(x)$ *denote the diagonal matrix in* $M_{|X|}(\mathbb{C})$ *defined by*

$$(E_i^*)_{yy} = \begin{cases} 1 & \partial(x,y) = i, \\ 0 & \partial(x,y) \neq i \end{cases} \qquad (y \in X)$$

E_i^* *is called the i-th dual idempotent of* Γ *with respect to* x*.*

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Definition 3.2

Terwilliger algebra $T := T(x)$ *of* Γ*, with respect to* x*, is a subalgebra of* $M_{|X|}(\mathbb{C})$*, generated by the adjacency matrix of* Γ *and the dual idempotents.*

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- $T-$module is a vector subspace W of V, which is invariant for every $t \in T$:

$$tW \subseteq W \text{ for all } t \in T.$$

Preliminaries

Definition 3.3

Let W denote an irreducible T-module. Then, W is an orthogonal direct sum of the nonvanishing spaces among $E_0^* W, E_1^* W, \ldots, E_{D(X)}^* W$. We define

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- *endpoint of W* $\quad r = min\{i|0 \leq i \leq \varepsilon, E_i^*W \neq 0\}$
- *diameter of W* $\quad d = |\{i|0 \leq i \leq \varepsilon, E_i^*W \neq 0\}| - 1$

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In addition,

- Irreducible T-module W is called thin, whenever

$$\dim E_i^*W \leq 1 \qquad \text{for each } 0 \leq i \leq \varepsilon.$$

Preliminaries

Matrices L, F, R

Define $L = L(x), F = F(x)$, and $R = R(x)$ in $M_{|X|}(\mathbb{C})$ by

$$L = \sum_{i=1}^{\varepsilon} E^*_{i-1} A E^*_i, \qquad F = \sum_{i=0}^{\varepsilon} E^*_i A E^*_i, \qquad R = \sum_{i=0}^{\varepsilon-1} E^*_{i+1} A E^*_i.$$

We refer to L, F, and R as the *lowering*, *flat*, and *raising* matrices with respect to x, respectively.

- Note that

$$F_{(z,y)} = \begin{cases} 1 & \partial(z,y) = 1 \text{ and } \partial(x,z) = \partial(x,y) \\ 0 & \text{otherwise} \end{cases}$$

  and

$$R_{(z,y)} = \begin{cases} 1 & \partial(z,y) = 1 \text{ and } \partial(x,z) = \partial(x,y) + 1 \\ 0 & \text{otherwise} \end{cases}$$

  .
- Moreover, $L, F, R \in T$, $F = F^\top$, $R = L^\top$, and $A = L + F + R$.

Uniform structure for a bipartite graph

Definition 3.4

Assume $\Gamma = (X, \mathcal{R})$ is bipartite. Fix a vertex $x \in X$. Define the following partial order $\leq$ on X:

$$\textit{for all} \;\; y, z \in X, \;\; \textit{let} \;\; y \leq z \quad \textit{whenever} \; \partial(x,y) + \partial(y,z) = \partial(x,z).$$

This allows us to directly translate the definition of a uniform poset to the setting of bipartite graphs.

Uniform structure for a bipartite graph

Definition 3.5

A parameter matrix $U = (e_{ij})_{1 \le i,j \le \varepsilon}$ *is defined to be a tridiagonal matrix with entries in* $\mathbb{C}$*, satisfying the following properties:*

- $e_{ii} = 1$ $(1 \le i \le \varepsilon)$,
- $e_{i,i-1} \neq 0$ *for* $2 \le i \le \varepsilon$ *or* $e_{i-1,i} \neq 0$ *for* $2 \le i \le \varepsilon$*, and*
- *the principal submatrix* $(e_{ij})_{s \le i, j \le t}$ *is nonsingular for* $1 \le s \le t \le \varepsilon$.

For convenience we write $e_i^- := e_{i,i-1}$ *for* $2 \le i \le \varepsilon$ *and* $e_i^+ := e_{i,i+1}$ *for* $1 \le i \le \varepsilon - 1$*. We also define* $e_1^- := 0$ *and* $e_\varepsilon^+ := 0$.

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- Let Γ be a bipartite graph. A uniform structure of Γ with respect to x is a pair (U, f) where $f = \{f_i\}_{i=1}^{\varepsilon}$ is a vector in $\mathbb{C}^\varepsilon$, such that

  $$e_i^- RL^2 + LRL + e_i^+ L^2 R = f_i L$$

  is satisfied on $E_i^* V$ for $1 \le i \le \varepsilon$

Uniform structure for a bipartite graph

Theorem 1 (P. Terwilliger- 1990)

Let $\Gamma = (X, \mathcal{R})$ be a bipartite graph and fix $x \in X$. Let $T = T(x)$ denote the corresponding Terwilliger algebra. Assume that Γ admits a uniform structure with respect to x. Then, the following assertions hold:

(i) *Every irreducible T-module is thin.*

(ii) *The isomorphism class of any irreducible T-module W is determined by its endpoint and its diameter.*

Graphs that support a uniform structure

Definition 3.6

Consider $\Gamma = (X, \mathcal{R})$: a non-bipartite graph, fix $x \in X$ and let

$$\mathcal{R}_f = \mathcal{R} \setminus \{yz \mid \partial(x,y) = \partial(x,z)\}.$$

We define $\Gamma_f = \Gamma_f(x)$ to be the graph with vertex set X and edge set $\mathcal{R}_f$, and we observe that Γ_f is bipartite and connected.

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- The graph Γ supports a uniform structure with respect to x, if Γ_f admits a uniform structure with respect to x.

Some observations

Let $\varepsilon = \varepsilon(x)$ and let $T_f = T_f(x)$ be the Terwilliger algebra of Γ_f. Then,

- since X is also the vertex set of Γ_f, we observe that V is also the standard module for Γ_f.
- the flat matrix of Γ_f is the zero matrix and we have $A_f = L + R$.
- for $0 \leq i \leq \varepsilon$,The i-th dual idempotents of Γ_f with respect to x is equal to E_i^*, and we have $T_f =< L, R, E_i^{*}{}_{i=0}^{\varepsilon} >$.

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Lemma 1 (Connection between a T-module and a T_f-module)

Let W denote a T-module. Then,

- *W is also a T_f-module.*
- *If W is a thin irreducible T-module, then W is a thin irreducible T_f-module.*

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☞Here we must mention that the following might happen.

- W is irreducible as a T-module, but reducible as a T_f-module.
- W and W' are non-isomorphic as T-modules, but they are isomorphic as T_f-modules.

⇒ Both of them happen in the case of *Doob graphs*.

Distance regular graphs

We know

- The Terwilliger algebra of the graph Γ and its modules,
- uniform structure for bipartite graphs,
- The graph Γ *supports a uniform structure with respect to* x, if Γ_f admits a uniform structure with respect to x.

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☞ **Distance regular graphs with classical parameters**.

Distance regular graphs

Definition 3.7

- *The graph* Γ *is* *distance-regular* *whenever, for all integers* $0 \leq h, i, j \leq D$ *and all* $x, y \in X$ *with* $\partial(x, y) = h$*, the number* $p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$ *is independent of the choice of* x, y*. The constants* (p^h_{ij}) *are known as the intersection numbers of* Γ*. For convenience,*
  $c_i := p^i_{1\,i-1}$ $(1 \leq i \leq D)$,
  $a_i := p^i_{1i}$ $(0 \leq i \leq D)$,
  $b_i := p^i_{1\,i+1}$ $(0 \leq i \leq D-1)$,
  $k_i := p^0_{ii}$ $(0 \leq i \leq D)$.
- Γ *is bipartite iff* $a_i = 0$ *for all* $0 \leq i \leq D$.

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- Γ *is bipartite iff* $a_i = 0$ *for all* $0 \leq i \leq D$.

Definition 3.8

A distance regular graph Γ *is called a* *near polygon* *whenever* $a_i = a_1 c_i$ *for* $1 \leq i \leq D-1$ *and* Γ *does not contain the complete multipartite graph* $K_{1,1,2}$ *as an induced subgraph.*

Definition 3.9 (Distance-regular graphs with classical parameters)

The graph Γ *is said to have classical parameters* (D, q, α, β) *whenever the intersection numbers of* Γ *satisfy*

$$\begin{cases} c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) & (1 \leq i \leq D), \\ b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) & (0 \leq i \leq D-1) \end{cases}$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + q + q^2 + \cdots + q^{j-1}.$$

Note that q *is an integer and* $q \notin \{0, -1\}$.

Distance regular graphs

Definition 3.10

Let $\Gamma = (X, \mathcal{R})$ *denote a distance-regular non-bipartite graph with diameter* $D \geq 3$*, intersection numbers* $b_i\,(0 \leq i \leq D-1)$*,* $c_i\,(1 \leq i \leq D)$*, and eigenvalues* $\theta_0 > \theta_1 > \ldots > \theta_D$*. The graph* Γ *is tight whenever the equality holds in*

$$\left(\theta_1 + \frac{b_0}{a_1+1}\right)\left(\theta_D + \frac{b_0}{a_1+1}\right) \geq -\frac{b_0 a_1 b_1}{(a_1+1)^2}.$$

Distance regular graphs

From now on we,

- Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular non-bipartite graph with diameter $D \geq 3$, intersection numbers $b_i \, (0 \leq i \leq D-1)$, $c_i \, (1 \leq i \leq D)$, and eigenvalues $\theta_0 > \theta_1 > \ldots > \theta_D$.
- Fix $x \in X$, and let $T = T(x)$ be the Terwilliger algebra of Γ and $E_i^* \, (0 \leq i \leq D)$ be the dual idempotents of Γ with respect to x.
- Let L, F, and R denote the corresponding lowering, flat, and raising matrices, respectively.
- Let $T_f = T_f(x)$ be the Terwilliger algebra of Γ_f. Note that T_f is generated by the matrices L, R, and E_i^* $(0 \leq i \leq D)$.

Part 2

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- DRGs with classical parameters of negative type.

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- Distance-regular graphs with classical parameters with $q = 1$.

DRGs with classical parameters of negative type

DRGs with classical parameters of negative type that support a uniform structure

DRGs with classical parameters of Negative type that support a uniform structure

$q \leq -2$

Let Γ be a distance-regular graph with classical parameters of negative type.

Question. Which graphs, Γ, in this category support a uniform structure?

We split the analysis of this question into three cases:

DRGs with classical parameters of Negative type that support a uniform structure

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Question. Which graphs, Γ, in this category support a uniform structure?

We split the analysis of this question into three cases:

- Case 1. Γ has intersection number $a_1 \neq 0$ and is not a near polygon.
- Case 2. Γ has intersection number $a_1 = 0$.
- Case 3. Γ is a near polygon.

Case 1. Γ has intersection number $a_1 \neq 0$ and is not a near polygon.

Proposition 1 (Š.Miklavič - 2009)

Assume that Γ is of negative type with $a_1 \neq 0$ and it is not a near polygon. Then, the following statements hold.

- *Up to isomorphism there is a unique irreducible module with endpoint 1 which is non-thin.*
- *Let W denote a non-thin irreducible T-module with endpoint 1. Pick a non-zero $w \in E_1^*W$. Then, the following vectors form a basis for W:*

$$E_i^* A_{i-1} w \quad (1 \leq i \leq D), \qquad E_i^* A_{i+1} w \quad (2 \leq i \leq D-1). \tag{1}$$

Case 1.

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]

Assume that Γ is of negative type with $a_1 \neq 0$ and it is not a near polygon. Then, Γ does not support a uniform structure with respect to x.

Case 1.

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]

Assume that Γ is of negative type with $a_1 \neq 0$ and it is not a near polygon. Then, Γ does not support a uniform structure with respect to x.

Proof.

- Let W denote a non-thin irreducible T-module with endpoint 1 (which is unique),
- pick a non-zero $w \in E_1^* W$ (W is also a T_f-module),
- let $W' \subseteq W$ be an irreducible T_f-module which contains w,
- using the action of L and R on the basis from Proposition 1, we observe that the vectors Rw and $LR^2 w$ are linearly independent.
- W' is non-thin,
- by Theorem 1, Γ does not support a uniform structure.

Case 2. Γ has intersection number $a_1 = 0$.

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]
Assume that Γ is of negative type with $a_1 = 0$. Then, Γ does not support a uniform structure with respect to x.

CASE 3. Γ IS A NEAR POLYGON.

We first recall following results for distance-regular graphs of negative type with $a_1 \neq 0$ and $c_2 > 1$.

Theorem 2 (Chih-wen Weng - 1999)

Assume Γ has classical parameters (D, q, α, β) where $D \geq 4$. Suppose $q \leq -2$, $a_1 \neq 0$, and $c_2 > 1$. Then, one of the following hold.

- *Γ is the dual polar graph ${}^2A_{2D-1}(-q)$.*
- *Γ is Hermitian forms graph $Her_{-q}(D)$.*
- *$\alpha = (q-1)/2$, $\beta = -(1+q^D)/2$, and $-q$ is a power of an odd prime.*

Corollary 1

Assume Γ has classical parameters (D, q, α, β). Suppose Γ is a regular near polygon with $q \leq -2$. Then, either Γ is the dual polar graph ${}^2A_{2D-1}(-q)$ or $D = 3$.

Case 3. Γ is a near polygon.

Therefore, we have the following result.

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]
Let Γ denote the dual polar graph ${}^2A_{2D-1}(-q)$. Then,

$$-\frac{q^4}{q^2+1}RL^2 + LRL - \frac{q^{-2}}{q^2+1}L^2R = (-q)^{2D-1}L \quad \text{(C. Worawannotai - 2013)}$$

is satisfied on E_i^*V for $1 \leq i \leq D$. Therefore, Γ supports a uniform structure with respect to x, where $e_i^- = -q^4/(q^2+1)$ $(2 \leq i \leq D)$, $e_i^+ = -q^{-2}/(q^2+1)$ $(1 \leq i \leq D-1)$, and $f_i = (-q)^{2D-1}$ $(1 \leq i \leq D)$.

Distance-regular graphs with classical parameters $(D, \alpha, \beta, q = 1)$

Distance-regular graphs with classical parameters with $q = 1$

We have the following classification for DRGs with classical parameters with $q = 1$.

Theorem 3 (Theorem 6.1.1 - Brouwer, Cohen, and Neumaier)

Let Γ denote a distance-regular graph with classical parameters with $q = 1$. Then, Γ is one of the following graphs:

- *Johnson graph $J(n, D)$, $n \geq 2D$, (tight: $n = 2D$)*
- *Gosset graph, (tight)*
- *Hamming graph $H(D, n)$,*
- *Halved cube $\frac{1}{2}H(n, 2)$, (tight: n even)*
- *Doob graph $D(n, m)$, $n \geq 1$, $m \geq 0$.*

We analyze each of these families in order to see which one admits a uniform structure.

Distance-regular graphs with classical parameters with $q = 1$

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]
Let Γ denote a tight graph with classical parameters with $q = 1$. Then, Γ does not support a uniform structure with respect to x.

Corollary 2

If Γ is one of the following graphs,

1. *Johnson graph $J(2D, D)$,*
2. *Gosset graph,*
3. *Halved cube $\frac{1}{2}H(n, 2)$ with n even,*

then, Γ does not support a uniform structure with respect to x.

JOHNSON GRAPHS $J(n, D)$ WITH $n > 2D$

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]
Let $\Gamma = J(n, D)$ with $n \geq 2D$. Then, Γ does not support a uniform structure.

Hamming graph $H(D, n)$ with $n \geq 3$

Theorem [B. Fernández, R.Maleki, Š.Miklavič, G. Monzillo - 2023]
Let Γ denote the Hamming graph $H(D, n)$ with $n \geq 3$. Then,

$$-\frac{1}{2}RL^2 + LRL - \frac{1}{2}L^2R = (n-1)L$$

is satisfied on E_i^*V for $1 \leq i \leq D$ and Γ supports a uniform structure with respect to x, where $e_i^- = -\frac{1}{2}$ $(2 \leq i \leq D)$, $e_i^+ = -\frac{1}{2}$ $(1 \leq i \leq D-1)$, and $f_i = n-1$ $(1 \leq i \leq D)$.

Halved cubes $\frac{1}{2}H(n,2)$ with n odd.

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]
Let Γ denote the Halved cube $\frac{1}{2}H(n,2)$ with n odd, $n \geq 7$. Recall that $D = \lfloor \frac{n}{2} \rfloor = (n-1)/2$. Then,

$$e_i^- RL^2 + LRL + e_i^+ L^2 R = f_i L$$

is satisfied on $E_i^* V$ for $1 \leq i \leq D$, where

$$e_i^- = \frac{4i-1-2D}{6-8i+4D} \ (2 \leq i \leq D) \qquad e_i^+ = \frac{4i-5-2D}{6-8i+4D} \ (1 \leq i \leq D-1)$$

$$f_i = -(4i-5)(4i-1) + (16i-12)D - 4D^2 \ \ (1 \leq i \leq D).$$

Therefore, Γ supports a uniform structure with respect to x.

Doob graphs $D(n, m)$ where $n \geq 1$, $m \geq 0$

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]

Let Γ denote the Doob graph $D(n, m)$ with $n \geq 1$, $m \geq 0$ and $D = 2n + m \geq 3$. Then,

$$-\frac{1}{2}RL^2 + LRL - \frac{1}{2}L^2R = 3L$$

is satisfied on E_i^*V for $1 \leq i \leq D$ and Γ supports a strongly uniform structure with respect to x, where $e_i^- = -\frac{1}{2}\,(2 \leq i \leq D)$, $e_i^+ = -\frac{1}{2}\,(1 \leq i \leq D-1)$, and $f_i = 3\,(1 \leq i \leq D)$.

Summary of the results

Non-bipartite distance-regular graphs with classical parameters (D, q, α, β) with $q \leq 1$.

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$$\begin{cases} \Gamma \text{ has intersection number } a_1 \neq 0 \text{ and is not a near polygon } \times \\ \Gamma \text{ has intersection number } a_1 = 0 \ \times \\ \\ \Gamma \text{ is a near polygon} \\ \end{cases}$$

Summary of the results

Non-bipartite distance-regular graphs with classical parameters (D, q, α, β) with $q \leq 1$.

- $q \leq -2$ (negative type)

$$\begin{cases} \Gamma \text{ has intersection number } a_1 \neq 0 \text{ and is not a near polygon } \times \\ \Gamma \text{ has intersection number } a_1 = 0 \ \times \\ \Gamma \text{ is a near polygon } \begin{cases} \Gamma \text{ is a dual polar graph, } D \geq 4 \ \checkmark \\ \Gamma \text{ has diameter } D = 3 \ ? \end{cases} \end{cases}.$$

Summary of the results

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Part 3

Distance-regular graphs with classical parameters $(D, \alpha, \beta, q \geq 2)$

Local graph

From now on we let Γ be a distance-regular graph with classical parameters (D, α, β, q), where $D \geq 4$, and $q \geq 2$. Fix $x \in X$ and assume that Γ supports a uniform structure with respect to x. Moreover, we assume that every irreducible T-module with endpoint one is thin.

- Let $\Delta = \Delta(x)$ be the subgraph of induced on the set of vertices adjacent to x in Γ. We refer to Δ as the local graph of Γ with respect to x.
- If W is any thin irreducible T-module with endpoint 1, then $E_1^* W$ is a one-dimensional eigenspace for $E_1^* A E_1^*$, whose eigenvalue η is called the *local eigenvalue of W*. $\Rightarrow \eta \in \{\eta_2, \eta_3, \ldots, \eta_k\}$, so $\tilde{\theta}_1 \leq \eta \leq \tilde{\theta}_D$.

Local Graph

Remark 1

Let W be a thin irreducible T-module with endpoint 1, it is known that the local eigenvalue of W belongs to $\{-q-1, \beta-\alpha-1, -1, \alpha\frac{q^{D-1}-1}{q-1}-1\}$.

Now, we further assume that $\alpha \neq 0$ and we have the following.

Theorem 4

Δ *is a connected strongly regular graph with eigenvalues*

$$\{a_1, \alpha\frac{q^{D-1}-1}{q-1}-1, -q-1\}$$

Local Graph

Theorem 5 (B. Fernández, R.Maleki, Š.Miklavič, G. Monzillo - 2023)

Assume $\alpha \neq 0$. *Let* $(n = b_0, k = a_1, \lambda, \mu)$ *denote the parameters of* Δ. *Then,*

$$\beta = \alpha \frac{q^{D+1} - 1}{q - 1} - q.$$

In particular,

$$n = b_0 = \frac{(q^D - 1)(\alpha q^{D+1} - q^2 + q - \alpha)}{(q-1)^2}, \quad k = a_1 = \frac{(q+1)(\alpha q^D - q - \alpha + 1)}{q - 1},$$

$$\lambda = \frac{\alpha q^D + \alpha q^2 - q^2 - \alpha q - q - \alpha + 2}{q - 1}, \quad \mu = \alpha(q+1).$$

A key Tool

Theorem 6 (A. Neumaier, 1979)

Let G be a strongly regular graph with parameters (n, k, λ, μ) and eigenvalues $k > r > s$. Assume that $s < -1$ is integral. Then, at least one of the following conditions must hold:

1. $r \leq \frac{s(s+1)(\mu+1)}{2} - 1$;
2. $\mu = s^2$ *(in which case G is a Steiner graph derived from a Steiner 2-system in which each line contains s points);*
3. $\mu = s(s+1)$ *(in which case G is a Latin square graph derived from an s-net).*

Two feasible parameter sets

Lemma 2 (B. Fernández, R.Maleki, Š.Miklavič, G. Monzillo - 2023)

Let Δ be the local graph of Γ with eigenvalues $a_1, \alpha\frac{q^{D-1}-1}{q-1} - 1$, and $-q-1$. Then the case (1) in Neumaier's Theorem cannot happen.

Two feasible parameter sets

Lemma 2 (B. Fernández, R.Maleki, Š.Miklavič, G. Monzillo - 2023)

Let Δ be the local graph of Γ with eigenvalues $a_1, \alpha\frac{q^{D-1}-1}{q-1} - 1$, and $-q-1$. Then the case (1) in Neumaier's Theorem cannot happen.

- Cases (2) and (3) are both feasible, and this leads us to the following obsevation.

$$(D, q, \alpha, \beta) = (D, q, q, \frac{q^2(q^D - 1)}{q - 1}$$

or

$$(D, q, \alpha, \beta) = (D, q, q + 1, \frac{q^{D+1}(q + 1) - q^2 - 1}{q - 1})$$

More Results

Theorem 7 (B. Fernández, R.Maleki, Š.Miklavič, G. Monzillo - 2023)

The family of distance regular graphs with classical parameters $(D, q, \alpha, \beta) = (D, q, q+1, \frac{q^{D+1}(q+1)-q^2-1}{q-1})$ *does not exist.*

Proof.

First of all, for $D \geq 6$ the intersection number p^6_{33} is an integer only for $q = 2, 4$. next, the multiplicity of the second eigenvalue is not an integer. □

More Results

Theorem 8 (B. Fernández, R.Maleki, Š.Miklavič, G. Monzillo - 2023)

Let $D \not\equiv 0 \pmod 6$. *The family of distance regular graphs with classical parameters* $(D, q, \alpha, \beta) = (D, q, q, \frac{q^2(q^D-1)}{q-1})$ *does not exist.*

Proof.

First of all, the multiplicity of the second eigenvalue is an integer only for D even. Second, the multiplicity of the third eigenvalue is an integer only when $D \equiv 0 \pmod 6$. □

Main Theorem

Theorem 9 (B. Fernández, R.Maleki, Š.Miklavič, G. Monzillo - 2023)

We have that either $\alpha = 0$, or $D \equiv 0 \pmod 6$ and Γ has classical parameters

$$\left(D, q, q, \frac{q^2(q^D - 1)}{q - 1}\right).$$

Remark 2

Computational results show that for $D \leq 3000$, the valency k_D and the multiplicity f_D of the eigenvalue θ_D are NOT integers.

A CONJECTURE

Conjecture 1

There exists NO distance-regular graph with classical parameters

$$\left(D, q, q, \frac{q^2(q^D - 1)}{q - 1}\right)$$

with $q \geq 2$ *and* $D \geq 4$.

Remark 3

Assuming the aforementioned conjecture is true, we have distance regular graphs with classical parameters only when $\alpha = 0$.


Thank
you!