Thin distance-regular graphs with classical parameters $(D, q, q, \frac{q^t-1}{q-1} - 1)$ with $t > D$ are the Grassmann graphs

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July 10, 2023

Definitions and notations

Let $\Gamma = (X, R)$ be a finite, simple, undirected, connected graph.

- $x \sim y$ if $xy \in R$; $x \nsim y$ if $xy \notin R$.
- **Distance** $\partial(x,y)$: the length of a shortest path connecting x and y.
- **Diameter** $D := D(\Gamma) = \max\{\partial(x,y) \mid x,y \in X\}$.
- $\Gamma_i(x) = \{y \in X \mid \partial(x,y) = i\}$ for a vertex x $(0 \leq i \leq D)$.
- **Regular** with valency k: $|\Gamma_1(x)| = k$ for all vertices in Γ.

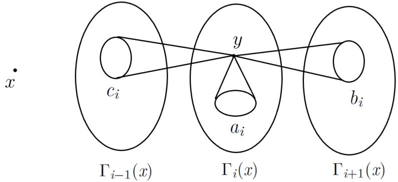
Distance-regular graphs(DRGs)

Distance-regular graph

A graph Γ is called **distance-regular** (DR) if there are constants a_i, b_i, c_i $(0 \leq i \leq D = D(\Gamma))$ s.t. for any $x, y \in X$, if $\partial(x,y) = i$ then

$$c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|,$$
$$a_i = |\Gamma_i(x) \cap \Gamma_1(y)|,$$
$$b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|.$$

- Γ is regular with valency $k = b_0$.
- $a_i + b_i + c_i = b_0 = k$.
- **intersection array**: $\{b_0 = k, b_1, \ldots, b_{D-1}; c_1 = 1, c_2, \ldots, c_D\}$.



DRGs with classical parameters

Classical parameters

A distance-regular graph Γ of diameter D has **classical parameters** (D,b,α,β) if the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right),$$

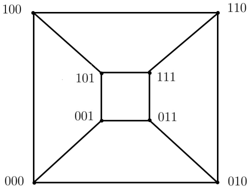
$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right),$$

where $\begin{bmatrix} j \\ 1 \end{bmatrix}_b = 1 + b + b^2 + \cdots b^{j-1}$ for $j \geq 1$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$.

- $b \neq 0, -1$.
- The parameters of the most of DRGs of diameter at least 3 with classical parameters are uniquely determined by the intersection array (see [BCN, Corollay 6.2.2]).

Hamming graphs

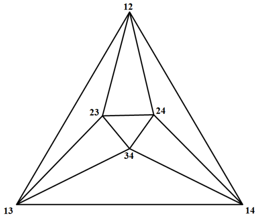
- $q \geq 2$, $D \geq 1$ integers.
- $\Omega = \{0, \ldots, q-1\}$.
- **Hamming graph** $H(D,q)$ has vertex set Ω^D.
- $x \sim y$ if they differ exactly one position.
- Diameter is D.
- $H(D,2)$=D-cube.
- DRG with $c_i = i$, $b_i = (D-i)(q-1)$.
- $(D,b,\alpha,\beta) = (D,1,0,q-1)$.



$H(3,2)$

Johnson graphs

- $1 \leq D \leq N$ integers.
- $\Omega = \{1, \ldots, N\}$.
- **Johnson graph** $J(N, D)$ has vertex set $\binom{\Omega}{D}$.
- $x \sim y$ if $|x \cap y| = D - 1$.
- $J(N, D) \simeq J(N, N - D)$, diameter $\min\{D, N - D\}$.
  $\Rightarrow$ w.l.o.g., assume that $N \geq 2D$.
- DRG with $c_i = i^2$, $b_i = (D - i)(N - D - i)$.
- $(D, b, \alpha, \beta) = (D, 1, 1, N - D)$.



$J(4, 2)$

Grassmann graphs

- $q \geq 2$ prime power, $1 \leq D \leq N-1$ integers.
- $\Omega = \mathbb{F}_q^N$: N-dimensional vector space over $\mathbb{F}_q$.
- **Grassmann graph** $J_q(N,D)$ has vertex set $\begin{bmatrix}\Omega \\ D\end{bmatrix}_q$, i.e., D-dimensional subspace of Ω.
- $x \sim y$ if $\dim(x \cap y) = D-1$.
- $J_q(N,D) \simeq J_q(N,N-D)$, diameter $\min\{D, N-D\}$.
  $\Rightarrow$ w.l.o.g., assume that $N \geq 2D$.
- DRG with $c_i = \left(\begin{bmatrix} i \\ 1\end{bmatrix}_q\right)^2$, $b_i = q^{2i+1}\begin{bmatrix} D-i \\ 1\end{bmatrix}_q \begin{bmatrix} N-D-i \\ 1\end{bmatrix}_q$.
- $(D,b,\alpha,\beta) = (D,q,q,\begin{bmatrix} N-D+1 \\ 1\end{bmatrix}_q - 1)$.

For a distance-regular graph with diameter D, define its intersection array by $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$.
One is interested whether there exists a unique DRG with a given intersection array. In this case we say that the DRG is determined by its parameters.

For examples:

- The Petersen graph is determined by its intersection array $\{3,2;1,1\}$.
- Hamming graph $H(D,q)$ [1] is determined by its intersection array unless $q = 4$, $D \geq 2$, in which case there are so-called Doob graphs.
- Johnson graph $J(N,D)$ $(N \geq 2D)$ [2] is determined by its intersection array unless $(N,D) = (8,2)$, in which case there are so-called Chang graphs.

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[1] Y. Egawa. *J. Combin. Theory Ser. A*, 31:108-125, 1981.
[2] P. Terwilliger. *Discrete Math.*, 58:175-189, 1986.

The Grassmann graphs

- In 1995, Metsch [3] showed that the Grassmann graph $J_q(N,D)$ $(3 \leq D \leq \frac{N}{2})$ is characterized by their intersection array except for the following:
  (1) $N = 2D$ or $N = 2D+1$, $q \geq 2$;
  (2) $N = 2D+2$ and $q \in \{2,3\}$;
  (3) $N = 2D+3$ and $q = 2$.
- In 2018, Gavrilyuk and Koolen [4] solved the case $N = 2D$, $q \geq 2$ with large enough D.
- In 2005, Van Dam and Koolen [5] discovered Twisted Grassmann graphs $\widetilde{J}_q(2D+1,D)$ that have the same intersection array as Grassmann graphs $J_q(2D+1,D)$, so $J_q(2D+1,D)$ is not determined by its intersection array.

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[3] K. Metsch. *European J. Combin.*, 16: 639–644, 1995.

[4] A. Gavrilyuk and J. Koolen. *Arxiv:1806.02652v1*, 2018.

[5] E. van Dam, J. Koolen. *Invent. Math.*, 162:189-193, 2005.

Our work

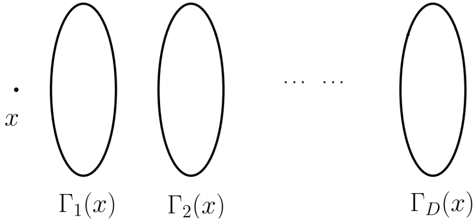
- We [6] showed that the Grassmann graph $J_q(N,D)$ $(2D+1 \leq N \leq 2D+3,\ q \geq 2)$ with large enough diameter is characterized by their intersection array if they are thin .

Remark: Twisted Grassmann graphs $\widetilde{J}_q(2D+1,D)$ are not thin.

[6] X. Liang, Y-Y. Tan and J. Koolen. *Electron J. Combin.*, 2021.

Terwilliger algebra

Let $\Gamma = (X, R)$ be a distance-regular graph with diameter D and A be its adjacency matrix (i.e., $A_{xy} = 1$ if $x \sim y$; 0 otherwise).



Fix a **base vertex** $x \in X$. Define $E_i^* = E_i^*(x) \subseteq \mathsf{Mat}_X(\mathbb{C})$ by

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \; y \in \Gamma_i(x), \\ 0 & \text{if } \; y \notin \Gamma_i(x). \end{cases}$$

- $\mathcal{T} = \mathcal{T}(x) = \langle A, E_0^*, E_1^*, \ldots, E_D^* \rangle$: **Terwilliger algebra w.r.t** x .

Irreducible $\mathcal{T}$-modules

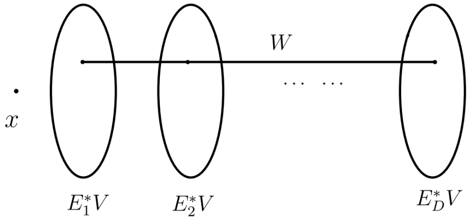
Let $\mathcal{T} = \mathcal{T}(x)$ be the Terwilliger algebra w.r.t x of Γ.

- $V = \mathbb{C}^X$ that is endowed with the Hermitian inner product.
- $\mathcal{T}$**-module** $W \subseteq V$ s.t. $Tw \in W$ for $\forall\ T \in \mathcal{T}$, $\forall\ w \in W$.
- $\mathcal{T}$-module W is called **irreducible** if it is non-zero, and contains no $\mathcal{T}$-submodule besides $0, W$.

Irreducible $\mathcal{T}$-modules

Consider Γ w.r.t the ordering $E_0^*, E_1^* \ldots, E_D^*$, where $E_i^* = E_i^*(x)$.
Let W be an irreducible $\mathcal{T}$-module of Γ.

- **endpoint** of W: $\min\{i \mid E_i^* W \neq 0\}$.
- **diameter** of W: $|\{i \mid E_i^* W \neq 0\}| - 1$.
- W is **thin** if $\dim E_i^* W \leq 1$ for all i $(0 \leq i \leq D)$.
- Γ is ***i*-thin** if each irreducible $\mathcal{T}(x)$-module of endpoint at most i is thin for all $x \in X$.
- Γ is **thin** if it is i-thin for all i $(0 \leq i \leq D)$.



Our results

For a natural number $q \geq 2$, define a function $\chi(q)$ by:

$$\chi(q) = \begin{cases} 13 & \text{if } q = 2, \\ 10 & \text{if } q = 3, \\ 9 & \text{if } q = 4, \\ 8 & \text{if } q \in \{5,6,7\}, \\ 7 & \text{if } q \geq 8. \end{cases}$$

Corollary [Liang, Koolen, Tan, 2021]

Let Γ be a thin distance-regular graph with classical parameters $(D, q, q, \frac{q^t-1}{q-1} - 1)$ with $q \geq 2$, $t > D$ integers. If $D \geq \chi(q)$, then Γ is the Grassmann graph $J_q(D+t-1, D)$.

Our results

Theorem [Liang, Koolen, Tan, 2021]

Let Γ be a 1-thin distance-regular graph with classical parameters $(D,q,q,\frac{q^t-1}{q-1}-1)$ with $q \geq 2$, $t > D$ integers. Assume further that Γ is μ-graph-regular with parameter ℓ. If $D \geq \chi(q)$, then Γ is the Grassmann graph $J_q(D+t-1,D)$.

Theorem [Terwilliger note]

Let Γ be a thin distance-regular graph with classical parameters with diameter $D \geq 5$. Then Γ is μ-graph-regular.

- A regular graph Γ is called **μ-graph-regular** (with parameter ℓ) if each subgraph induced on $\Gamma_1(x) \cap \Gamma_1(y)$ for any two vertices x,y with $\partial(x,y) = 2$ is regular with valency ℓ.

Partial linear spaces and point graphs

- A **partial linear space** is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ is a finite set (called the **point set**), $\mathcal{L}$ is a finite set (called the **line set**), and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ is the **incidence relation** such that
  - every line is incident with at least two points ;
  - any two distinct points lie on at most one line.
- The **point graph** of $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a graph defined with $\mathcal{P}$ as its vertex set, with two points being adjacent, if they are collinear.

Back

Partial linear spaces and point graphs

Theorem 9.3.9 [BCN, 1989]

Let $(\mathcal{P}, \mathcal{L}, \in)$ be a partial linear space such that for some integer $q \geq 2$:

(1) each line has at least $q^2 + q + 1$ points;

(2) each point is on more than $q + 1$ lines;

(3) if $P \in \mathcal{P}$, $l \in \mathcal{L}$ and $\partial(P, l) = 1$, then there are exactly $q + 1$ lines on P meeting l;

(4) if the points P and P' have distance 2 in the point graph Γ, then there are precisely $q + 1$ lines l on P such that $\partial(P', l) = 1$;

(5) the point graph Γ of $(\mathcal{P}, \mathcal{L}, \in)$ is connected.

Then q is a prime power, and $(\mathcal{P}, \mathcal{L}, \in) \simeq \left(\begin{bmatrix} \Omega \\ D \end{bmatrix}_q, \begin{bmatrix} \Omega \\ D+1 \end{bmatrix}_q, \subseteq \right)$ for some integer N, $\Omega = \mathbb{F}_q^N$ and $3 \leq D \leq \frac{N}{2}$. In particular, $\Gamma = J_q(N, D)$.

Back

Known results

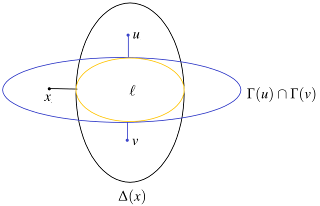
- The local graph $\Delta(x)$ at a vertex x of a graph Γ is the subgraph of Γ induced by $\Gamma_1(x)$.
- The local graph of a Grassmann graph $J_q(N,D)$ is isomorphic to the q-clique extension of $\left(\begin{bmatrix} N-D \\ 1 \end{bmatrix}_q \times \begin{bmatrix} D \\ 1 \end{bmatrix}_q \right)$-grid.
- The spectrum of the q-clique extension of the $(t_1 \times t_2)$-grid is $\{[q(t_1+t_2-1)-1]^1, [q(t_1-1)-1]^{t_2-1}, [q(t_2-1)-1]^{t_1-1}, [-1]^{(q-1)t_1t_2}, [-q-1]^{(t_1-1)(t_2-1)}\}$.

More definitions

- A k-regular graph Γ with v vertices is called **edge-regular** with parameters (v,k,a) if any two adjacent vertices have exactly a common neighbors; called **co-edge-regular** with parameter (v,k,c) if any two distinct non-adjacent vertices have exactly c common neighbors.

Lemma

Let Γ be a graph that is edge-regular with parameters (v,k,a) and μ-graph-regular with parameter ℓ. Then any local graph of Γ is co-edge-regular with parameters (k,a,ℓ). Back



Grand cliques in G

Let G be a graph that is cospectral with the q-clique extension of the $(t_1 \times t_2)$-grid, where $q \geq 2$, $t_1 > 2t_2 > 2$ are integers.

Result 1 [Liang, Koolen, Tan, 2021]

For any clique C of G, we have $|C| \leq qt_1$. If equality holds, then every vertex outside C has exactly q neighbors in C. Back

Assume futher that G is co-edge-regular with parameters (v, k, c).

- We call a maximal clique in G a **grand clique**, if it contains at least $\frac{19}{36}k$ vertices.

Result 2 [Liang, Koolen, Tan, 2021]

If $t_2 > \frac{36(q+1)^6}{q^2}$, then any vertex of G lies on a unique grand clique. Back

Result 3 [Liang, Koolen, Tan, 2021]

If G has a clique of size qt_1, then G is the q-clique extension of the $(t_1 \times t_2)$-grid. Back

Proof of the main theorem

Let $\Gamma = (X,R)$ be a 1-thin distance-regular graph with classical parameters $(D,b,\alpha,\beta) = \left(D,q,q,\begin{bmatrix} D+e+1 \\ 1 \end{bmatrix}_q - 1\right)$, where $q \geq 2$ and $e \in \{1,2,3\}$ are integers and $D \geq \chi(q)$.

- Assume further that Γ is μ-graph-regular with parameter ℓ.
- The local graph $\Delta(x)$ at any x of Γ is co-edge-regular with parameters (k,a_1,ℓ). Lemma
- Set $t_1 = \begin{bmatrix} D+e \\ 1 \end{bmatrix}_q$, $t_2 = \begin{bmatrix} D \\ 1 \end{bmatrix}_q$.
- $\Delta(x)$ is cospectral with the q-clique extension of the $(t_1 \times t_2)$-grid.

Proof of the main theorem

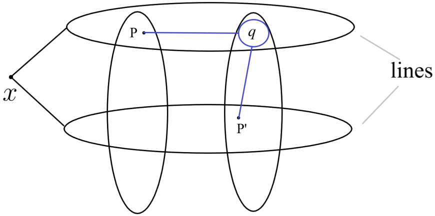
- There exists a Delsarte clique in Γ, say C. ($|C| = qt_1 + 1$)
- For any $x \in C$, $\Delta(x)$ is the q-clique extension of the $(t_1 \times t_2)$-grid. ▸ Result 3
- For any neighbor y of x, $\Delta(y)$ is again the q-clique extension of the $(t_1 \times t_2)$-grid.
- As Γ is connected, any local graph is the q-clique extension of the $(t_1 \times t_2)$-grid.

Proof of the main theorem

- A maximal clique is called a **line** of Γ if it contains at least $\frac{19}{36}a_1 + 1$ vertices.
- Let $\mathcal{L}$ be the set consisting of all lines in Γ.
- As $D \geq \chi(q)$, for any two adjacent vertices $x, y \in V$, there exists a unique line $l \in \mathcal{L}$ such that $x, y \in l$. ▸ Result 2
- $(X, \mathcal{L}, \in)$ is a partial linear space such that Γ is its point graph. ▸ Partial linear spaces

Proof of the main theorem

- Γ is the point graph of the partial linear space $(X, \mathcal{L}, \in)$, where $\mathcal{L}$ is the set of Delsarte cliques of Γ.
- Every edge lies in a unique Delsarte clique and any vertex outside a Delsarte clique C has either $q+1$ or none neighbors in C. Result 1
- Γ is the Grassmann graph $J_q(2D+e, D)$. Theorem 9.3.9



Thank you for your attention!