My conjectures in spectral graph theory

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Spectral notation

- n - number of vertices; m - number of edges
- $\overline{G}$ - complement of G
- A - adjacency matrix
- $\mu = \mu_1 \geq, ..., \geq \mu_n$ eigenvalues of A $(\sum_i \mu_i = tr(A) = 0)$

$$s^+ = \sum_{+ve} \mu_i^2 \text{ and } s^- = \sum_{-ve} \mu_i^2 \text{ so } s^+ + s^- = \sum_i \mu_i^2 = tr(A^2) = 2m$$

s^+ and s^- called *positive and negative square energies*

- $\omega(G)$ - clique number; $\alpha(G) = \omega(\overline{G})$ - independence number
- $\chi(G)$ - chromatic number
- $\chi_v(G) = \vartheta'(\overline{G})$ - vector chromatic number = Schrijver variant of Lovász theta function of complement
- $\chi_f(G) = \lim_{b\to\infty} \frac{\chi_b(G)}{b}$ - fractional chromatic number
- $\omega(G) \leq \chi_v(G) \leq \vartheta(\overline{G}) \leq \chi_f(G) \leq \chi(G)$
- (n^+, n^-, n^0) - numbers of positive, negative and zero eigenvalues (known as inertia)

Some applications of spectral graph theory

- Spectral proofs of non-spectral results eg characterisation of Moore graphs with diameter 2 (Hoffman, Singleton) and proof of Sensitivity Conjecture in computer science (Hao Huang)
- Spectral bounds for non-spectral (often NP-hard) parameters, eg

$$1 + \frac{\mu_1}{|\mu_n|} \leq \chi(G) \leq 1 + \mu$$

- Non-spectral bounds for spectral parameters eg

$$\mu \geq \frac{2m}{n} = d \text{ with equality only for regular graphs}$$

Proven conjecture 1 - lower bound for $\omega(G)$

- $\frac{2m}{2m-\mu^2} \leq \chi(G)$ - Elphick PhD (1981)
- $\frac{2m}{2m-\mu^2} \leq \omega(G)$ - Conjecture in Edwards and Elphick (1983)
- $\frac{2m}{2m-\mu^2} \leq \omega(G)$ - Proved by Nikiforov (2002) and cited 200x
- This provides a spectral proof of the concise version of Turán's Theorem (1941) since:

$$\frac{4m^2}{n^2} \leq \mu^2 \leq \frac{2m(\omega-1)}{\omega} \Rightarrow \frac{2m}{n} = d \leq \frac{n(\omega-1)}{\omega}$$

- Proof tools of Nikiforov: 1/3 page using Motzkin-Straus inequality (1965), which states that for any $n-$vector with $\sum_{i=1}^{n} x_i = 1$ and $x_i \geq 0$:

$$\sum_{ij \in E} x_i x_j \leq \frac{\omega-1}{2\omega}.$$

Also use Rayleigh quotient, normalized principal eigenvector and Cauchy-Schwarz.

Proven conjecture 2 - Introducing s^+ and s^-

- $1 + \max\left(\frac{s^+}{s^-}, \frac{s^-}{s^+}\right) \le \chi(G)$ - Conj. by Wocjan/Elphick (2013)
- $1 + \max\left(\frac{s^+}{s^-}, \frac{s^-}{s^+}\right) \le \chi_v(G)$ - Conjecture by Wocjan, Elphick, Anekstein (2023)
- $1 + \max\left(\frac{s^+}{s^-}, \frac{s^-}{s^+}\right) \not\le \omega(G)$ eg 5-cycle
- $1 + \max\left(\frac{s^+}{s^-}, \frac{s^-}{s^+}\right) \le \chi(G)$ - Proved by Ando/Lin (2015)
- $1 + \max\left(\frac{s^+}{s^-}, \frac{s^-}{s^+}\right) \le \chi_v(G)$ - Proved by Coutinho/Spier (2023)
- $\frac{2m}{2m-\mu^2} \le \frac{2m}{2m-s^+} = 1 + \frac{s^+}{s^-}$ - my inspiration for the conjecture
- Proof tools of Coutinho/Spier: spectral decomposition, semidefinite programs, positive semidefinite matrices, Cauchy-Schwarz (4 pages)

Proven conjecture 3 - lower bound for $\chi_v(G)$

- δ_1 = largest eigenvalue of signless Laplacian $Q = D + A$
- λ_1 = largest eigenvalue of Laplacian $L = D - A$
- $1 + \frac{\mu_1}{\mu_1 - \delta_1 + \lambda_1} \leq \chi(G)$ - proved by Kolotilina (2010)
- $\chi_v(G) \leq \chi(G)$ for all graphs; $\chi_v(G) \ll \chi(G)$ for some graphs
- $1 + \frac{\mu_1}{\mu_1 - \delta_1 + \lambda_1} \leq \chi_v(G)$ - conjecture by Elphick, experimental evidence by Anekstein, proof by Wocjan (2023)
- Exact for example for strongly regular, distance regular and symmetric graphs
- Inspiration: Galtman proved that $1 + \frac{\mu_1}{|\mu_n|} \leq \chi_v(G)$ and $|\mu_n| = \mu_1 - \delta_1 + \lambda_1$ for regular graphs
- Proof tools: Majorization, Gram matrices, Perron-Frobenius theorem, spectral decomposition, Schur products, stochastic matrices, correlation matrices (3 pages)

Proven conjecture 4 - an inertial bound

- $1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) \le \chi_f(G) \le \chi(G)$ - Conjecture by Elphick/Wocjan (2017)
- $1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) \le \chi(G)$ - Proven by Elphick/Wocjan (2017)
- Inspiration was that $\chi_f(G) \ge n/\alpha(G)$ and $\alpha(G) \le \min\left(n^+ + n^0, n^- + n^0\right)$. Also complete $q-$partite graphs have $n^+ = 1$ and $n^- = q - 1$.
- Interesting that n^0 is required in upper bound for $\alpha(G)$ but not required in lower bound for $\chi(G)$.
- Proof tools: Hermitian matrices, unitary matrices, rank of positive semidefinite matrices, orthogonal projectors, matrix algebra (2 pages)

Unproven conjecture 1 - a Nordhaus-Gaddum bound

- $n-1 \leq \mu(G)+\mu(\overline{G}) < \sqrt{2}n$ - proof by Nosal (1970)
- $\mu(G)+\mu(\overline{G}) \leq 4n/3-1$ - conjecture by Nikiforov (2007) and best possible
- $\mu(G)+\mu(\overline{G}) \leq 1.3660n-1$ - proof by Csikvári (2009)
- $\mu(G)+\mu(\overline{G}) \leq 4n/3-1$ - proof by Terpai (2011), using graphons and analysis
- $\sqrt{s^+(G)}+\sqrt{s^+(\overline{G})} < \sqrt{2}n$ - proof by Elphick and Aouchiche (2017)
- $\sqrt{s^+(G)}+\sqrt{s^+(\overline{G})} \leq 4n/3-1$ - conjecture by Elphick and Aouchiche (2017) and best possible
- No counter-examples found using AGX software or in Wolfram Mathematica database of graphs with up to 40 vertices
- Very hard to prove so perhaps search for a counter-example using AI

Unproven conjecture 2 - an inertial bound for $\chi_f(G)$

- $1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) \leq \chi_f(G)$ - Conj. by Elphick/Wocjan (2017)
- $1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) \not\leq \chi_v(G)$ - Noted by Elphick/Wocjan (2017)
- This bound is exact for Kneser graphs, and $\chi_f(G)$ can be defined in terms of homomorphisms from G to Kneser graphs
- Very surprised if false
- Combine proof for $\chi(G)$ due to Elphick and Wocjan with proof that $1 + \max\left(\frac{s^+}{s^-}, \frac{s^-}{s^+}\right) \leq \chi_f(G)$ due to Guo and Spiro (2023)

Unproven conjecture 3 - lower bound for $\omega(G)$

- For $G \neq K_n$, $\mu_1^2 + \mu_2^2 \leq \frac{2m(\omega-1)}{\omega}$ - Conjecture by Bollobás and Nikiforov (2007)
- Proved for triangle-free graphs by Lin, Ning and Wu (2021) and for regular graphs by Zhang (2024)
- In 2023, Elphick, Linz and Wocjan conjectured that for all G

$$\sum_{i=1}^{\ell} \mu_i^2 \leq \frac{2m(\omega-1)}{\omega} \text{ where } \ell = \min(n^+, \omega)$$

  They proved this conjecture for triangle-free, weakly perfect and Kneser graphs and Kumar and Pragada (2024) proved for graphs with at most $O(m^{1.5-\epsilon})$ triangles.
- Proof by Zhang uses that $\mu_2 > 0$, so may be possible to generalise his proof for regular graphs
- Liu and Ning (2023) placed these conjectures 3rd and 4th in their review of "Unsolved Problems in spectral graph theory"

Unproven conjecture 4 - bound for s^+ and s^-

- $\mu^2 \leq 2m - n + 1$ - Proved by Hong (1988)
- For any *connected* graph, Elphick, Farber, Goldberg and Wocjan (2016) conjectured that

  $\max(s^+, s^-) \leq 2m - n + 1$ equivalent to: $\min(s^+, s^-) \geq n - 1$

  They proved this for bipartite, barbell and regular graphs and for almost all graphs. For regular graphs, using Ando/Lin and Brooks theorem:

  $$\min(s^+, s^-) \geq \frac{2m}{\chi(G)} \geq \frac{2m}{\Delta} = \frac{2m}{d} = n$$

  with separate treatment for complete graphs and odd cycles.
- In general, if G has κ components, then $\min(s^+, s^-) \geq n - \kappa$
- $s^+ = s^- = n - 1$ for trees and $s^- = n - 1$ for complete graphs
- Liu and Ning (2023) placed this conjecture 1st in their review of over 20 "Unsolved Problems in spectral graph theory"

Unproven conjecture 4 - bound for s^+ and s^- continued

- I regard as my best conjecture to date because simple to state and straightforward to prove for regular graphs and almost all graphs. Easy counter-examples for disconnected graphs. I would be very surprised if false and conjecture does not "look" hard at first sight.
- Abiad, de Lima, Desai, Guo, Hogben, Madrid (2023) gave the name positive and negative square energies to s^+ and s^- and developed new tools, but made modest progress in proving for new graph classes
- $s^+ \gg s^-$ for almost all graphs so could treat as 2 conjectures
- $s^+(G) \ngeq s^+(G - e)$ and $s^-(G) \ngeq s^-(G - e)$ - otherwise fairly trivial proof using spanning trees
- I think hard because A is an irreducible matrix for connected graphs and little is known, apart from Perron-Frobenius theorem, about spectrum of irreducible matrices.

Unproven conjecture 5 - a weaker conjecture for s^+ and s^-

- Elphick and Linz (2024), conjectured that for *all* graphs

$$\min(s^+, s^-) \geq \max(n^+, n^-).$$

- This conjecture is weaker than Unproven Conjecture 4 and there is no counter-example in Wolfram Mathematica database of graphs with $n \leq 100$
- Let $E(G) = \sum_{i=1}^{n} |\mu_i|$ denote the energy of G. Simple, using Cauchy-Schwarz, that $s^+ \geq E^2/4n^+$ and $s^- \geq E^2/4n^-$.
- Aouchiche and Hansen conjectured in 2010 that:

$$\frac{E(G)}{2} = \sum_{\mu_i > 0} \mu_i = \sum_{\mu_i < 0} |\mu_i| \geq \max(n^+, n^-). \qquad (1)$$

- Akbari *et al* (2009) proved that $E(G) \geq n^+ + n^-$ using AM-GM inequality and that $\prod_{\mu_i \neq 0} |\mu_i| \geq 1$.
- However $\prod_{\mu_i > 0} \mu_i \not\geq 1$ and $\prod_{\mu_i < 0} |\mu_i| \not\geq 1$, so need a new approach to prove (1).

A final thought

- In 1940, Coulson proved that

$$E(G) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left[n - x \frac{d}{dx} \ln \phi(G; ix) \right] dx \qquad (2)$$

  where $\phi(G; x) = \det(xI_n - A(G))$ is the characteristic polynomial of G and $i = \sqrt{-1}$.
- The book *Graph Energy* by Li, Shi and Gutman provides a fairly straightforward proof of (2).
- This formula can be used to calculate graph energy without knowing a graph's eigenvalues.
- Could this approach help to prove $E(G) \geq 2 \max(n^+, n^-)$?
- Are there Coulson-type equalities for square energies?
- Could such equalities help to prove Unproven Conjectures 4 or 5?

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