A Primal-Dual Extension of Goemans and Williamson Algorithm to the Fractional Cut Covering Problem

N. Benedetto Proença [1] M. K. de Carli Silva[2] C. M. Sato[3] L. Tunçel[1]

[1]Department of Combinatorics and Optimization
University of Waterloo

[2]University of São Paulo

[3]Center of Mathematics, Computing and Cognition
Federal University of ABC Region

AGT Seminar 24 April 2023

Laplacian

We consider the Laplacian of a graph G as a *linear transformation*: $\mathcal{L}_G \colon \mathbb{R}^E \to \mathbb{S}^V$

$$\mathcal{L}_G(w) := \sum_{ij \in E} w_{ij}(e_i - e_j)(e_i - e_j)^\mathsf{T} \in \mathbb{S}^V$$

It thus has an *adjoint* $\mathcal{L}_G^* \colon \mathbb{S}^V \to \mathbb{R}^E$

$$\mathcal{L}_G^*(Z)_{ij} := Z_{ii} + Z_{jj} - 2Z_{ij} \text{ for every } ij \in E$$

The usual *Laplacian matrix* is $\mathcal{L}_G(\mathbb{1})$

Gram Matrices

Let $Y \in \mathbb{S}_+^V$.
Then $Y_{ij} = v_i^\mathsf{T} v_j$ for $v \colon V \to \mathbb{R}^V$. Hence

$$\begin{aligned} \mathcal{L}_G^*(Y)_{ij} &= Y_{ii} + Y_{jj} - 2Y_{ij} \\ &= v_i^\mathsf{T} v_i + v_i^\mathsf{T} v_i - 2 v_i^\mathsf{T} v_j \\ &= \|v_i - v_j\|^2 \end{aligned}$$

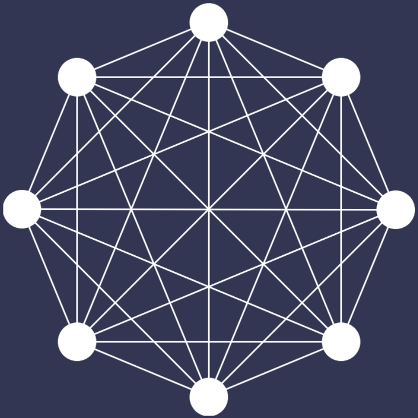
Gram Matrices

Let $Y \in \mathbb{S}_+^V$.
Then $Y_{ij} = v_i^\mathsf{T} v_j$ for $v \colon V \to \mathbb{R}^V$. Hence

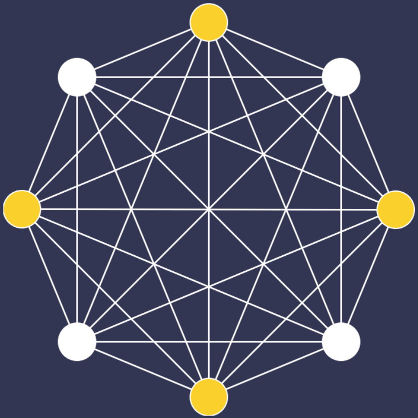
$$\begin{aligned}\mathcal{L}_G^*(Y)_{ij} &= Y_{ii} + Y_{jj} - 2Y_{ij} \\ &= v_i^\mathsf{T} v_i + v_i^\mathsf{T} v_i - 2v_i^\mathsf{T} v_j \\ &= \|v_i - v_j\|^2\end{aligned}$$

- Read "$Y_{ij} \leq \gamma$" as $v_i^\mathsf{T} v_j \leq \gamma$,
- Read "$\mathcal{L}_G^*(Y) \geq z$" as $\|v_i - v_j\|^2 \geq z_{ij}$ for every $ij \in E$

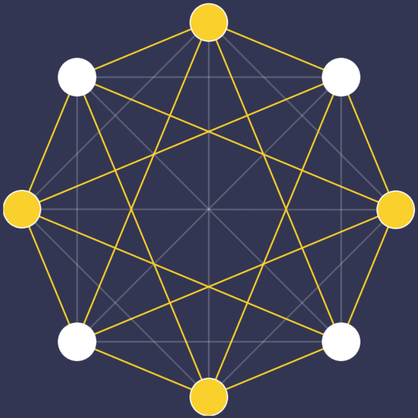
Cuts



Cuts



Cuts



Cut Covers

Figure: $\{0, 2, 4, 6\}$

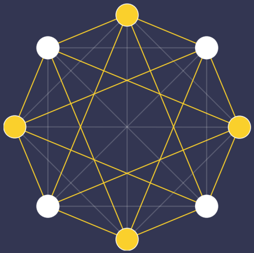


Figure: $\{2, 3, 6, 7\}$

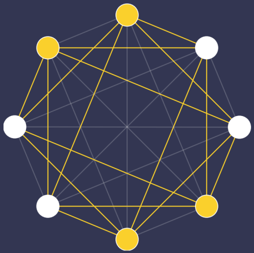
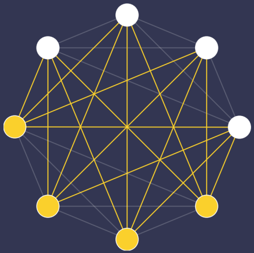


Figure: $\{4, 5, 6, 7\}$



Cut Covers

Figure: $\{0, 2, 4, 6\}$

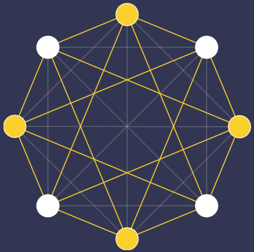


Figure: $\{2, 3, 6, 7\}$

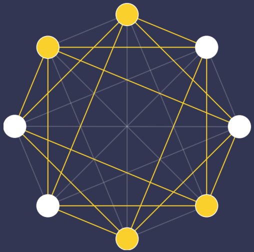
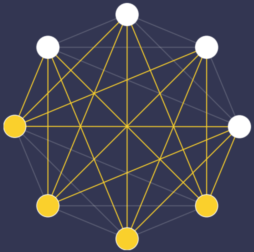


Figure: $\{4, 5, 6, 7\}$



$\mathrm{cc}(K_8) = 3,$

Cut Covers

Figure: $\{0, 2, 4, 6\}$

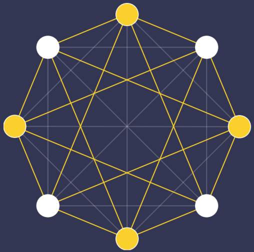


Figure: $\{2, 3, 6, 7\}$

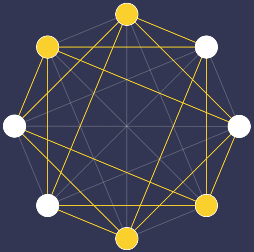
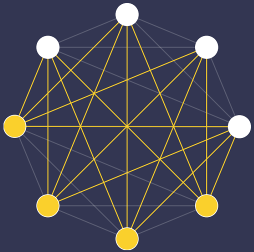


Figure: $\{4, 5, 6, 7\}$



$$cc(K_8) = 3, \ cc(G) = \lceil \lg \chi(G) \rceil$$

Fractional Cut-Covering Problem

Figure: $\{0, 2, 4, 6\}$

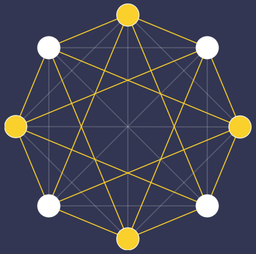


Figure: $\{2, 3, 6, 7\}$

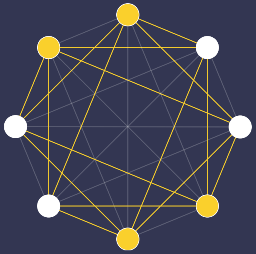
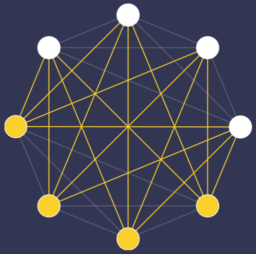


Figure: $\{4, 5, 6, 7\}$



Fractional Cut-Covering Problem

Figure: $\{0, 2, 4, 6\}$

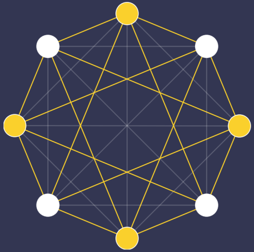


Figure: $\{2, 3, 6, 7\}$

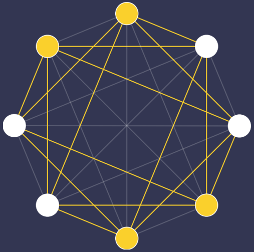
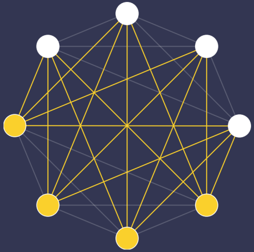


Figure: $\{4, 5, 6, 7\}$



$$\mathcal{S} := \{\, S \subseteq V : |S| = 4,\ 0 \in S \}$$

$$\frac{7}{4} = \frac{1}{2}\binom{8}{4}\binom{6}{3}^{-1} = \frac{|\mathcal{S}|}{|\{\, S \in \mathcal{S} : ij \in \delta(S)\}|}$$

Fractional Cut-Covering Problem

Figure: $\{0, 2, 4, 6\}$

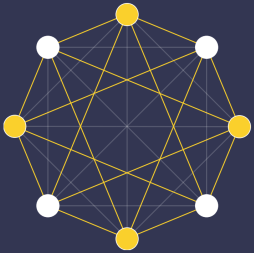


Figure: $\{2, 3, 6, 7\}$

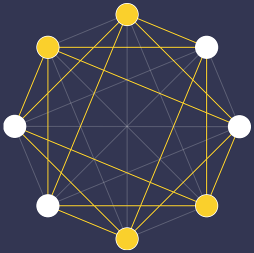
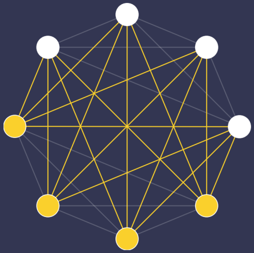


Figure: $\{4, 5, 6, 7\}$



$$\mathcal{S} := \{\, S \subseteq V : |S| = 4,\ 0 \in S \}$$

$$\frac{7}{4} = \frac{1}{2}\binom{8}{4}\binom{6}{3}^{-1} = \frac{|\mathcal{S}|}{|\{\, S \in \mathcal{S} : ij \in \delta(S)\}|} \geq \mathrm{fcc}(K_8)$$

The Weighted Fractional Cut-Covering Problem

$$\mathrm{fcc}(G) := \min\left\{ \mathbb{1}^\mathsf{T} y : y \in \mathbb{R}_+^{\mathcal{P}(V)}, \sum_{S \subseteq V} y_S \mathbb{1}_{\delta(S)} \geq \mathbb{1} \right\}$$

The Weighted Fractional Cut-Covering Problem

$$\mathrm{fcc}(G, z) := \min\left\{ \mathbb{1}^{\mathsf{T}} y : y \in \mathbb{R}_{+}^{\mathcal{P}(V)}, \sum_{S \subseteq V} y_S \mathbb{1}_{\delta(S)} \geq z \right\}$$

The Weighted Fractional Cut-Covering Problem

$$\operatorname{fcc}(G,z) \coloneqq \min\Big\{ \mathbb{1}^{\mathsf{T}} y : y \in \mathbb{R}_{+}^{\mathcal{P}(V)},\ \sum_{S \subseteq V} y_S \mathbb{1}_{\delta(S)} \geq z \Big\}$$

$$= \max\{ z^{\mathsf{T}} x : x \in \mathbb{R}_{+}^{E},\ \forall S \subseteq V,\ \mathbb{1}_{\delta(S)}^{\mathsf{T}} x \leq 1 \}$$

The Weighted Fractional Cut-Covering Problem

$$\operatorname{fcc}(G, z) := \min\left\{ \mathbb{1}^{\mathsf{T}} y : y \in \mathbb{R}_{+}^{\mathcal{P}(V)},\ \sum_{S \subseteq V} y_S \mathbb{1}_{\delta(S)} \geq z \right\}$$

$$= \max\{ z^{\mathsf{T}} x : x \in \mathbb{R}_{+}^{E},\ \operatorname{mc}(G, x) \leq 1 \}$$

Question

Can we *dualize* the celebrated approximation algorithm by Goemans and Williamson [1]?

Question

Can we *dualize* the celebrated approximation algorithm by Goemans and Williamson [1]?

$$\widetilde{\mathrm{mc}}(G, w) := \max\{\, \langle \tfrac{1}{4}\mathcal{L}_G(w), Y \rangle : \, Y \in \mathbb{S}^V_+, \, \mathrm{diag}(Y) = \mathbb{1} \}$$

$$\alpha_{\mathrm{GW}} \widetilde{\mathrm{mc}}(G, w) \le \mathrm{mc}(G, w) \le \widetilde{\mathrm{mc}}(G, w)$$

$$\alpha_{\mathrm{GW}} \approx 0.878$$

- $\mathrm{GW}(Y)$ samples subset of V for $Y \in \mathbb{S}^V_+$ with $\mathrm{diag}(Y) = \mathbb{1}$
- How to find the correct PSD matrix to sample from?

Previous Works

$$\chi_{\mathrm{vec}}(G) = \min\left\{\, 1 - \tfrac{1}{\gamma} : Y \in \mathbb{S}_+^V,\, \mathrm{diag}(Y) = \mathbb{1},\, \forall ij \in E,\; Y_{ij} \leq \gamma \right\}$$

Šámal [3], and Neto and Ben-Ameur [2] show that

$$2\left(1 - \frac{1}{\chi_{\mathrm{vec}}(G)}\right) \leq \mathrm{fcc}(G) \leq \frac{1}{\alpha_{\mathrm{GW}}} 2\left(1 - \frac{1}{\chi_{\mathrm{vec}}(G)}\right)$$

Previous Works

$$\chi_{\mathsf{vec}}(G) = \min\left\{\, 1 - \tfrac{1}{\gamma} : Y \in \mathbb{S}_+^V,\ \mathrm{diag}(Y) = \mathbb{1},\ \forall ij \in E,\ Y_{ij} \le \gamma \right\}$$

Šámal [3], and Neto and Ben-Ameur [2] show that

$$2\left(1 - \frac{1}{\chi_{\mathsf{vec}}(G)}\right) \le \mathrm{fcc}(G) \le \frac{1}{\alpha_{\mathrm{GW}}} 2\left(1 - \frac{1}{\chi_{\mathsf{vec}}(G)}\right)$$

- A polynomial-time $O(1)$-approximation algorithm for the *value* of $\mathrm{fcc}(G)$

Previous Works

$$\chi_{\mathsf{vec}}(G) = \min\left\{ 1 - \tfrac{1}{\gamma} : Y \in \mathbb{S}_+^V, \operatorname{diag}(Y) = \mathbb{1}, \forall ij \in E,\ Y_{ij} \leq \gamma \right\}$$

Šámal [3], and Neto and Ben-Ameur [2] show that

$$2\left(1 - \frac{1}{\chi_{\mathsf{vec}}(G)}\right) \leq \mathsf{fcc}(G) \leq \frac{1}{\alpha_{\mathrm{GW}}} 2\left(1 - \frac{1}{\chi_{\mathsf{vec}}(G)}\right)$$

- A polynomial-time $O(1)$-approximation algorithm for the *value* of $\mathsf{fcc}(G)$
- the upper bound is obtained via repeated sampling from $\mathsf{GW}(Y)$

This Presentation

Theorem

For any $\frac{1}{5}\alpha_{\mathrm{GW}} < \beta < \alpha_{\mathrm{GW}}$*, there exists a polynomial-time randomized algorithm producing* $y := \mathcal{A}(G, z) \in \mathbb{R}_+^{\mathcal{P}(V)}$ *with* $|\mathrm{supp}(y)| = O(\ln(n))$*,*

$$\mathbb{1}^\top y \leq \frac{1}{\beta} \mathrm{fcc}(G, z),$$

and such that $\sum_{S \subseteq V} y_S \mathbb{1}_{\delta(S)} \geq z$ *with high probability*

Why Weights?

A function $f\colon E(G) \to E(H)$ is *cut continuous* if for every $T \subseteq V(H)$ there exists $S \subseteq V(G)$ such that

$$f^{-1}(\delta(T)) = \delta(S)$$

Why Weights?

A function $f\colon E(G) \to E(H)$ is *cut continuous* if for every $T \subseteq V(H)$ there exists $S \subseteq V(G)$ such that

$$f^{-1}(\delta(T)) = \delta(S)$$

$$\text{if } P_f e_{ij} = e_{f(ij)}, \text{ then } P_f^{\mathsf{T}} \mathbb{1}_{\delta(T)} = \mathbb{1}_{\delta(S)}$$

Why Weights?

A function $f\colon E(G) \to E(H)$ is *cut continuous* if for every $T \subseteq V(H)$ there exists $S \subseteq V(G)$ such that

$$f^{-1}(\delta(T)) = \delta(S)$$

$$\text{if } P_f e_{ij} = e_{f(ij)}, \text{ then } P_f^{\mathsf{T}} \mathbb{1}_{\delta(T)} = \mathbb{1}_{\delta(S)}$$

$$\text{if } x \geq 0, \text{ mc}(G, x) \leq 1 \text{ then } P_f x \geq 0, \text{ mc}(H, P_f x) \leq 1$$

Why Weights?

A function $f\colon E(G) \to E(H)$ is *cut continuous* if for every $T \subseteq V(H)$ there exists $S \subseteq V(G)$ such that

$$f^{-1}(\delta(T)) = \delta(S)$$

$$\text{if } P_f e_{ij} = e_{f(ij)}, \text{ then } P_f^{\mathsf{T}} \mathbb{1}_{\delta(T)} = \mathbb{1}_{\delta(S)}$$

$$\text{if } x \geq 0,\, \mathrm{mc}(G,x) \leq 1 \text{ then } P_f x \geq 0,\, \mathrm{mc}(H, P_f x) \leq 1$$

$$\begin{aligned} \mathrm{fcc}(H,z) &= \max\{\, z^{\mathsf{T}} u : u \geq 0,\, \mathrm{mc}(H,u) \leq 1\} \\ &\geq \max\{\, z^{\mathsf{T}} P_f x, : x \geq 0,\, \mathrm{mc}(G,x) \leq 1\} \\ &= \mathrm{fcc}(G, P_f^{\mathsf{T}} z) \end{aligned}$$

Outline

1. Fractional Cut-Covering
2. An SDP Relaxation
3. Rounding and Sparsifying Optimal Solutions
4. The Algorithm
5. Discussion

Outline

The SDP

Recall that

$$\mathrm{fcc}(G, z) \quad = \max\{\, z^{\mathsf{T}} w : w \in \mathbb{R}^{E}_{+},\ \mathrm{mc}(G, w) \le 1\}$$

The SDP

Recall that

$$\begin{aligned}\operatorname{fcc}(G,z) &= \max\{\, z^{\mathsf{T}} w : w \in \mathbb{R}^{E}_{+},\ \operatorname{mc}(G,w) \le 1\} \\ &\ge \max\{\, z^{\mathsf{T}} w : w \in \mathbb{R}^{E}_{+},\ \widetilde{\operatorname{mc}}(G,w) \le 1\} \\ &=: \widetilde{\operatorname{fcc}}(G,z)\end{aligned}$$

The SDP

Recall that

$$\begin{aligned} \mathsf{fcc}(G,z) &= \max\{\, z^{\mathsf{T}} w : w \in \mathbb{R}^{E}_{+},\ \mathsf{mc}(G,w) \le 1\} \\ &\ge \max\{\, z^{\mathsf{T}} w : w \in \mathbb{R}^{E}_{+},\ \widetilde{\mathsf{mc}}(G,w) \le 1\} \\ &=: \widetilde{\mathsf{fcc}}(G,z) \end{aligned}$$

Using that

$$\alpha_{\mathrm{GW}} \widetilde{\mathsf{mc}}(G,w) \le \mathsf{mc}(G,w) \le \widetilde{\mathsf{mc}}(G,w)$$

we get that

$$\widetilde{\mathsf{fcc}}(G,z) \le \mathsf{fcc}(G,z) \le \tfrac{1}{\alpha_{\mathrm{GW}}} \widetilde{\mathsf{fcc}}(G,z)$$

Duality and Duality

Note

$$\widetilde{\mathrm{mc}}(G, w) = \min\{\, \mathbb{1}^\mathsf{T} x : x \in \mathbb{R}^V,\ \tfrac{1}{4}\mathcal{L}_G(w) \preceq \mathrm{Diag}(x)\}$$

Duality and Duality

Note

$$\widetilde{\mathrm{mc}}(G, w) = \min\{\, \mathbb{1}^\mathsf{T} x : x \in \mathbb{R}^V,\ \tfrac{1}{4}\mathcal{L}_G(w) \preceq \mathrm{Diag}(x)\}$$

$$\widetilde{\mathrm{fcc}}(G, z) \quad := \max\{\, z^\mathsf{T} w : w \in \mathbb{R}^E_+,\ \widetilde{\mathrm{mc}}(G, w) \leq 1\}$$

Duality and Duality

Note

$$\widetilde{\mathrm{mc}}(G,w) = \min\{\, \mathbb{1}^\mathsf{T} x : x \in \mathbb{R}^V,\ \tfrac{1}{4}\mathcal{L}_G(w) \preceq \mathrm{Diag}(x)\}$$

$$\begin{aligned}\widetilde{\mathrm{fcc}}(G,z) &\coloneqq \max\{\, z^\mathsf{T} w : w \in \mathbb{R}^E_+,\ \widetilde{\mathrm{mc}}(G,w) \leq 1\} \\ &= \max\left\{\, z^\mathsf{T} w : \begin{array}{c} w \in \mathbb{R}^E_+,\ x \in \mathbb{R}^V \\ \mathbb{1}^\mathsf{T} x \leq 1,\ \tfrac{1}{4}\mathcal{L}_G(w) \preceq \mathrm{Diag}(x) \end{array} \right\}\end{aligned}$$

Duality and Duality

Note

$$\widetilde{\mathrm{mc}}(G,w) = \min\{\, \mathbb{1}^\mathsf{T} x : x \in \mathbb{R}^V,\ \tfrac{1}{4}\mathcal{L}_G(w) \preceq \mathrm{Diag}(x)\}$$

$$\begin{aligned}
\widetilde{\mathrm{fcc}}(G,z) \quad &:= \max\{\, z^\mathsf{T} w : w \in \mathbb{R}^E_+,\ \widetilde{\mathrm{mc}}(G,w) \le 1\} \\
&= \max\left\{ z^\mathsf{T} w : \begin{array}{c} w \in \mathbb{R}^E_+,\ x \in \mathbb{R}^V \\ \mathbb{1}^\mathsf{T} x \le 1,\ \tfrac{1}{4}\mathcal{L}_G(w) \preceq \mathrm{Diag}(x) \end{array} \right\} \\
&= \min\left\{ \mu : \begin{array}{c} \mu \in \mathbb{R}_+,\ Y \in \mathbb{S}^V_+ \\ \mathrm{diag}(Y) = \mu\mathbb{1},\ \tfrac{1}{4}\mathcal{L}^*_G(Y) \ge z \end{array} \right\}
\end{aligned}$$

Duality and Duality

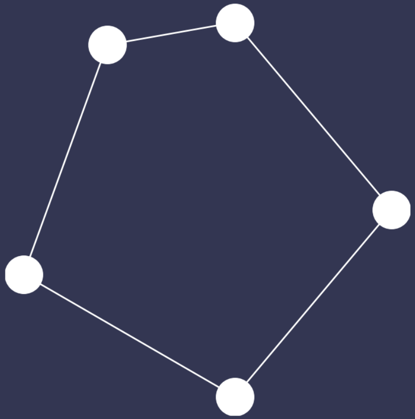
Note

$$\widetilde{\mathrm{mc}}(G,w) = \min\{\, \mathbb{1}^{\mathsf{T}}x : x \in \mathbb{R}^{V},\ \tfrac{1}{4}\mathcal{L}_G(w) \preceq \mathrm{Diag}(x)\}$$

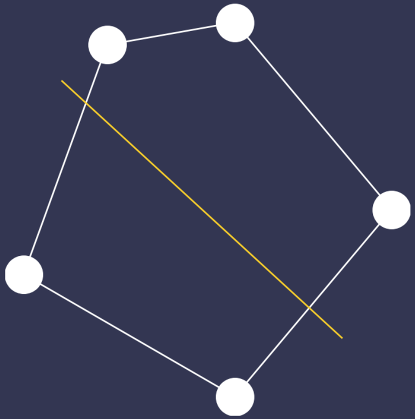
$$\begin{aligned}
\widetilde{\mathrm{fcc}}(G,z) &:= \max\{\, z^{\mathsf{T}}w : w \in \mathbb{R}^{E}_{+},\ \widetilde{\mathrm{mc}}(G,w) \le 1\} \\
&= \max\left\{ z^{\mathsf{T}}w : \begin{array}{c} w \in \mathbb{R}^{E}_{+},\ x \in \mathbb{R}^{V} \\ \mathbb{1}^{\mathsf{T}}x \le 1,\ \tfrac{1}{4}\mathcal{L}_G(w) \preceq \mathrm{Diag}(x) \end{array} \right\} \\
&= \min\left\{ \mu : \begin{array}{c} \mu \in \mathbb{R}_{+},\ Y \in \mathbb{S}^{V}_{+} \\ \mathrm{Diag}(Y) = \mu\mathbb{1},\ \tfrac{1}{4}\mathcal{L}^{*}_{G}(Y) \ge z \end{array} \right\}
\end{aligned}$$

Outline

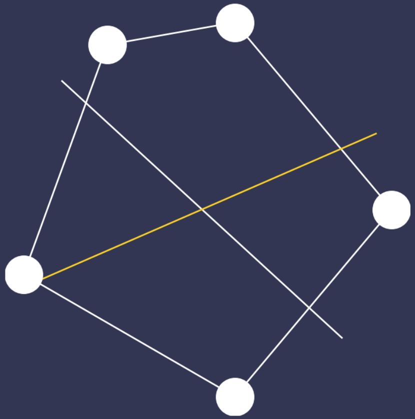
Simulating the Algorithm



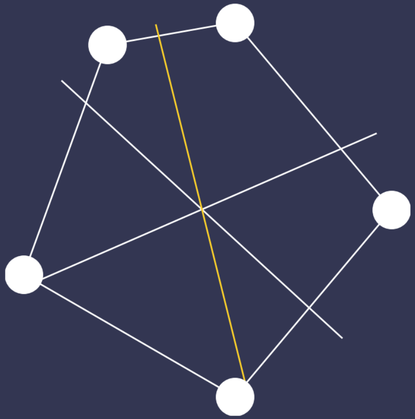
Simulating the Algorithm



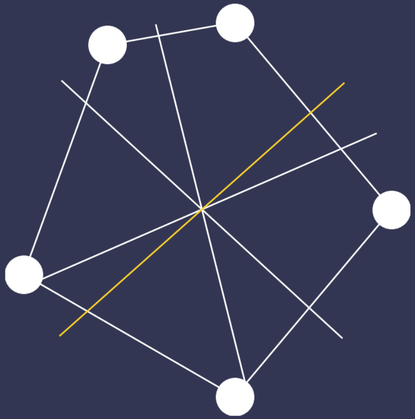
Simulating the Algorithm



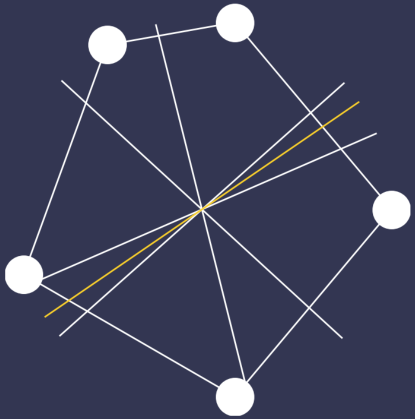
Simulating the Algorithm



Simulating the Algorithm



Simulating the Algorithm

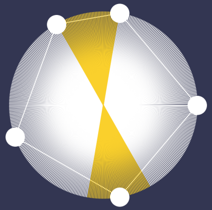


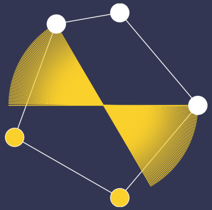
Rounding Solutions

$$\widetilde{\mathrm{fcc}}(G,z) = \min\left\{ \mu : \begin{array}{c} \mu \in \mathbb{R}_+,\ Y \in \mathbb{S}^V_+ \\ \mathrm{Diag}(Y) = \mu \mathbb{1},\ \frac{1}{4}\mathcal{L}^*_G(Y) \geq z \end{array} \right\}$$

Rounding Solutions

$$\widetilde{\mathrm{fcc}}(G,z) = \min\left\{ \mu : \begin{array}{c} \mu \in \mathbb{R}_+,\ Y \in \mathbb{S}_+^V \\ \mathrm{Diag}(Y) = \mu\mathbb{1},\ \frac{1}{4}\mathcal{L}_G^*(Y) \geq z \end{array} \right\}$$





Rounding Solutions

$$\widetilde{\text{fcc}}(G, z) = \min\left\{ \mu : \begin{array}{c} \mu \in \mathbb{R}_+, \; Y \in \mathbb{S}^V_+ \\ \text{Diag}(Y) = \mu \mathbb{1}, \; \frac{1}{4}\mathcal{L}^*_G(Y) \geq z \end{array} \right\}$$

$$y_S \propto \text{Prob}\big(\text{GW}(\mu^{-1} Y) = S\big) \text{ covers } z \text{ and } \mathbb{1}^\mathsf{T} y \leq \frac{1}{\alpha_{\text{GW}}} \mu$$

Rounding Solutions

$$\widetilde{\mathrm{fcc}}(G, z) = \min\left\{ \mu : \begin{array}{c} \mu \in \mathbb{R}_+, \; Y \in \mathbb{S}_+^V \\ \mathrm{Diag}(Y) = \mu \mathbb{1}, \; \frac{1}{4}\mathcal{L}_G^*(Y) \geq z \end{array} \right\}$$

$$y_S \propto \mathrm{Prob}\big(\mathrm{GW}(\mu^{-1} Y) = S\big) \text{ covers } z \text{ and } \mathbb{1}^\mathsf{T} y \leq \frac{1}{\alpha_{\mathrm{GW}}}\mu$$

Refines idea from Neto and Ben-Ameur [2]

Outline

The Algorithm

Fix $\beta \in (\frac{1}{5}\alpha_{\mathrm{GW}}, \alpha_{\mathrm{GW}})$

Set

$$\overline{\sigma} := \frac{1}{2}\frac{\alpha_{\mathrm{GW}} - \beta}{\beta}$$

$$\overline{\varepsilon} := \frac{1}{5}\left(\frac{\alpha_{\mathrm{GW}}}{\beta} - (1 + \overline{\sigma})\right)$$

$$C := \frac{75\sqrt{60}\pi}{4}\frac{\alpha_{\mathrm{GW}}^2\sqrt{\beta}}{(\alpha_{\mathrm{GW}} - \beta)^{5/2}}$$

The Algorithm

Fix $\beta \in (\frac{1}{5}\alpha_{\mathrm{GW}}, \alpha_{\mathrm{GW}})$

Set $\overline{\sigma} := \overline{\sigma}(\beta)$

$\overline{\varepsilon} := \overline{\varepsilon}(\beta)$

$C := C(\beta)$

The Algorithm

procedure ApproxFcc(G, z)
  $z_e \leftarrow \max(z_e, \frac{1}{2}\overline{\varepsilon}\|z\|_\infty)$ **for each** $e \in E$
  $Y \leftarrow$ Approx-fccSDP-Solve$(G, z, \overline{\sigma}\|z\|_\infty)$
  $y \leftarrow 0 \in \mathbb{R}_+^{\mathcal{P}(V)},\ \hat{z} \leftarrow 0 \in \mathbb{R}_+^E$
  repeat $\lceil C \ln(|V|) \rceil$ **times**
    $S \leftarrow \mathrm{GW}(Y)$
    $y_S \leftarrow y_S + 1,\ \hat{z} \leftarrow \hat{z} + \mathbb{1}_{\delta(S)}$
  end
  $\gamma \leftarrow \max\{ z_e/\hat{z}_e : e \in E,\ \hat{z}_e \neq 0\}$
  return γy
end procedure

The Algorithm

procedure ApproxFcc(G, z)
  $z_e \leftarrow \max(z_e, \frac{1}{2}\overline{\varepsilon}\|z\|_\infty)$ **for each** $e \in E$
  $Y \leftarrow$ Approx-fccSDP-Solve$(G, z, \overline{\sigma}\|z\|_\infty)$
  $y \leftarrow 0 \in \mathbb{R}_+^{\mathcal{P}(V)},\ \hat{z} \leftarrow 0 \in \mathbb{R}_+^E$
  repeat $\lceil C \ln(|V|) \rceil$ **times**
    $S \leftarrow \mathrm{GW}(Y)$
    $y_S \leftarrow y_S + 1,\ \hat{z} \leftarrow \hat{z} + \mathbb{1}_{\delta(S)}$
  end
  $\gamma \leftarrow \max\{ z_e/\hat{z}_e : e \in E,\ \hat{z}_e \neq 0\}$
  return γy
end procedure

Outline

Eigenvectors

$$\widetilde{\mathrm{mc}}(G,w) = \min\left\{ \tfrac{1}{n}\lambda_{\max}\left(\tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u)\right) : u \in \mathbb{R}^V,\ u^\mathsf{T}\mathbb{1} = 0 \right\}$$

Given $z \in \mathbb{R}^E_+$, we compute

$$Y \in \mathbb{S}^V_+,\ \mathrm{diag}(Y) = \mu\mathbb{1}, \tfrac{1}{4}\mathcal{L}^*_G(Y) \geq z$$
$$w \geq 0,\ u^\mathsf{T}\mathbb{1} = 0,\ \tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u) \preceq \tfrac{1}{n}I$$

Eigenvectors

$$\widetilde{\mathrm{mc}}(G, w) = \min\left\{ \tfrac{1}{n}\lambda_{\max}\big(\tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u)\big) : u \in \mathbb{R}^V,\ u^\mathsf{T}\mathbb{1} = 0 \right\}$$

Given $z \in \mathbb{R}^E_+$, we compute

$$Y \in \mathbb{S}^V_+,\ \mathrm{diag}(Y) = \mu\mathbb{1}, \tfrac{1}{4}\mathcal{L}^*_G(Y) \geq z$$
$$w \geq 0,\ u^\mathsf{T}\mathbb{1} = 0,\ \tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u) \preceq \tfrac{1}{n}I$$

Eigenvectors

$$\widetilde{\mathrm{mc}}(G, w) = \min\left\{ \tfrac{1}{n}\lambda_{\max}\left(\tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u)\right) : u \in \mathbb{R}^V,\ u^\mathsf{T}\mathbb{1} = 0 \right\}$$

Given $z \in \mathbb{R}^E_+$, we compute

$$Y \in \mathbb{S}^V_+,\ \mathrm{diag}(Y) = \mu\mathbb{1},\ \tfrac{1}{4}\mathcal{L}^*_G(Y) \geq z$$
$$w \geq 0,\ u^\mathsf{T}\mathbb{1} = 0,\ \tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u) \preceq \tfrac{1}{n}I$$
$$(\tfrac{1}{n}I - (\tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u)))Y = 0$$

Eigenvectors

$$\widetilde{\mathrm{mc}}(G, w) = \min\left\{ \tfrac{1}{n}\lambda_{\max}\left(\tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u)\right) : u \in \mathbb{R}^V,\ u^\mathsf{T}\mathbb{1} = 0 \right\}$$

Given $z \in \mathbb{R}^E_+$, we compute

$$Y \in \mathbb{S}^V_+,\ \mathrm{diag}(Y) = \mu\mathbb{1}, \tfrac{1}{4}\mathcal{L}^*_G(Y) \geq z$$
$$w \geq 0,\ u^\mathsf{T}\mathbb{1} = 0,\ \tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u) \preceq \tfrac{1}{n}I$$
$$\tfrac{1}{n}Y = \left(\tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u)\right)Y$$

Eigenvectors

$$\widetilde{\mathrm{mc}}(G,w) = \min\left\{ \tfrac{1}{n}\lambda_{\max}\big(\tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u)\big) : u \in \mathbb{R}^V,\ u^{\mathsf{T}}\mathbb{1} = 0 \right\}$$

Given $z \in \mathbb{R}^E_+$, we compute

$$Y \in \mathbb{S}^V_+,\ \mathrm{diag}(Y) = \mu\mathbb{1}, \tfrac{1}{4}\mathcal{L}_G^*(Y) \geq z$$
$$w \geq 0,\ u^{\mathsf{T}}\mathbb{1} = 0,\ \tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u) \preceq \tfrac{1}{n}I$$
$$\tfrac{1}{n}Y = \big(\tfrac{1}{4}\mathcal{L}_G(w) + \mathrm{Diag}(u)\big)Y$$

- Y encodes a geometric representation of G
- Y encodes eigenvectors of a "Laplacian"

Optimization and Homomorphisms

If $f\colon E(G) \to E(H)$ is cut continuous, then

$$\mathrm{fcc}(G, P_f^{\mathsf{T}} z) \leq \mathrm{fcc}(H, z)$$

If $f\colon V(G) \to V(H)$ is a graph homomorphism, then

$$\chi_f(G, P_f^{\mathsf{T}} z) \leq \chi_f(H, z)$$

and

$$\vartheta(\overline{G}, P_f^{\mathsf{T}} z) \leq \vartheta(\overline{H}, z)$$

Conclusion

- A weighted version of fcc allows for the use of convex optimization techniques

Conclusion

- A weighted version of fcc allows for the use of convex optimization techniques
- Duality theory of convex optimization extends Goemans and Williamson's celebrated approximation algorithm to the fractional cut-covering setting

Conclusion

- A weighted version of fcc allows for the use of convex optimization techniques
- Duality theory of convex optimization extends Goemans and Williamson's celebrated approximation algorithm to the fractional cut-covering setting
- Computing either of $\{\widetilde{\mathrm{mc}}, \widetilde{\mathrm{fcc}}\}$ implicitly computes the other one

Conclusion

- A weighted version of fcc allows for the use of convex optimization techniques
- Duality theory of convex optimization extends Goemans and Williamson's celebrated approximation algorithm to the fractional cut-covering setting
- Computing either of $\{\widetilde{\mathrm{mc}}, \widetilde{\mathrm{fcc}}\}$ implicitly computes the other one
- Given either of $w, z \in \mathbb{R}^E_+$, one can compute the triplet (w, z, ρ), as well as (mostly) combinatorial certificates that

$$\alpha_{\mathrm{GW}}\rho \leq \mathrm{mc}(G, w) \leq \rho \leq \mathrm{fcc}(G, z) \leq \tfrac{1}{\alpha_{\mathrm{GW}}}\rho$$

Thank You

References I

[1] M. X. Goemans and D. P. Williamson. "Improved Approximation Algorithms for Maximum Cut and Satisfiability Problems Using Semidefinite Programming". In: *Journal of the Association for Computing Machinery* 42.6 (1995), pages 1115–1145.

[2] J. Neto and W. Ben-Ameur. "On Fractional Cut Covers". In: *Discrete Applied Mathematics. The Journal of Combinatorial Algorithms, Informatics and Computational Sciences* 265 (2019), pages 168–181.

[3] R. Šámal. "Cubical Coloring—Fractional Covering by Cuts and Semidefinite Programming". In: *Discrete Mathematics & Theoretical Computer Science. DMTCS.* 17.2 (2015), pages 251–266.