Eigenvalues for Stochastic Matrices with a Prescribed Stationary Distribution

Steve Kirkland

Department of Mathematics
University of Manitoba

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Stochastic matrices and Markov chains

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Associated with T is a *Markov chain*, i.e. a sequence of nonnegative vectors $x(k), k = 1, 2, 3, \ldots$ satisfying $x(k+1)^\top = x(k)^\top T$ and $x(k)^\top \mathbf{1} = 1, k \in \mathbb{N}$. Evidently $x(k+1)^\top = x(1)^\top T^k$, for each $k \in \mathbb{N}$.

A square entrywise nonnegative matrix M is *primitive* if, for some $k \in \mathbb{N}$, M^k has all positive entries. Equivalently, the directed graph of M is strongly connected, and the gcd of the cycle lengths is 1.

Examples: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ .25 & .75 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$

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Spectral theory for stochastic matrices

Theorem (Perron–Frobenius 1907, 1912)

Let T be an $n \times n$ primitive stochastic matrix, and denote the eigenvalues of T by $1 \equiv \lambda_1, \lambda_2, \ldots, \lambda_n$. Then a) $|\lambda_j| < 1, j = 2, \ldots, n$; b) there is a unique left eigenvector w of corresponding to the eigenvalue 1 such that $w^\top \mathbf{1} = 1$; and c) w has all positive entries.

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It now follows that $T^k \to \mathbf{1}w^\top$ as $k \to \infty$. Hence, our Markov chain $x(k)$ converges the stationary distribution w as $k \to \infty$, regardless of the initial distribution $x(1)$.

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More on eigenvalues

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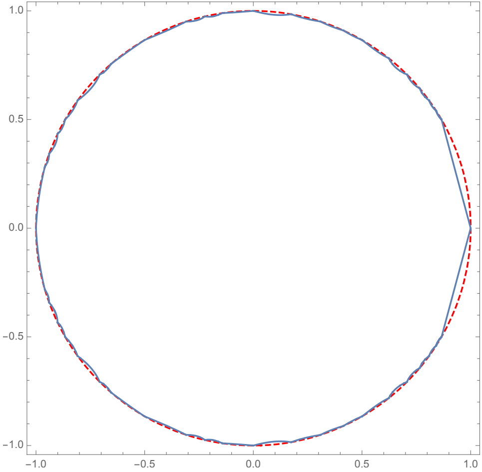
The region for $n = 12$



The stationary distribution

It turns out that the entries in the entries in the stationary distribution can be understood in terms of the sums of weights of certain spanning directed trees in the directed graph associated with T. This is the Markov chain matrix tree theorem.

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A simple example

Suppose we have $w \in \mathbb{R}^3$ with $0 < w_1 \leq w_2 \leq w_3$. A typical stochastic matrix T having $w^\top$ as the stationary distribution has the form

$$T = \begin{bmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ \frac{(1-a)w_1 - cw_2}{w_3} & \frac{(1-d)w_2 - bw_1}{w_3} & \frac{w_3+(a+b-1)w_1+(c+d-1)w_2}{w_3} \end{bmatrix},$$

where necessarily all entries are nonnegative. Observe that for such a T we have $trace(T) = a + d + \frac{w_3+(a+b-1)w_1+(c+d-1)w_2}{w_3} \geq \frac{w_3-w_1-w_2}{w_3} = \frac{2w_3-1}{w_3}$. It now follows that if λ is a non-real eigenvalue of T then $Re(\lambda) \geq \frac{w_3-1}{2w_3}$. For example, if $w_3 > \frac{1}{2}$ then any non-real eigenvalue of any stochastic matrix with $w^\top$ as a left Perron vector necessarily has real part strictly greater than $-\frac{1}{2}$.

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A Karpelevič-type region

Suppose that w is a positive vector whose entries sum to 1.

Consider the following sets:
$\mathcal{S}(w) = \{T \in M_n(\mathbb{R}) | T \geq 0, T\mathbf{1} = \mathbf{1}, w^\top T = w^\top\}$,
$\sigma_{\mathcal{S}}(w) = \{\lambda | \lambda$ is an eigenvalue of some $T \in \mathcal{S}(w)\}$.

Observe that $\mathcal{S}(w)$ includes all irreducible stochastic matrices having $w^\top$ as the stationary distribution. In fact the irreducible members of $\mathcal{S}(w)$ form a dense subset.

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Easy observations

Suppose that $w \in \mathbb{R}^n, w > 0, w^\top \mathbf{1} = 1$.

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Elements of $\sigma_{\mathcal{S}}(w)$ for any w

Theorem

Suppose that $n \geq 2$*. Then* $\bigcap_{w \in \mathbb{R}^n, w > 0, w^\top \mathbf{1} = 1} \sigma_{\mathcal{S}}(w) = [0,1]$.

The fact that $[0,1] \subseteq \sigma_{\mathcal{S}}(w)$ for any w follows from observation iii).

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Intersection of $\sigma_{\mathcal{S}}(w)$ with the unit circle

Theorem

Suppose that $n \geq 2$, that $2 \leq k \leq n$, and that $w \in \mathbb{R}^n$ with $w > 0, w^\top \mathbf{1} = 1$. We have $e^{\frac{2\pi j}{k}} \in \sigma_{\mathcal{S}}(w)$ for some $j = 1, \ldots, k-1$ that is relatively prime to k, if and only if there is a collection of non-empty disjoint subsets $S_1, \ldots, S_k \subseteq \{1, \ldots, n\}$ such that the values $\sum_{l \in S_i} w_l, i = 1, \ldots, k$, are all equal.

Corollary

Consider a vector $w \in \mathbb{R}^n$ with $w > 0, w^\top \mathbf{1} = 1$. Suppose that for any pair of non-empty disjoint subsets $S_1, S_2 \in \{1, \ldots, n\}, \sum_{j \in S_1} w_j \neq \sum_{k \in S_2} w_k$. Then for any stochastic matrix T such that $w^\top T = w^\top$, the only eigenvalue of T of unit modulus is 1. In particular, if such a T is irreducible, it is necessary primitive.

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Reversible stochastic matrices

Suppose that A is a symmetric nonnegative matrix such that $A\mathbf{1} = w$. Set $W =$diag(w). Then $T = W^{-1}A$ is stochastic, and is known as a *reversible* stochastic matrix.

The reversible stochastic matrices form an important subfamily.

Define $\mathcal{R}(w) = \{T \in M_n(\mathbb{R}) | T \geq 0, T\mathbf{1} = \mathbf{1}, w^\top T = w^\top, T \text{ is reversible}\}$ and $\sigma_{\mathcal{R}}(w) = \{\lambda | \lambda \in \sigma(T) \text{ for some } T \in \mathcal{R}(w)\}$.

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Extreme points

An extreme point of a convex set is one that can't be written as a nontrivial convex combination of other members of that set.

Theorem

Suppose that $n \geq 2$, and that $w \in \mathbb{R}^n$ with $w > 0, w^\top \mathbf{1} = 1$. There is an extreme point T of $\mathcal{R}(w)$ such that $\underline{\lambda}(w)$ is an eigenvalue of T.

Idea: Let $\lambda_{\min}(T)$ denote the smallest eigenvalue of a reversible stochastic matrix T. If $T_1, T_2 \in \mathcal{R}(w)$, then $\lambda_{\min}(tT_1 + (1-t)T_2) \geq t\lambda_{\min}(T_1) + (1-t)\lambda_{\min}(T_2)$. Then $\lambda_{\min}(\bullet)$ is a concave function, so it takes its minimum at an extreme point of $\mathcal{R}(w)$.

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More on extreme points

Brualdi has characterised the extreme points of the convex set of symmetric nonnegative matrices with prescribed positive row sum vector. That leads to the following. (For a reversible stochastic matrix T, let $G(T)$ denote its graph.)

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Suppose that $w \in \mathbb{R}^n, w > 0, w^\top \mathbf{1} = 1$. A matrix $T \in \mathcal{R}(w)$ is an extreme point of $\mathcal{R}(w)$ if and only if each connected component of $G(T)$ is either a tree or a unicyclic graph whose unique cycle has odd length (possibly a loop).

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Upshot

Theorem

Suppose that $w \in \mathbb{R}^n, w > 0, w^\top \mathbf{1} = 1$, and for each pair of non-empty disjoint subsets S_1, S_2 of $\{1, \ldots, n\}$, we have $\sum_{l \in S_1} w_l \neq \sum_{l \in S_2} w_l$. There is a $T \in \mathcal{R}(w)$ such that i) $\lambda_{\min}(T) = \underline{\lambda}(w)$ and ii) $G(T)$ is a tree with a loop.

Idea: Show that if $T \in \mathcal{R}(w)$ is a minimizer for $\lambda_{\min}$ then $G(T)$ is connected. Also show that if $G(T)$ has an odd cycle of length $2k+1$, then there is a $\tilde{T} \in \mathcal{R}(w)$ such that $\lambda_{\min}(\tilde{T}) \leq \lambda_{\min}(T)$ and $\tilde{T}$ has a cycle of length $2k-1$.

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Suppose that $w \in \mathbb{R}^n, w > 0, w^\top \mathbf{1} = 1$, *and for each pair of non-empty disjoint subsets* S_1, S_2 *of* $\{1, \ldots, n\}$, *we have* $\sum_{l \in S_1} w_l \neq \sum_{l \in S_2} w_l$. *There is a* $T \in \mathcal{R}(w)$ *such that i)* $\lambda_{\min}(T) = \underline{\lambda}(w)$ *and ii)* $G(T)$ *is a tree with a loop.*

Idea: Show that if $T \in \mathcal{R}(w)$ is a minimizer for $\lambda_{\min}$ then $G(T)$ is connected. Also show that if $G(T)$ has an odd cycle of length $2k+1$, then there is a $\tilde{T} \in \mathcal{R}(w)$ such that $\lambda_{\min}(\tilde{T}) \leq \lambda_{\min}(T)$ and $\tilde{T}$ has a cycle of length $2k-1$.

Upshot

Theorem

Suppose that $w \in \mathbb{R}^n, w > 0, w^\top \mathbf{1} = 1$, *and for each pair of non-empty disjoint subsets* S_1, S_2 *of* $\{1, \ldots, n\}$, *we have* $\sum_{l \in S_1} w_l \neq \sum_{l \in S_2} w_l$. *There is a* $T \in \mathcal{R}(w)$ *such that i)* $\lambda_{\min}(T) = \underline{\lambda}(w)$ *and ii)* $G(T)$ *is a tree with a loop.*

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A bound

Theorem

Suppose that $w \in \mathbb{R}^n, w > 0, w^\top \mathbf{1} = 1$. *Set*

$$\gamma = \frac{1}{\max_j w_j} \min \left\{ \sum_{p \in S_1} w_p - \sum_{q \in S_2} w_q \middle| S_1, S_2 \in \{1, \ldots, n\}, S_1 \cap S_2 = \emptyset, \sum_{p \in S_1} w_p \geq \sum_{q \in S_2} w_q \right\}.$$

Then $\underline{\lambda}(w) \geq -(1 - \gamma^{n-1})^{\frac{1}{n-1}}$.

Idea: Suppose T attains $\underline{\lambda}(w)$ as an eigenvalue, and $G(T)$ is a tree with a loop. The smallest positive entry in T is at least γ, and T^{n-1} has a column with all entries at least γ^{n-1}.

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$\underline{\lambda}(w)$ for $n = 2$

Theorem

Suppose that $w \in \mathbb{R}^2$ with $w > 0, w^\top \mathbf{1} = 1$, and $w_1 < w_2$. Then $\underline{\lambda}(w) = -\frac{w_1}{w_2}$.

Idea: There's just one extreme point that fits the bill:

$$\begin{bmatrix} 0 & 1 \\ \frac{w_1}{w_2} & \frac{w_2 - w_1}{w_2} \end{bmatrix}.$$

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$\underline{\lambda}(w)$ for $n = 3$, part 1

Theorem

Suppose that $w \in \mathbb{R}^3$ with $w > 0, w^\top \mathbf{1} = 1$, and $w_1 < w_2 < w_3$, and $w_3 > \frac{1}{2}$. Then

$$\underline{\lambda}(w) = \begin{cases} \frac{w_3 - 1}{w_3} & \text{if } w_1 + 3w_2 \geq 1, \\ \frac{-1}{2}\left(\frac{w_2 - w_1}{w_3} + \sqrt{\frac{(w_2 - w_1)^2}{w_3^2} + \frac{4w_1(1-2w_2)}{w_2 w_3}} \right) & \text{if } w_1 + 3w_2 < 1. \end{cases}$$

Idea: There are two candidate matrices to consider:

$$T_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{w_1}{w_3} & \frac{w_2}{w_3} & \frac{2w_3 - 1}{w_3} \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{w_1}{w_2} & 0 & \frac{w_2 - w_1}{w_2} \\ 0 & \frac{w_2 - w_1}{w_3} & \frac{w_1 + w_3 - w_2}{w_3} \end{bmatrix}.$$

Now determine when $\lambda_{\min}(T_1)$ is $<, =, >$ $\lambda_{\min}(T_2)$.

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$\underline{\lambda}(w)$ for $n = 3$, part 2

Theorem

Suppose that $w \in \mathbb{R}^3$ *with* $w > 0, w^\top \mathbf{1} = 1$, *and* $w_1 < w_2 < w_3 < \frac{1}{2}$. *Let*

$$x_1^- = \frac{-1}{2}\left(\frac{w_2 - w_1}{w_3} + \sqrt{\frac{(w_2 - w_1)^2}{w_3^2} + \frac{4w_1(1 - 2w_2)}{w_2 w_3}}\right),$$

$$x_2^- = \frac{-1}{2}\left(\frac{w_3 - w_2}{w_1} + \sqrt{\frac{(w_3 - w_2)^2}{w_1^2} + \frac{4w_2(1 - 2w_3)}{w_1 w_3}}\right).$$

Then $\underline{\lambda}(w) = \min\{x_1^-, x_2^-\}$.

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There are three candidate matrices:

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T_3 is not a contender, as $\lambda_{\min}(T_2) \leq \lambda_{\min}(T_3)$. Now choose the smaller of $\lambda_{\min}(T_1), \lambda_{\min}(T_2)$.

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In fact . . .

For each $w_2 \in (\frac{1}{4}, \frac{1}{2})$ with $w_2 \neq \frac{1}{3}$, there is a unique $w_1^* \in (\frac{1}{2} - w_2, w_2)$ such that

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It turns out that for fixed w_2, w_1^* is a root of an unpleasant quartic whose coefficients are polynomials in w_2.

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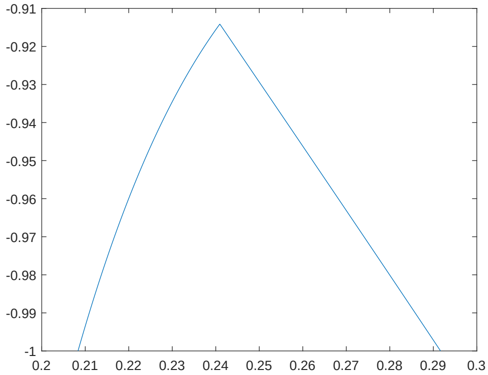
Plot of $\underline{\lambda}(w)$ for $w_2 = \frac{7}{24}, w_1 \in [\frac{1}{2} - w_2, w_2]$



Final thoughts

The case $w = \frac{1}{n}\mathbf{1}$ is the subject of the Perfect-Mirsky conjecture, open since 1965.

Considering $G(T)$ is a great help in dealing with the reversible variant of the problem.

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