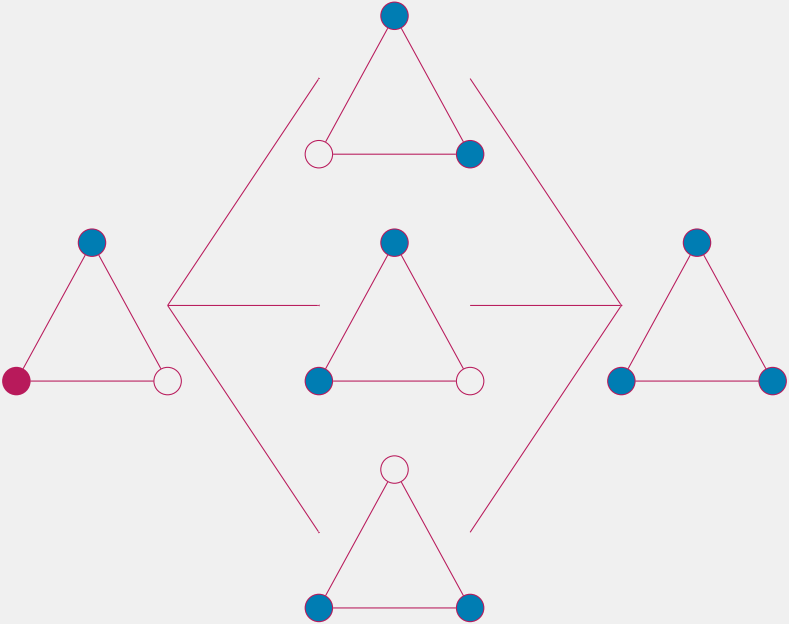
Quotient graphs & stochastic matrices

Frederico Cançado, Gabriel Coutinho



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- Recall: Birkhoff's Theorem.
- Weaker than isomorphism: for X, Y k-regular

$$A_X J = kJ = JA_Y.$$

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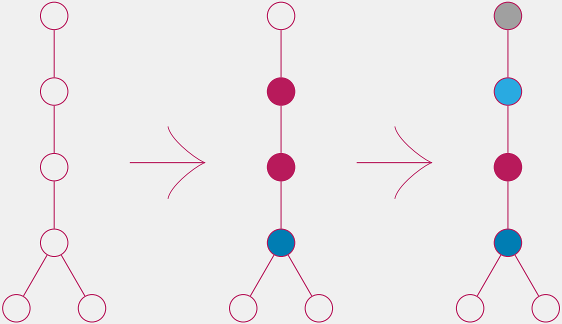

Ramana, Schreinerman and Ullman (adapted)

For two graphs X and Y on the same number of vertices, the following are equivalent:

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- X and Y have in common the coarsest equitable partition.

Equitable partitions

- We say that a partition π is equitable if for every pair of cells r, s we have

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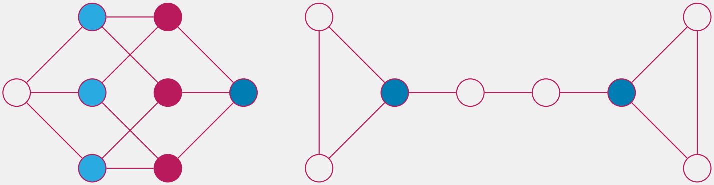
- We observe that a partition is invariant in $1WL$ iff it is equitable.



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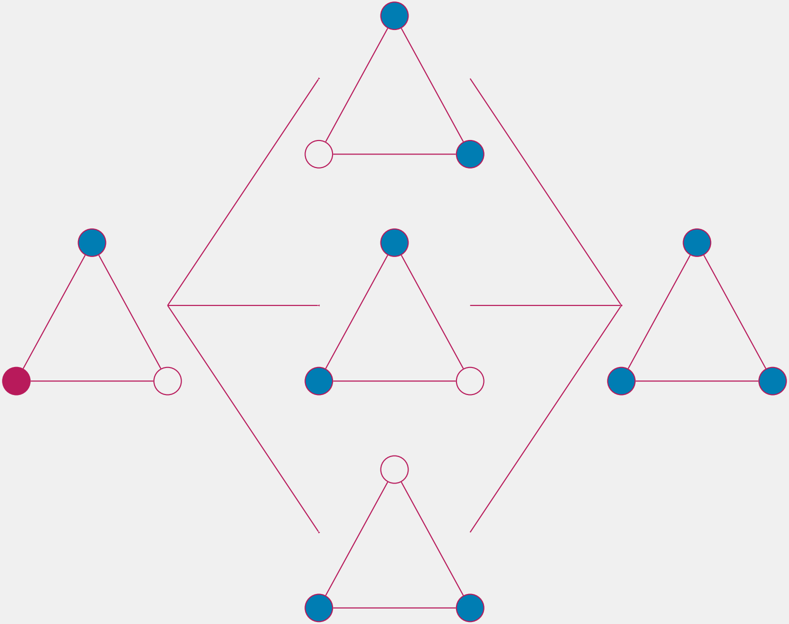
- The equitable partitions forms a lattice with respect to this ordering.



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- We can define a new weighted digraph called the unsymmetrized quotient defined on the partition classes by:

$$w(rs) = N(a, C_s), \qquad \text{for some } a \in C_r.$$

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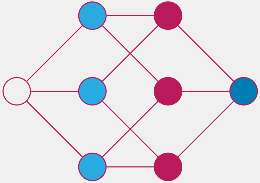

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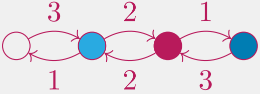

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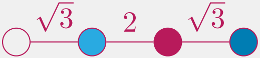

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Given two graphs X, Y, the following are equivalent:

- There are equitable partitions such that $X/\pi \simeq Y/\nu$;
- $X/\pi \simeq Y/\nu$ for their coarsest equitable partitions;
- X and Y have a common cover;
- There is a matrix M with constant row sum and constant column sum such that $A_X M = M A_Y$. - New!

Quotients and lattice - New!

There is an order preserving lattice isomorphism

$$\varphi\colon \{\pi \in \Pi(X)\colon \nu \leq \pi\} \to \Pi(X/\nu).$$

for which is also valid:

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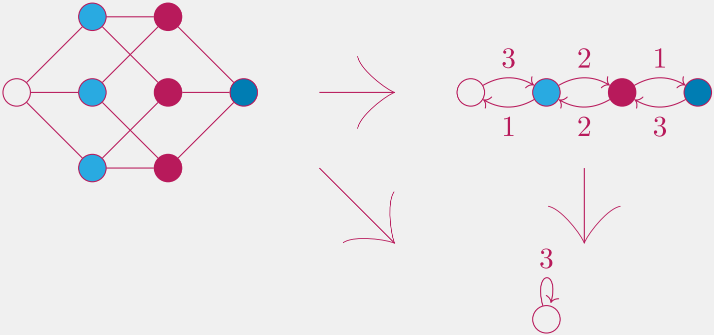

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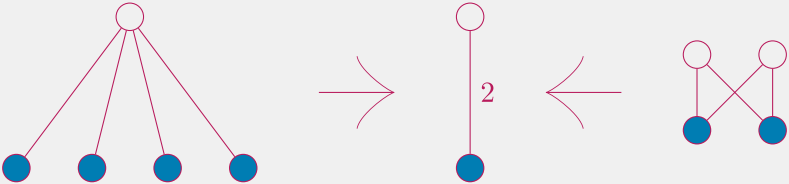

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- $\widetilde{X/\pi} \simeq \widetilde{Y/\nu}$ for some π, ν;
- there is M such that $A_X M = M A_Y$ and both MM^T and $M^T M$ are doubly stochastic.

Conclusion

Quotien	Combinatorics	Matricial
$X/\pi \simeq Y/\nu$ $\lvert V(X)\rvert = \lvert V(Y)\rvert$	1-WL	$A_X M = M A_Y$ M doubly-stochastic
$X/\pi \simeq Y/\nu$	common cover	$A_X M = M A_Y$ M line, column regular
$\widetilde{X/\pi} \simeq \widetilde{Y/\nu}$	???	$A_X M = M A_Y$ $M M^T$, $M^T M$ stochastic

Thank you!

Beamer theme by Jose Manoel Calderon Trilla