The spectra of lift of digraphs and factored lifts of graphs

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joint work with

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Outline

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Introduction

Notation:

- Let Γ be a (di)graph with vertex set $V = V(\Gamma)$ and arc set $E = E(\Gamma)$. Given a (finite) group G with generating set S, a **voltage assignment** of Γ is a mapping $\alpha : E \to S$.

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- The pair (Γ, α) (or Γ for short) is a **voltage digraph**.
- The **lifted digraph** or **lift** Γ^α is the digraph with vertex set $V(\Gamma^\alpha) = V \times G$ and arc set $E(\Gamma^\alpha) = E \times G$, where there is an arc from vertex (u, g) to vertex (v, h) if and only if $uv \in E$ and $h = g\alpha(uv)$.

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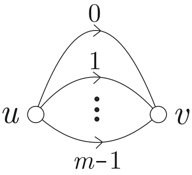
- **Example.** $(\boldsymbol{B}(z))_{uv} = 1 + z + \cdots + z^{m-1}$.



Introduction: Example: The Alegre digraph

$$\boldsymbol{B}(z) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & z^4 & z^4 & 0 & 0 \\ 0 & 0 & 0 & z+z^4 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ z & 0 & 0 & 0 & 1 \end{pmatrix}$$

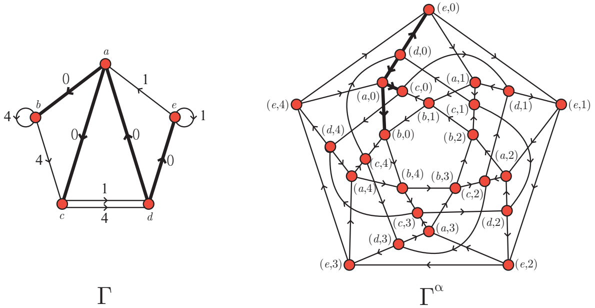


Figure: The base digraph on the group $\mathbb{Z}_5$ (right) and the Alegre digraph as a lift (left).

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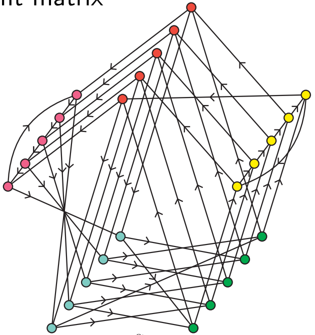
- **Example: The Alegre digraph.** It has a regular partition with five sets each of five vertices, with quotient matrix

$$\boldsymbol{B}(1) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Γ^{α}

Outline

The spectrum of lift digraphs for cyclic groups

- **Lemma (Godsil, 1993).** Let Γ be a base digraph with a given voltage assignment α on the **cyclic group** G. Let $\boldsymbol{B}$ be the polynomial matrix of its lift Γ^{α}. Then,

$$\mathrm{sp}\boldsymbol{B}(1) \subset \mathrm{sp}\Gamma^{\alpha}.$$

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- **Theorem (D., Fiol, Miller, Ryan, Širáň, 2019).** Let $\Gamma = (V, E)$ be a base digraph with a voltage assignment α in $\mathbb{Z}_k$. The **spectrum of the lift Γ^{α}** is

$$\mathrm{sp}\,\Gamma^{\alpha} = \bigcup_{z \in \omega^i} \mathrm{sp}(\boldsymbol{B}(z)),$$

where ω^i is the set of all k-th roots of unity.

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- From the previous theorem, evaluating them at the 5-th roots of unity ω^i for $i = 0, 1, 2, 3, 4$, we get:

$z \setminus \lambda(z)$	$0^{[2]}$	i	$-i$	$z + \frac{1}{z}$
1	$0^{[2]}$	i	$-i$	2
ω	$0^{[2]}$	i	$-i$	$\frac{1}{2}(1 + \sqrt{5})$
ω^2	$0^{[2]}$	i	$-i$	$\frac{1}{2}(1 - \sqrt{5})$
ω^3	$0^{[2]}$	i	$-i$	$\frac{1}{2}(1 + \sqrt{5})$
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Table: The eigenvalues of the Alegre digraph.

- The complete spectrum of Alegre digraph is

$$\mathrm{sp}\Gamma^{\alpha} = \left\{2, 0^{[10]}, i^{[5]}, -i^{[5]}, \frac{1}{2}(1+\sqrt{5})^{[2]}, \frac{1}{2}(1-\sqrt{5})^{[2]}\right\}.$$

Outline

Lift digraphs for Abelian groups

- If G is an Abelian group, say $G = \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_n}$, with $m = |G| = \prod_{i=1}^{n} k_i$, the entries of the **polynomial matrix $\boldsymbol{B}$** can be replaced by polynomials with n variables $z_1, \ldots, z_n$. That is,

$$(\boldsymbol{B})_{uv} = \sum_{i_1,\ldots,i_n} \alpha_{i_1,\ldots,i_n} z_1^{i_1} \cdots z_n^{i_n},$$

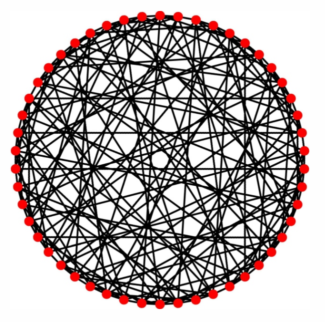
where

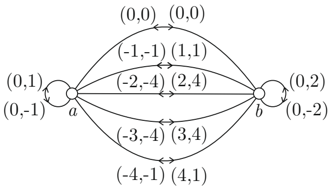
$$\alpha_{i_1,\ldots,i_n} = \begin{cases} 1 & \text{if } \exists\, uv \in E : \alpha(uv) = (g_{i_1}, \ldots, g_{i_n}) \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Lift digraphs for Abelian groups

Example: The Hoffman-Singleton graph (HS)

- It was known that the Hoffman-Singleton graph can be constructed as a lift of a base digraph on two vertices, with voltages in the group $\mathbb{Z}_5 \times \mathbb{Z}_5$.





Weisstein, Eric W. "Hoffman-Singleton Graph." From MathWorld–A Wolfram Web Resource.
https://mathworld.wolfram.com/Hoffman-SingletonGraph.html

Lift digraphs for Abelian groups

Example: The Hoffman-Singleton graph

- The polynomial matrix of the base digraph of HS is

$$\boldsymbol{B}(w,z)=\begin{pmatrix} w+\frac{1}{w} & 1+zw+z^2w^4+z^3w^4+z^4w^4 \\ 1+\frac{1}{zw}+\frac{1}{z^2w^4}+\frac{1}{z^3w^4}+\frac{1}{z^4w^4} & w^2+\frac{1}{w^2}. \end{pmatrix}.$$

Lift digraphs for Abelian groups

Example: The Hoffman-Singleton graph

$z \backslash w$	1	ω	ω^2	ω^3	ω^4
1	$7, -3$	$2, -3$	$2, -3$	$2, -3$	$2, -3$
ω	$2, 2$	$2, -3$	$2, -3$	$2, -3$	$2, -3$
ω^2	$2, 2$	$2, -3$	$2, -3$	$2, -3$	$2, -3$
ω^3	$2, 2$	$2, -3$	$2, -3$	$2, -3$	$2, -3$
ω^4	$2, 2$	$2, -3$	$2, -3$	$2, -3$	$2, -3$

Table: The eigenvalues of the HS graph.

$$\operatorname{sp} \mathrm{HS} = \{7^{[1]}, 2^{[28]}, -3^{[21]}\}.$$

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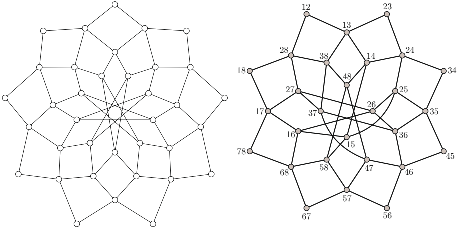

Figure: $F_2(C_9)$ $F_2(C_8)$

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- Let $V(\Gamma)$ and $A(\Gamma)$ denote the vertex and arc set of Γ.
- Let $\boldsymbol{A}$ be the adjacency matrix of Γ.
- For a **group** G and a **subgroup** $H < G$, let G/H and $[G : H]$ denote the set of all right cosets of H in G and the index of H in G, respectively.

A combined base graph

- A **combined voltage assignment** on a graph Γ in a group G consists of a pair of functions (α, ω), where α is an ordinary voltage assignment $\alpha : A(\Gamma) \to G$ (in the case of graphs, with the property that mutually reverse arcs receive mutually inverse voltages), and ω assigns to every vertex $v \in V(\Gamma)$ a subgroup $\omega(v) < G$.

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- The graph Γ, together with a combined voltage assignment (α, ω), is called the **combined base graph.**

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- The **vertex set** of the factored lift is the set $V^{(\alpha,\omega)} = \{(v, H) \mid v \in V(\Gamma) \text{ and } H \in G/\omega(v)\}$. For every vertex $v \in V(\Gamma)$, one has $[G : \omega(v)]$ vertices in the factored lift.

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- Let $a = uv \in A(\Gamma)$ be an arc emanating from a vertex u and terminating at a vertex v, with a voltage $\alpha(a) \in G$. For each such **arc** $a = uv$, there is an arc in the factored lift, emanating from a vertex (u, H) for some $H \in G/\omega(u)$ and terminating at a vertex (v, K) for some $K \in G/\omega(v)$ if and only if $H\alpha(a) \cap K \neq \emptyset$.

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- **Example.** If $G = \mathbb{Z}_{12}$, $H = 3\mathbb{Z}_{12} = \{0, 3, 6, 9\}$, $K = 4\mathbb{Z}_{12} = \{0, 2, 4\}$, and $\alpha(uv) = 0$, then, in the factored graph, vertex (u, H) is adjacent to the vertices (v, K), $(v, K + 3)$, $(v, K + 6)$, and $(v, K + 9)$.

Outline

Factored lifts on cyclic groups: Example 1

The **Johnson graph** $J(4,2)$ is a factored lift on the group $G = \mathbb{Z}_4 = \{0,1,2,3\}$ (see figure (a)).

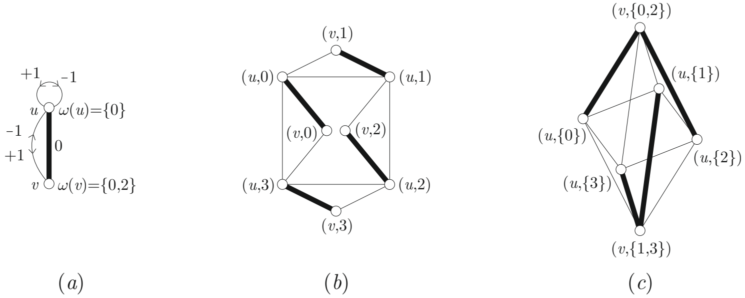
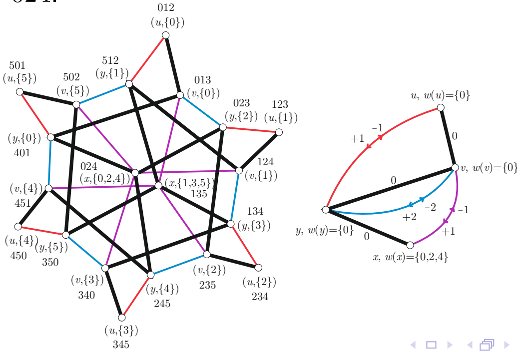


Figure: (a) The combined base graph on $\mathbb{Z}_4$ of $J(4,2)$. (b) The standard lift of the graph in (a). (c) The factored lift of the graph in (a), that is, the Johnson graph $J(4,2)$ (or octahedron graph).

Factored lifts on cyclic groups: Example 2

- Consider the **token graph** $F_3(C_6)$ with $G = \mathbb{Z}_6$ and $H = \{0, 2, 4\} < G$.
- There are orbits with 6 vertices and one orbit with 2 vertices.
- Each of these vertices is a representative of one orbit. We take the simplified notation $\{i, j, k\} = ijk$ and $u = 012$, $v = 013$, $y = 014$, and $x = 024$.



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- Given a combined base graph $(\Gamma,(\alpha,\omega))$, its **associated base graph** (Γ,α^+) has the same vertices as $(\Gamma,(\alpha,\omega))$, all of them associated to the trivial group. Besides, each arc uv in $(\Gamma,(\alpha,\omega))$ with $\omega(u)=H=\{h_1,\ldots,h_r\}$ and $\alpha(uv)=i$, becomes the r arcs $h_1+i,\ldots,h_r+i$.

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- Let $\boldsymbol{B}(z)$ be the polynomial matrix of the associated base graph (Γ, α^+), where, for each arc uv with voltage $\alpha^+(uv) = i$, the (u, v)-entry of $\boldsymbol{B}(z)$ has a term z^i.

Factored lifts on cyclic groups

Theorem

Let $\Gamma^{(\alpha,\omega)}$ *be a* **factored lift graph associated with the combined voltage graph** Γ *on the vertex set* $V(\Gamma) = \{u_1, \ldots, u_n\}$*. Let* $\boldsymbol{A}$ *be the adjacency matrix of* $\Gamma^{(\alpha,\omega)}$*, and* $\boldsymbol{B}(z)$ *the polynomial* $n \times n$ *matrix of its associated base graph* (Γ, α^+)*. Let* $\boldsymbol{f} = (f_1, \ldots, f_n)$ *be a* λ*-eigenvector of* $\boldsymbol{B}(z)$ *with* $z = \zeta^r = e^{i\frac{r2\pi}{n}}$ *satisfying the following condition:*
P1. *Let* $o(r) = \frac{n}{\gcd(n,r)}$ *and* $n_i = [\mathbb{Z}_n : \omega(u_i)]$*. For every* $u_i \in V(\Gamma)$*, either* $f_i = 0$*, or* $o(r)$ *divides* n_i*.*
Then, there exists a corresponding λ*-eigenvector of the factored graph* $\Gamma^{(\alpha,\omega)}$*.*

Factored lifts on cyclic groups: Example 2

To find the associated base graph and its voltages, we reason as follows:

(u) Vertex $u = 012$ is adjacent to $512 = y + 1$ and $013 = v$.
Therefore, $\alpha(uy) = +1$ and $\alpha(uv) = 0$.

(v) Vertex $v = 013$ is adjacent to $513 = x + 1$, $023 = y + 2$, $012 = u$, and $014 = y$.
Therefore, $\alpha(vx) = +1$, $\alpha(\underline{vy}) = +2$, $\alpha(vu) = 0$, and $\alpha(vy) = 0$.

(y) Vertex $y = 014$ is adjacent to $514 = v - 2$, $024 = x$, $013 = v$, and $015 = u - 1$. Therefore, $\alpha(\underline{yv}) = -2$, $\alpha(yx) = 0$, $\alpha(yv) = 0$, and $\alpha(yu) = -1$.

(x) Vertex $x = 024$ is adjacent to $124 = v + 1$, $245 = y + 2$, $014 = y$, $034 = v + 3$, $502 = v - 1$, and $023 = y - 2$.
Thus, $\boldsymbol{B}(xv) = z + z^3 + z^{-1}$ and $\boldsymbol{B}(xy) = 1 + z^2 + z^{-2}$.

Factored lifts on cyclic groups: Example 2

- The polynomial matrix is

$$\boldsymbol{B}(z) = \begin{pmatrix} 0 & 1 & z & 0 \\ 1 & 0 & 1+z^2 & z \\ z^{-1} & 1+z^{-2} & 0 & 1 \\ 0 & z^{-1}+z+z^3 & 1+z^2+z^{-2} & 0 \end{pmatrix}.$$

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- As expected, such a matrix is obtained from the base matrix of the combined base graph, namely

$$\begin{pmatrix} 0 & 1 & z & 0 \\ 1 & 0 & 1+z^2 & z \\ z^{-1} & 1+z^{-2} & 0 & 1 \\ 0 & z^{-1} & 1 & 0 \end{pmatrix}$$

by multiplying the last row by $1+z^2+z^{-2}$.

Factored lifts on cyclic groups: Example 2

$\zeta = e^{i\frac{2\pi}{6}}$, $z = \zeta^r$	$\lambda_{r,1}$	$\lambda_{r,2}$	$\lambda_{r,3}$	$\lambda_{r,4}$
$\mathrm{sp}(\boldsymbol{B}(\zeta^0))$	4	0	-2	-2
$\mathrm{sp}(\boldsymbol{B}(\zeta^1)) = \mathrm{sp}(\boldsymbol{B}(\zeta^5))$	2	0^*	-1	-1
$\mathrm{sp}(\boldsymbol{B}(\zeta^2)) = \mathrm{sp}(\boldsymbol{B}(\zeta^4))$	1	1	0^*	-2
$\mathrm{sp}(\boldsymbol{B}(\zeta^3))$	2	2	0	-4

Table: All the eigenvalues of the matrices $\boldsymbol{B}(\zeta^r)$, which yield the eigenvalues of the 3-token graph $F_3(C_6)$ plus four 0's (those marked with '*').

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The **spectrum** of $F_3(C_6)$ is

$$\mathrm{sp}\, F_3(C_6) = \{4^{[1]}, 2^{[4]}, 0^{[2]}, -2^{[4]}, -4^{[1]}\},$$

indicating that we are dealing with a bipartite graph.

Outline

Factored lifts on non-cyclic groups

- Let $\Gamma = (V, A)$ be a base graph of order k with a combined voltage assignment (α, β) in a group G. Let ρ be a complex **irreducible representation** of G in $\mathbb{C}^d$ of dimension $d = d(\rho)$.

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- We introduce a $dk \times dk$ complex **block matrix** $\boldsymbol{B}(\rho)$ with entries

$$\boldsymbol{B}_{u,v}(\rho) = \rho(G_u) \sum_{a \in \overrightarrow{uv}} \rho(\alpha(a)),$$

where each block is a $d \times d$ matrix.

Factored lifts on non-cyclic groups

Theorem

Let (α, ω) be a combined voltage assignment on a graph $\Gamma = (V, A)$ with k vertices in a group G and let $n_u = |G : G_u|$ for every $u \in V$, and with the order of the factored lift $\Gamma^{(\alpha,\omega)}$ equal to $k^\omega = \sum_{u \in V} n_u$. Let G have order n, with ν conjugacy classes, and let $\{\rho_r : \ r = 0, 1, \ldots, \nu - 1\}$ be a complete set of complex irreducible representations of G, of dimensions $d(\rho_r) = d_r$, so that $\sum_{r=0}^{\nu-1} d_r^2 = n$. Let $\mathcal{B}$ be the **multiset of eigenvalues**

$$\mathcal{B} = \bigcup_{r=0}^{\nu-1} d_r \ \mathrm{sp}(\boldsymbol{B}(\rho_r)).$$

of cardinality $\sum_{r=0}^{\nu-1} k d_r^2 = kn$. Then, the following statements hold:

- *(i) The multiset $\mathcal{B}$ contains at most $k^\omega = \sum_{u \in V} n_u$ non-zero eigenvalues.*
- *(ii) The spectrum of the factored lift $\Gamma^{(\alpha,\omega)}$ is the multiset $\mathcal{B} \setminus \mathcal{Z}$, where $\mathcal{Z}$ is a multiset containing $kn - k^\omega$ zeros.*

Factored lifts on non-cyclic groups: Example

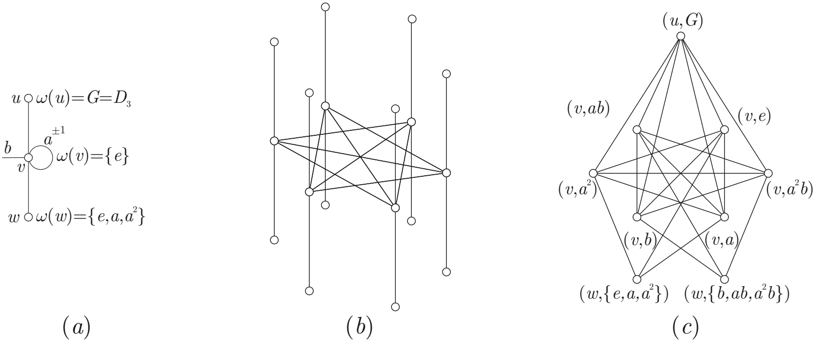

Figure: (a) The combined base graph on the dihedral group $\boldsymbol{D}_3$. (b) The standard lift of the graph in (a). (c) The factored lift of the graph in (a).

Factored lifts on non-cyclic groups: Example

- The **dihedral group** D_3: $D_3 = \langle a, b \mid a^3 = b^2 - (ab)^2 = e \rangle$ of order 6.

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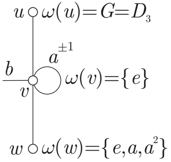
- The matrix $\boldsymbol{B} = \boldsymbol{B}(\Gamma)$ is the 3×3 matrix:

$$\boldsymbol{B} = \begin{pmatrix} 0 & e & 0 \\ e & a + a^{-1} + b & e \\ 0 & e & 0 \end{pmatrix}.$$



Factored lifts on non-cyclic groups: Example

	e	a	a^2	b	ab	a^2b
ρ_0	1	1	1	1	1	1
ρ_1	1	1	1	-1	-1	-1
ρ_2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$	$\begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^{-2} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \zeta^2 \\ \zeta^{-2} & 0 \end{pmatrix}$

Table: Irreducible representations of $D_3 = \langle a, b \,|\, a^3 = b^2 = (ab)^2 = e \rangle$, where $\zeta = e^{2\pi i/3}$.

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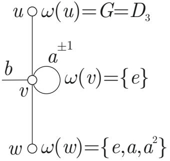
Table: Irreducible representations of $D_3 = \langle a, b \mid a^3 = b^2 = (ab)^2 = e \rangle$, where $\zeta = e^{2\pi i/3}$.

- $\rho_0(G_u) = 6$,
  $\rho_1(G_u) = 0$,
  $\rho_0(G_v) = \rho_1(G_v) = 1$,
  $\rho_0(G_w) = \rho_1(G_w) = 3$.



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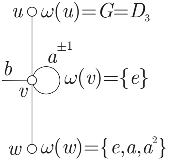

- $\rho_0(G_u) = 6$,
  $\rho_1(G_u) = 0$,
  $\rho_0(G_v) = \rho_1(G_v) = 1$,
  $\rho_0(G_w) = \rho_1(G_w) = 3$.
- For example, $\boldsymbol{B}(\rho_1)_{u,v} = 1$,
  $\boldsymbol{B}(\rho_1)_{v,v} = \rho_1(G_v)(\rho_1(a) + \rho_1(a^{-1}) + \rho_1(b)) = 1$.

Factored lifts on non-cyclic groups: Example

- The 3×3 matrices $\boldsymbol{B}(\rho_0)$ and $\boldsymbol{B}(\rho_1)$ are

$$\boldsymbol{B}(\rho_0) = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 3 & 1 \\ 0 & 3 & 0 \end{pmatrix}, \qquad \boldsymbol{B}(\rho_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix}.$$

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- Similarly, $\boldsymbol{B}(\rho_2)_{u,v}$ gives

$$\boldsymbol{B}(\rho_2) = \left(\begin{array}{cc|cc|cc} 0 & 0 & 1+\zeta+\zeta^2 & 1+\zeta+\zeta^2 & 0 & 0 \\ 0 & 0 & 1+\zeta^{-1}+\zeta^{-2} & 1+\zeta^{-1}+\zeta^{-2} & 0 & 0 \\ \hline 1 & 0 & \zeta+\zeta^{-1} & 1 & 1 & 0 \\ 0 & 1 & 1 & \zeta+\zeta^{-1} & 0 & 1 \\ \hline 0 & 0 & 1+\zeta+\zeta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\zeta^{-1}+\zeta^{-2} & 0 & 0 \end{array}\right).$$

Factored lifts on non-cyclic groups: Example

$\boldsymbol{B}(\rho_0)$			$\boldsymbol{B}(\rho_1)$		
$3(1+\sqrt{5})/2$	0	$3(1-\sqrt{5})/2$	$(1+\sqrt{13})/2$	0	$(1-\sqrt{13})/2$

$\boldsymbol{B}(\rho_2)$					
0	0	0^*	0^*	0^*	-2

Table: Eigenvalues of the matrices $\boldsymbol{B}(\rho_r)$ for $r = 0, 1, 2$.

Factored lifts on non-cyclic groups: Example

$\boldsymbol{B}(\rho_0)$			$\boldsymbol{B}(\rho_1)$		
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Table: Eigenvalues of the matrices $\boldsymbol{B}(\rho_r)$ for $r = 0, 1, 2$.

The spectrum of Γ^α is

$$\operatorname{sp}\Gamma^\alpha = \{3(1 \pm \sqrt{5})/2, (1 \pm \sqrt{13})/2, 0^4\}.$$

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