Constructing Cospectral Graphs

– some "prehistory"

History: on the one hand:

Spectral graph theory started in the 80s, when Cheeger's inequality was used as a means for constructing sparse and balanced cuts in a graph. In the 2000s, our field moved on from studying specific eigenvalues to studying the whole spectrum of the Laplacian matrix with fast Laplacian solvers. To obtain fast Laplacian solvers, we needed to sparsify graphs, for which we exploited concentration phenomena of random matrices. In the 2010s, improvements to these tools led to improvements on a wide variety of problems, like maximum flow, travelling salesman (both symmetric and asymmetric), and random spanning tree generation. In this talk, we briefly survey this chain of events and suggest some future directions.

???

On the other:

Spectra of Graphs: Theory and Application

Dragoš M. Cvetković, Michael Doob, Horst Sachs
Academic Press, 1980 - Mathematics - 368 pages

what did they find to write about?

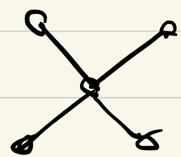
Von Collatz, L. and Sinogowitz, U. (1957) Spektren Endlicher Grafen. Abhandlungen aus dem Mathematischen Seminar der Universitat Hamburg, 21, 63-77. http://dx.doi.org/10.1007/BF02941924

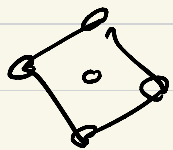
Cospectral

Question:

The adjacency matrices of isomorphic graphs are permutation equivalent, hence similar. Converse?

No!





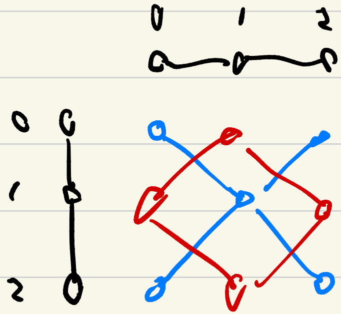
$t^3(t^2-4)$

$$\phi(K_{a,b}) = (t^2 - ab)\,t^{a+b-2} \qquad (a,b \geq 1)$$

More interesting examples?

Direct product

$X \times Y$, $A(X \times Y) = A(X) \otimes A(Y)$



M

$$\begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & C^T \\ C & 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & B^T \otimes C^T \\ 0 & 0 & B^T \otimes C & 0 \\ 0 & B \otimes C^T & 0 & 0 \\ B \otimes C & 0 & 0 & 0 \end{bmatrix} \sim \left[\begin{array}{cc|cc} 0 & B^T \otimes C & 0 & 0 \\ B \otimes C^T & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & B^T \otimes C^T \\ 0 & 0 & B \otimes C & 0 \end{array}\right]$$

N

$$M^2 = \begin{bmatrix} B^TB \otimes CC^T & 0 \\ 0 & BB^T \otimes C^TC \end{bmatrix}, \qquad N^2 = \begin{bmatrix} B^TB \otimes C^TC & 0 \\ 0 & BB^T \otimes CC^T \end{bmatrix}$$

If R & S are s.t. RS & SR are both defined, then RS & SR have the same nonzero eigenvalues, with the same multiplicities

$\Rightarrow CC^T$ & C^TC are cospectral, but for zeros

Connected Cospectral

Partitioned tensor product

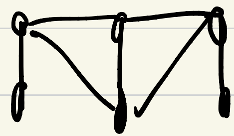
Let $L = \begin{pmatrix} U & R \\ S & V \end{pmatrix}$, $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $H' = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$ be partitioned matrices, where A & D are square. Define the partitioned tensor product

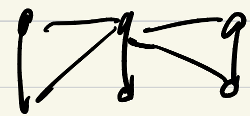
$$L \hat{\otimes} H = \begin{pmatrix} U \otimes A & R \otimes B \\ S \otimes C & V \otimes D \end{pmatrix}, \quad L \hat{\otimes} H' = \begin{pmatrix} U \otimes D & R \otimes C \\ S \otimes B & V \otimes D \end{pmatrix}$$

Theorem If $L = \begin{pmatrix} I_m & R \\ S & I_n \end{pmatrix}$, then $L \hat{\otimes} H$ and $L \hat{\otimes} H'$ are cospectral if $m=n$, or if A & D are cospectral. $\square$

G & M

Knuth

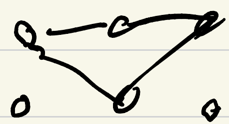


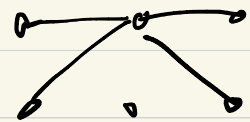


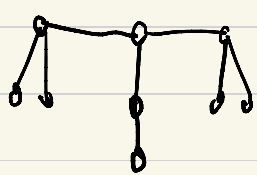
$$L = \left[\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right]$$

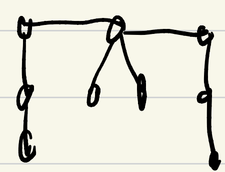
$$H = \begin{pmatrix} A & B \\ B^T & A \end{pmatrix}, \quad H' = \begin{pmatrix} A & B^T \\ B & A \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$



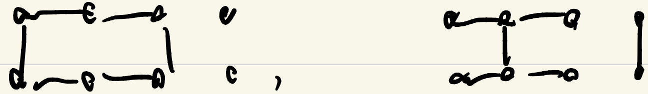


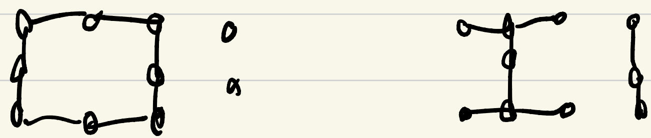


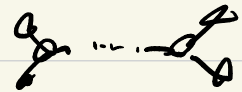


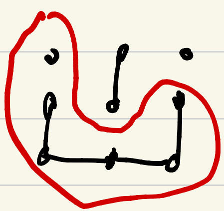
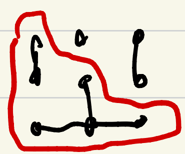
$x\, K_{3,3} - t^2(t^2-1)(t^3-7t)$

$x\, P_{12} - t^2(t^3-4t)(t^2-1)$





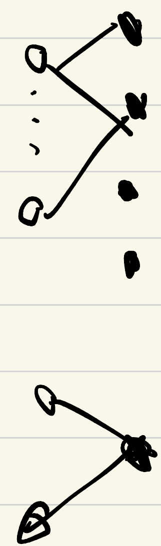
⇒ spectral radius of  is 2.

smallest cospectral forests

$X \subseteq Y$

bipartite graph



Subdivision graphs

If B is the vertex-edge incidence matrix of X, then

$$BB^T = \Delta + A$$ unsigned Laplacian

$$B^TB = 2I + L$$ adjacency matrix of line graph

Further

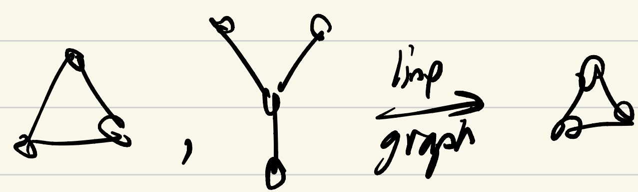
$$S = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

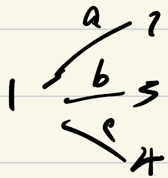
is the adjacency matrix of the subdivision graph of X.

We have

$$S^2 = \begin{pmatrix} BB^T & 0 \\ 0 & B^TB \end{pmatrix} = \begin{pmatrix} \Delta + A & 0 \\ 0 & 2I + L \end{pmatrix}$$

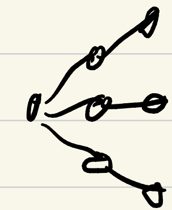
Recall that if M is $k \times l$ and N is $l \times k$, then MN & NM have the same non-zero eigenvalues with the same multiplicities.

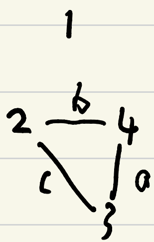




$$B_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

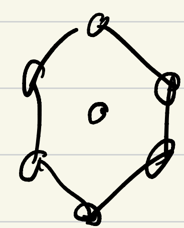
$$S_1 = \left[\begin{array}{c|ccc} & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ \hline 1\ 1\ 0\ 0 & & & \\ 1\ 0\ 1\ 0 & & 0 & \\ 1\ 0\ 0\ 1 & & & \end{array}\right]$$





$$B_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$S_2 = \left[\begin{array}{c|ccc} & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ & 1 & 0 & 1 \\ & 1 & 1 & 0 \\ \hline 0\ 0\ 1\ 1 & & & \\ 0\ 1\ 0\ 1 & & 0 & \\ 0\ 1\ 1\ 0 & & & \end{array}\right]$$



$$[S_1] \overset{?}{=} S_2$$

orthogonal

$$\left(\frac{1}{2} J_4 - I_4\right) B_1 = B_2$$

$$\left(\frac{1}{2} J - I_4\right)^2 = I$$

Cospectral, cospectral complements

Assume $A = \begin{pmatrix} C & B \\ B^T & D \end{pmatrix}$ with C & D square, C regular.

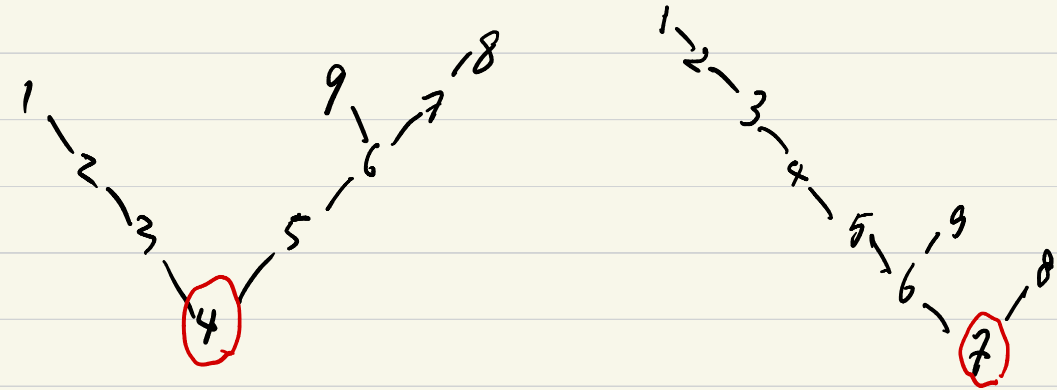
Suppose B is $\ell \times m$, with ℓ even and that each column of B is $\underset{\sim}{0}$, $\underset{\sim}{1}$ or has exactly half its entries equal to 1. Define $Q = \underbrace{\frac{2}{\ell} J_\ell - I_\ell}$. Then $Q = Q^T$ and $Q^2 = I$ and

$$\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} C & B \\ B^T & D \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \overset{=C}{QCQ} & \overset{\alpha}{QB} \\ B^T Q & D \end{pmatrix}$$

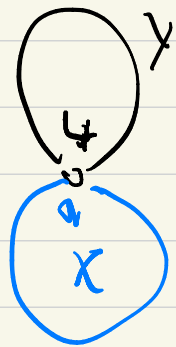
$Q \underset{\sim}{1} = \underset{\sim}{1}$

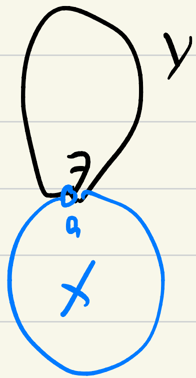
Trees

Schwenk:



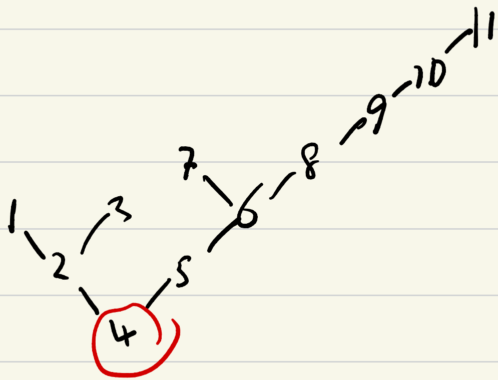
Key: $S \setminus 4$ & $S \setminus 7$ are cospectral

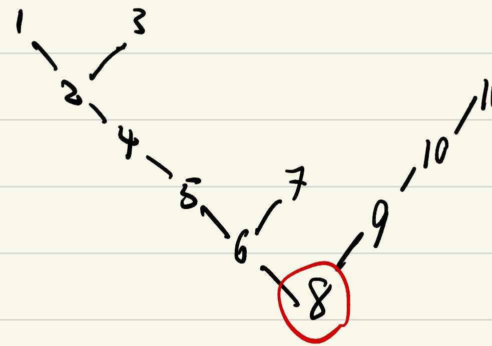




$$\phi(X \cdot Y, t) = \phi(X, a)\,\phi(Y) + \phi(X)\,\phi(Y, b) - t\,\phi(X, a)\,\phi(Y, b)$$

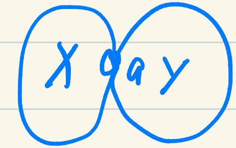
Complements





Key: $S \setminus 4$ and $S \setminus 8$ cospectral, cospectral complements

Lemma $W(X \circ Y, t)$ is determined by



$W(X)$, $W(Y)$ all walks

$W_a(X)$, $W_a(Y)$ all walks starting at a

$C_a(X)$, $C_a(Y)$ closed walks on a $\square$

Observe that $S \setminus 4 \cong S \setminus 8$.

Let $W(X,t)$ be the generating function for all walks in X. So

$$W(X,t) = \sum_{m \geq 0} \underset{\sim}{1}^T A^m \underset{\sim}{1}\, t^m .$$

Theorem If X & Y are cospectral, their complements are cospectral if & only if $W(X,t) = W(Y,t)$.

Proof

$$\phi(\bar{X}, t) = \det(tI - (J - I - A))$$

$$= \det((t+1)I + A - J)$$

$$= \det((t+1)I + A)\,\det(I - (\cdots)^{-1}\mathbf{1}\mathbf{1}^T)$$

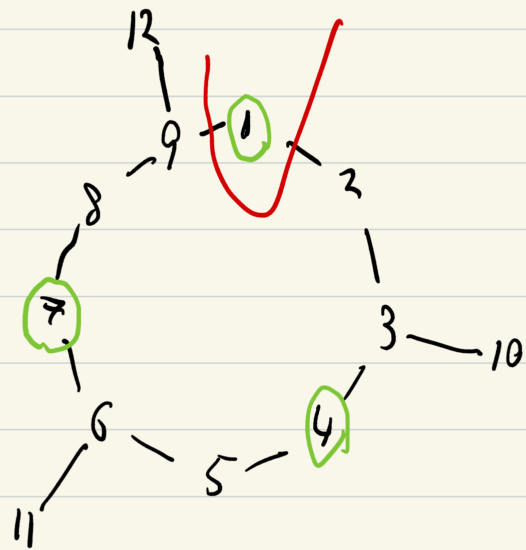
$$= \det((t+1)I + A)\,\det(I - \mathbf{1}^T((t+1)I + A)^{-1}\mathbf{1})$$

$$= \det((t+1)I + A)\,(1 - \mathbf{1}^T(\quad)^{-1}\mathbf{1})$$

$(-1)^n \phi(X, -t-1)$

all walks

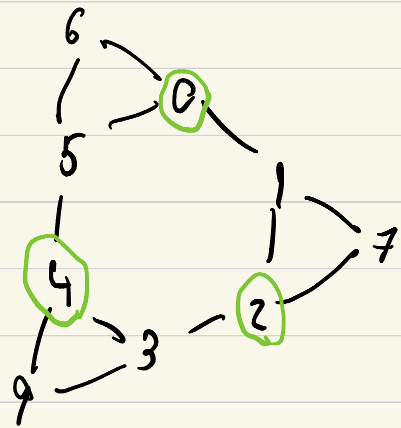
X



Herndon & Ellezy

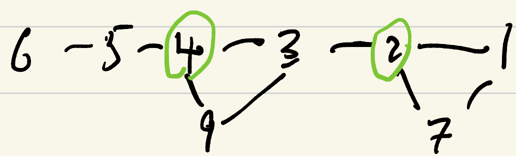
$X \setminus 1$ is a tree, vertices 4 & 7 are pseudosimilar.

S :

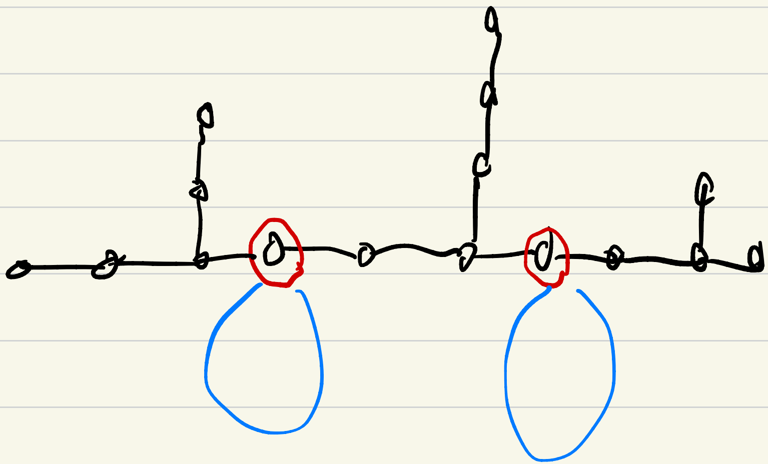


Vertices 2 & 4 are pseudosimilar in $S \setminus 0$.

$S \setminus 0$



McKay



complement

Laplacian

distance matrix

normalized Laplacian

G & M: (1976–1978)

Products of graphs & their spectra.

Constructing cospectral graphs.

The spectrum of a graph. (with D. A. Holton)

Partitioned tensor products & their spectra.

Some computational results on the spectra of graphs

D.E. Knuth: Partitioned tensor products & their spectra. (1995)

W.C. Herndon & M.L. Ellezy Jr: Isospectral graphs and molecules. (1975)

K. A. McAvaney: A note on limbless trees (1974)