Quantum walk state transfer on a hypercube

Martin Štefaňák and Stanislav Skoupy

Department of Physics, Faculty of Nuclear Sciences and Physical Engineering
Czech Technical University in Prague, Czech Republic

FACULTY OF NUCLEAR SCIENCES AND PHYSICAL ENGINEERING CTU IN PRAGUE

Outline

Outline

Introduction

- State transfer between vertices of a graph by discrete time quantum walk
- Evolution — coin (acts locally on vertices) and shift (propagation along arcs)
- Utilize quantum walk search — vertices are marked with a different coin
- Only local modification of dynamics, the rest of the graph is passive
- Initialize the walk on one marked vertex — sender
- Evolve to reach the second (receiver) with high probability
- Run-time of the order of $O(\sqrt{N})$
- Analysis — dimensional reduction — find (approximate) invariant subspace

Outline

Notation

- Vertices of the hypercube labeled by n-bit strings $\vec{x} = x_1 \dots x_n$, $x_i = 0, 1$
- Position basis states — $|\vec{x}\rangle$
- Unit vector in direction d — $\vec{e}_d = e_1^d \dots e_n^d$, $e_j^d = \delta_{j,d}$
- Basis states of the coin — $|d\rangle$
- Marked vertex ($\vec{0}$ w.l.o.g.) has a loop — coin basis state $|0\rangle$
- Hilbert space of the walk can be written as a direct sum of local spaces

$$\mathcal{H}' = \bigoplus_{\vec{x}} \mathcal{H}_{\vec{x}}$$

$$\begin{aligned} \vec{x} \neq \vec{0} : \qquad & \mathcal{H}_{\vec{x}} = \mathrm{Span}\left\{|\vec{x}, d\rangle | d = 1, \dots n\right\} \\ \vec{x} = \vec{0} : \qquad & \mathcal{H}_{\vec{0}} = \mathrm{Span}\left\{|\vec{0}, d\rangle | d = 0, 1, \dots n\right\} \end{aligned}$$

Evolution operator of the search

- Evolution operator has the usual form — $U' = S' \cdot C'$
- Conditional shift operator S'

$$S' = \sum_{d=1}^{n} \sum_{\vec{x}} |\vec{x} \oplus \vec{e}_d, d\rangle\langle \vec{x}, d| + |\vec{0}, 0\rangle\langle \vec{0}, 0|$$

- Coin operator — Grover on non-marked vertices, and $-G'$ on the marked one

$$C' = (I_P - |\vec{0}\rangle\langle\vec{0}|) \otimes G - |\vec{0}\rangle\langle\vec{0}| \otimes G'$$

- Grover diffusion operator G

$$G = 2|s_C\rangle\langle s_C| - I_C, \quad |s_C\rangle = \frac{1}{\sqrt{n}} \sum_{d=1}^{n} |d\rangle$$

Evolution operator of the search

- Modified Grover coin with a weighted loop G' (Wong)

$$G' = 2|s_l\rangle\langle s_l| - I_0, \quad |s_l\rangle = \frac{1}{\sqrt{n+l}}\left(\sqrt{l}|0\rangle + \sum_{d=1}^{n}|d\rangle\right)$$

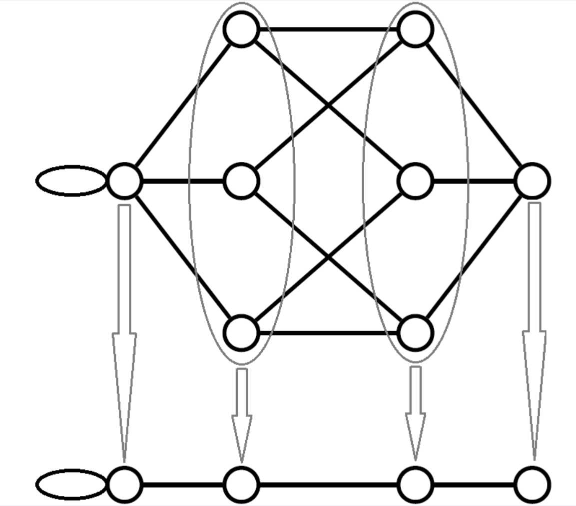
- Initial state of search — equal weight superposition of all states except for the loop

$$|\psi_0\rangle = \frac{1}{\sqrt{n2^n}}\sum_{d=1}^{n}\sum_{\vec{x}}|\vec{x}, d\rangle$$

- Target state of the search — loop at the marked vertex $|\vec{0}, 0\rangle$

Reduction to a walk on a line (Shenvi, Kempe, Whaley)

- Problem can be reduced to the walk on a finite line with a non-homogeneous coin



Reduction to a walk on a line

- Define $2n+1$ orthonormal basis states

$$|x,R\rangle = \frac{1}{\sqrt{(n-x)\binom{n}{x}}}\sum_{|\vec{x}|=x}\sum_{x_d=0}|\vec{x},d\rangle, \quad x=0,\ldots n-1$$

$$|x,L\rangle = \frac{1}{\sqrt{x\binom{n}{x}}}\sum_{|\vec{x}|=x}\sum_{x_d=1}|\vec{x},d\rangle, \quad x=1,\ldots n$$

$$|0,\circlearrowleft\rangle = |\vec{0},0\rangle$$

- Expression of the shift operator S'

$$S' = \sum_{x=0}^{n-1}(|x,R\rangle\langle x+1,L| + |x+1,L\rangle\langle x,R|) + |0,\circlearrowleft\rangle\langle 0,\circlearrowleft|$$

Reduction to a walk on a line

- Coin operator

$$C' = |0\rangle\langle 0| \otimes C_0' + \sum_{x=1}^{n} |x\rangle\langle x| \otimes C_x$$

- Position dependent coins $x \neq 0, n$ ($C_n = 1$)

$$C_x = \begin{pmatrix} \cos\theta_x & \sin\theta_x \\ \sin\theta_x & -\cos\theta_x \end{pmatrix}, \quad \cos\theta_x = 1 - \frac{2x}{n}, \quad \sin\theta_x = \frac{2}{n}\sqrt{x(n-x)}$$

- Coin at the marked vertex depends on the weight of the loop l

$$C_0' = \begin{pmatrix} -\frac{n-l}{n+l} & -\frac{2\sqrt{nl}}{n+l} \\ -\frac{2\sqrt{nl}}{n+l} & \frac{n-l}{n+l} \end{pmatrix}$$

Approximate invariant subspace

- Initial state $|\psi_0\rangle$ in terms of $\{|x, R\rangle, |x, L\rangle, |0, \circlearrowleft\rangle\}$

$$|\psi_0\rangle = \frac{1}{2^{\frac{n}{2}}}(|0, R\rangle + |n, L\rangle) + \frac{1}{2^{\frac{n}{2}}}\sum_{x=1}^{n-1}\left(\sqrt{\binom{n-1}{x-1}}|x, L\rangle + \sqrt{\binom{n-1}{x}}|x, R\rangle\right)$$

- $|\psi_0\rangle$ together with $|0, \circlearrowleft\rangle$ and $|\psi_1\rangle$ form approximate invariant subspace

$$|\psi_1\rangle = \frac{1}{c}\sum_{x=0}^{n/2-2}\frac{1}{\sqrt{2\binom{n-1}{x}}}(|x, R\rangle - |x+1, L\rangle), \quad \frac{1}{c} = \left(\sum_{x=0}^{n/2-2}\frac{1}{\binom{n-1}{x}}\right)^{-\frac{1}{2}} \approx 1 - \frac{1}{2n}$$

Approximate invariant subspace

- Exact eigenvector of U' with eigenvalue 1 for all values of l

$$|\alpha_1\rangle = \sqrt{\frac{l2^n}{n+l2^n}}|\psi_0\rangle - \sqrt{\frac{n}{n+l2^n}}|0,\circlearrowleft\rangle$$

- For small l we find "almost" eigenvectors of U'

$$\begin{aligned}|\alpha_2\rangle &= \sqrt{\frac{n}{n+l2^n}}|\psi_0\rangle + \sqrt{\frac{l2^n}{n+l2^n}}|0,\circlearrowleft\rangle \\ |\alpha_3\rangle &= |\psi_1\rangle \\ \langle\alpha_j|U'|\alpha_j\rangle &= 1 - O(l/n)\,,\ j = 2,3\end{aligned}$$

Evolution in the approximate invariant subspace

- Effective evolution operator in the approximate invariant subspace

$$U'_{ef} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{pmatrix}, \quad \omega \approx \sin\omega \approx \frac{(2n-1)\sqrt{l+\frac{n}{2^n}}}{\sqrt{2n}(l+n)}$$

- Rotation in the $|\alpha_2\rangle$, $|\alpha_3\rangle$ plane by an angle ω

Evolution in the approximate invariant subspace

- Initial and target state in terms of $|\alpha_j\rangle$

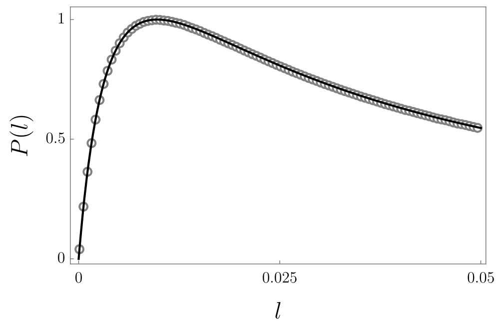
$$|\psi_0\rangle = \sqrt{\frac{l2^n}{n+l2^n}}|\alpha_1\rangle + \sqrt{\frac{n}{n+l2^n}}|\alpha_2\rangle$$

$$|0,\circlearrowleft\rangle = \sqrt{\frac{n}{n+l2^n}}|\alpha_1\rangle - \sqrt{\frac{l2^n}{n+l2^n}}|\alpha_2\rangle$$

- Maximal success probability achieved if we rotate from $|\alpha_2\rangle$ to $-|\alpha_2\rangle$

$$P(l) = |\langle 0,\circlearrowleft|\psi(T)\rangle|^2 \approx \frac{4nl2^n}{(n+l2^n)^2}$$

Success for 10-dim hypercube



Optimal weight for search on a hypercube

- Success probability reaches 1 for weight $l = n/2^n \equiv d/N$ (Wong, Høyer and Yu)
- For this value of l the states $|\alpha_{1,2}\rangle$ simplifies into

$$|\alpha_1\rangle = \frac{1}{\sqrt{2}}\left(|\psi_0\rangle - |0,\circlearrowleft\rangle\right), \quad |\alpha_2\rangle = \frac{1}{\sqrt{2}}\left(|\psi_0\rangle + |0,\circlearrowleft\rangle\right)$$

- Run-time of the search algorithm

$$T_1 = \frac{\pi}{\omega} \approx \frac{n\pi(1+2^{-n})}{2n-1}2^{\frac{n}{2}} \sim \frac{\pi}{2}2^{\frac{n}{2}} = O(\sqrt{N})$$

State transfer by search with a switch

- Utilize periodicity of the search for one vertex for state transfer
- Initialize at the sender vertex in the loop — $|\vec{s},0\rangle$
- Mark sender vertex
- Evolve for T_1 steps close to the initial state of the search $|\psi_0\rangle$
- Switch marking from the sender to the receiver vertex $\vec{r}$
- Evolve for T_1 steps close to the loop on the receiver vertex $|\vec{r},0\rangle$

Total run-time of the state transfer with a switch

$$T_2 = 2T_1 \sim \pi 2^{\frac{n}{2}}$$

Outline

Reduction to a walk on a line

- State transfer from $\vec{0}$ to $\vec{1}$ with both vertices marked simultaneously
- Reduction to a line again possible, loops at both ends
- Additional basis state — loop at the receiver vertex — $|n, \circlearrowleft\rangle = |\vec{1}, 0\rangle$
- Evolution operator — $U'' = C'' \cdot S''$
- Shift operator S''

$$S'' = \sum_{x=0}^{n-1} (|x, R\rangle\langle x+1, L| + |x+1, L\rangle\langle x, R|) + |0, \circlearrowleft\rangle\langle 0, \circlearrowleft| + |n, \circlearrowleft\rangle\langle n, \circlearrowleft|$$

- Position dependent coins, similar as for search

$$C'' = (|0\rangle\langle 0| + |n\rangle\langle n|) \otimes C_0' + \sum_{x=1}^{n-1} |x\rangle\langle x| \otimes C_x$$

Approximate invariant subspace

- Initial state of the state transfer — loop at the sender vertex — $|0, \circlearrowleft\rangle$
- Target state of the state transfer — loop at the receiver vertex — $|n, \circlearrowleft\rangle$
- Together with $|\psi_0\rangle, |\psi_1\rangle$ and $|\psi_2\rangle$ they form an approximate 5-dim invariant subspace

$$|\psi_2\rangle = \frac{1}{c}\sum_{x=0}^{n/2-2} \frac{1}{\sqrt{2\binom{n-1}{x}}}\left(|n-x, L\rangle - |n-x-1, R\rangle\right)$$

Approximate invariant subspace

- Exact eigenvector of U'' with eigenvalue 1

$$|\beta_1\rangle = \sqrt{\frac{l2^n}{l2^n+2n}}|\psi_0\rangle - \sqrt{\frac{n}{l2^n+2n}}(|0,\circlearrowleft\rangle + |n,\circlearrowleft\rangle)$$

- For small l, "almost" eigenvectors of U''

$$\begin{aligned}
|\beta_2\rangle &= \sqrt{\frac{2n}{l2^n+2n}}|\psi_0\rangle + \sqrt{\frac{l2^{n-1}}{l2^n+2n}}(|0,\circlearrowleft\rangle + |n,\circlearrowleft\rangle)\\
|\beta_3\rangle &= \frac{1}{\sqrt{2}}\left(|\psi_1\rangle - |\psi_2\rangle\right)\\
|\beta_4\rangle &= \frac{1}{\sqrt{2}}\left(|0,\circlearrowleft\rangle - |n,\circlearrowleft\rangle\right)\\
|\beta_5\rangle &= \frac{1}{\sqrt{2}}\left(|\psi_1\rangle + |\psi_2\rangle\right), \quad \langle\beta_j|U''|\beta_j\rangle = 1 - O(l/n),\ j = 2,\ldots 5
\end{aligned}$$

Evolution in the approximate invariant subspace

- Effective evolution operator in the approximate invariant subspace

$$U''_{ef} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\omega_1 & -\sin\omega_1 & 0 & 0 \\ 0 & \sin\omega_1 & \cos\omega_1 & 0 & 0 \\ 0 & 0 & 0 & \cos\omega_2 & -\sin\omega_2 \\ 0 & 0 & 0 & \sin\omega_2 & \cos\omega_2 \end{pmatrix}$$

- Two rotations
  - in the $|\beta_2\rangle, |\beta_3\rangle$ plane by an angle ω_1
  - in the $|\beta_4\rangle, |\beta_5\rangle$ plane by an angle ω_2

$$\omega_1 \approx \frac{\sqrt{2n\left(l + n2^{1-n}\right)}}{c(n+l)}, \quad \omega_2 \approx \frac{\sqrt{2ln}}{c(n+l)}$$

Evolution in the approximate invariant subspace

- Initial state in terms of $|\beta_j\rangle$

$$|0,\circlearrowleft\rangle = -\sqrt{\frac{n}{l2^n+2n}}|\beta_1\rangle + \sqrt{\frac{l2^{n-1}}{l2^n+2n}}|\beta_2\rangle + \frac{1}{\sqrt{2}}|\beta_4\rangle$$

- Target state in terms of $|\beta_j\rangle$

$$|n,\circlearrowleft\rangle = -\sqrt{\frac{n}{l2^n+2n}}|\beta_1\rangle + \sqrt{\frac{l2^{n-1}}{l2^n+2n}}|\beta_2\rangle - \frac{1}{\sqrt{2}}|\beta_4\rangle$$

- We have to rotate from $|\beta_4\rangle$ to $-|\beta_4\rangle$, and make a full circle in $|\beta_2\rangle, |\beta_3\rangle$ plane
- Optimal choice of the weight l — make the angles harmonic

$$l = \frac{2}{3}\frac{n}{2^n} \equiv \frac{2}{3}\frac{d}{N} \implies \omega_1 = 2\omega_2 \sim 2\frac{2}{\sqrt{3}}2^{-\frac{n}{2}}$$

Evolution in the approximate invariant subspace

- For $l = \frac{2}{3}\frac{n}{2^n}$ we find simpler form of $|\beta_{1,2}\rangle$

$$|\beta_1\rangle = \frac{1}{2}|\psi_0\rangle - \sqrt{\frac{3}{8}}(|0,\circlearrowleft\rangle + |n,\circlearrowleft\rangle), \quad |\beta_2\rangle = \frac{\sqrt{3}}{2}|\psi_0\rangle + \sqrt{\frac{1}{8}}(|0,\circlearrowleft\rangle + |n,\circlearrowleft\rangle)$$

- Eigenvectors of U''_{ef} corresponding to $\lambda_j^{(\pm)} = e^{\pm i\omega_j}$

$$|\omega_1^{(\pm)}\rangle = \frac{1}{\sqrt{2}}\left(|\beta_2\rangle \mp i|\beta_3\rangle\right), \quad |\omega_2^{(\pm)}\rangle = \frac{1}{\sqrt{2}}\left(|\beta_4\rangle \mp i|\beta_5\rangle\right)$$

- State after t steps

$$|\psi(t)\rangle = -\sqrt{\frac{3}{8}}|\beta_1\rangle + \frac{1}{4}(e^{i2\omega_2 t}|\omega_1^{(+)}\rangle + e^{-i2\omega_2 t}|\omega_1^{(-)}\rangle) + \frac{1}{2}(e^{i\omega_2 t}|\omega_2^{(+)}\rangle + e^{-i\omega_2 t}|\omega_2^{(-)}\rangle)$$

Fidelity of state transfer

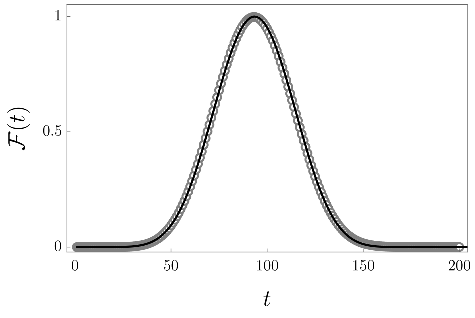
- Fidelity of state transfer into the loop at the receiver vertex at time t

$$\mathcal{F}(t) = |\langle n, \circlearrowleft | \psi(t)\rangle|^2 = \frac{1}{64}\left(3 + \cos(2\omega_2 t) - 4\cos(\omega_2 t)\right)^2$$

- Fidelity reaches unity for T_3

$$T_3 = \frac{\pi}{\omega_2} \sim \frac{\pi\sqrt{3}}{2} 2^{\frac{n}{2}} = \frac{\sqrt{3}}{2} T_2$$

Fidelity for 10-dim hypercube



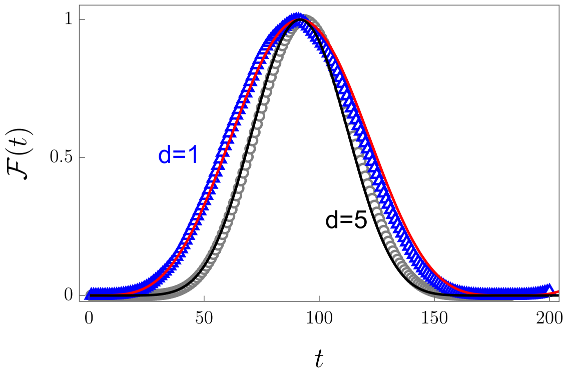
Faster than transfer with switch by a factor $\sqrt{3}/2$

Outline

State transfer between vertices of arbitrary distance d

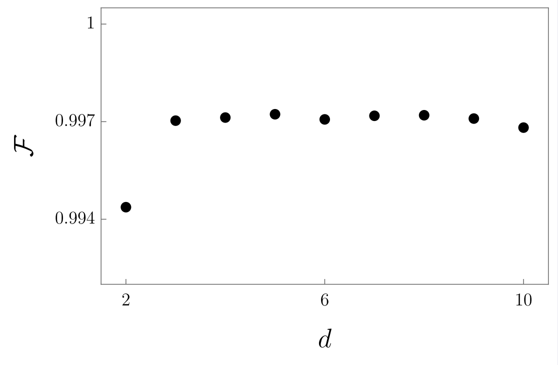
- Sender at $\vec{0}$, receiver at vertex $\vec{r}$ with Hamming weight d
- Weighted loops at both vertices
- We consider same weight $l = \frac{2}{3}\frac{n}{2^n}$ as for transfer to the antipode ($d = n$)
- Initial state — $|\vec{0}, 0\rangle$
- Target state — $|\vec{r}, 0\rangle$
- Reduction to a line is not possible, mostly numerical evidence
- For $d = 2, \ldots, n-1$ the fidelity behaves similarly as for $d = n$
- For $d = 1$ the peak is wider, but the maximum is reached at time close to T_3

State transfer on 10-dim hypercube

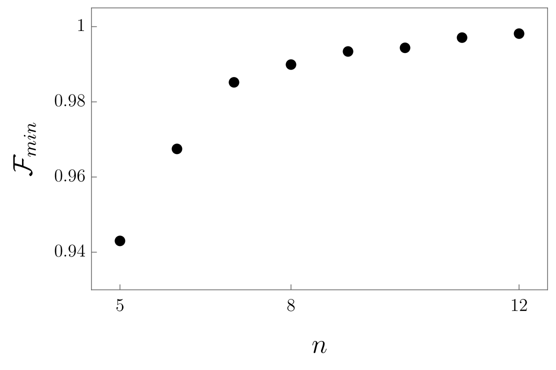


Fidelity of state transfer for $d \geq 2$

Fidelity for given distance d



Minimal fidelity as function of n



- Fidelity as a function of the distance d is almost constant, except for $d = 2$
- With increasing dimension of the hypercube n the minimal fidelity improves

Approximate invariant subspace for $d \geq 2$

- Approximation — state transfer evolves in 5-dim invariant subspace
- Relevant eigenvalues $1,\ e^{\pm i\omega_1},\ e^{\pm i\omega_2}$, phases $\omega_{1,2}$ close to formulas for the antipode
- Exact eigenvector corresponding to eigenvalue 1 for all distances d

$$|\gamma_1\rangle = \frac{1}{2}|\psi_0\rangle - \sqrt{\frac{3}{8}}(|\vec{0},0\rangle + |\vec{r},0\rangle)$$

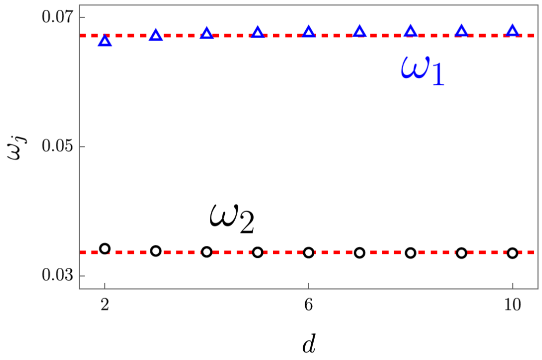
- "Almost" eigenvectors $|\gamma_2\rangle$ and $|\gamma_4\rangle$ constructed in a similar way as $|\beta_2\rangle$ and $|\beta_4\rangle$

$$|\gamma_2\rangle = \frac{\sqrt{3}}{2}|\psi_0\rangle + \sqrt{\frac{1}{8}}(|\vec{0},0\rangle + |\vec{r},0\rangle), \quad |\gamma_4\rangle = \frac{1}{\sqrt{2}}(|\vec{0},0\rangle - |\vec{r},0\rangle)$$

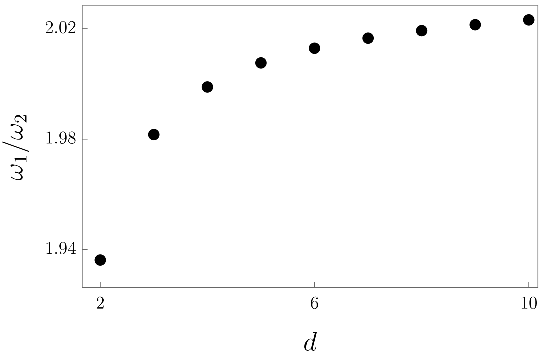
- $|\gamma_3\rangle$ and $|\gamma_5\rangle$ would require analogy of $|\psi_1\rangle$ and $|\psi_2\rangle$ — we do not have analytical form

Relevant phases for $d \geq 2$

Relevant phases for $n = 10$



Ratio of phases for $n = 10$



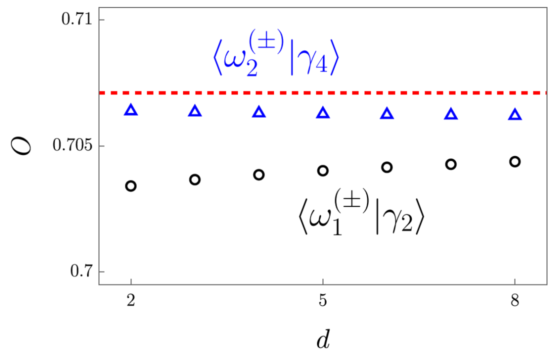
- Relevant phases are close to the values for $d = n$
- With increasing distance d the ratio approaches 2

Relevant eigenvectors for $d \geq 2$

- $|\gamma_2\rangle$ and $|\gamma_4\rangle$ in terms of eigenvectors $|\omega_1^{(\pm)}\rangle$, $|\omega_2^{(\pm)}\rangle$ — similar as for the antipode

$$|\gamma_2\rangle \approx \frac{1}{\sqrt{2}}(|\omega_1^{(+)}\rangle + |\omega_1^{(-)}\rangle)$$
$$|\gamma_4\rangle \approx \frac{1}{\sqrt{2}}(|\omega_2^{(+)}\rangle + |\omega_2^{(-)}\rangle)$$

Overlap with eigenvectors for $n = 8$



For $d \geq 2$ state transfer works in a similar way as for $d = n$

State transfer between nearest neighbours

- Larger projection of the initial and the target state onto $\ker(U'' - 1)$

$$|\gamma_0\rangle = \frac{x}{\sqrt{2}}(|\vec{0},0\rangle + |\vec{r},0\rangle) + y|\psi_0\rangle + \frac{z}{\sqrt{2}}(|\vec{0},d_r\rangle + |\vec{r},d_s\rangle)$$

- Amplitudes of the exact eigenstate

$$x = \sqrt{\frac{3a}{b}}, \quad a = 2^n + 2n - 4, \quad b = 3\cdot 2^n + 8n - 12 - 2^{2-n}$$

$$y = -\frac{2(n-1)}{\sqrt{ab}}, \quad z = -\frac{\sqrt{n2^{n+1}}(1-2^{1-n})}{\sqrt{ab}}$$

- For large n, y and z vanishes, x tends to 1 — approximation of the eigenvector

$$|\gamma_0\rangle \approx \frac{1}{\sqrt{2}}(|\vec{0},0\rangle + |\vec{r},0\rangle)$$

State transfer between nearest neighbours

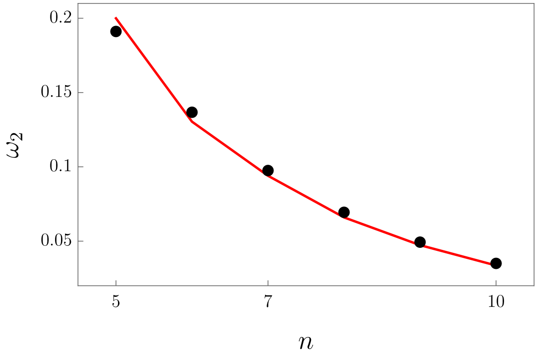
- For $d = 1$ only one pair of eigenvalues $\lambda_2^{(\pm)} = e^{\pm i\omega_2}$ is relevant

$$\omega_2 \sim \frac{2}{\sqrt{3}} 2^{-\frac{n}{2}}$$

- Initial and target states in terms of eigenstates

$$|\vec{0}, 0\rangle \approx \frac{x}{\sqrt{2}}|\gamma_0\rangle + \frac{1}{2}(|\omega_2^{(+)}\rangle + |\omega_2^{(-)}\rangle)$$
$$|\vec{r}, 0\rangle \approx \frac{x}{\sqrt{2}}|\gamma_0\rangle - \frac{1}{2}(|\omega_2^{(+)}\rangle + |\omega_2^{(-)}\rangle)$$

Comparison of ω_2 with numerics



State transfer between nearest neighbours

- State after t steps

$$|\psi(t)\rangle = \frac{x}{\sqrt{2}}|\gamma_0\rangle + \frac{1}{2}(e^{-i\omega_2 t}|\omega_2^{(+)}\rangle + e^{i\omega_2 t}|\omega_2^{(-)}\rangle)$$

- Fidelity of state transfer at time t

$$\mathcal{F}(t) = \frac{1}{4}(x^2 - \cos(\omega_2 t))^2$$

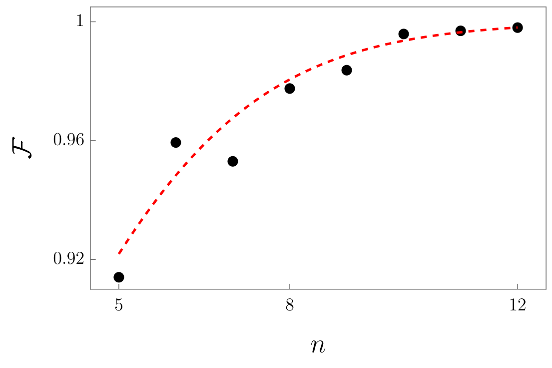
- Maximal fidelity at $T_3 = \pi/\omega_2$

$$\mathcal{F} = \frac{1}{4}(1 + x^2)^2$$

- For large n — $x = 1$

$$\mathcal{F}(t) = \sin^4\left(\frac{\omega_2 t}{2}\right)$$

Max fidelity for a given n



Conclusions

- Two approaches to state transfer on a hypercube
- Exploit search and switch marking from sender to receiver after one period
- Mark both sender and receiver with weighted loops
- In the second case the run-time is faster
- Analytical results for search, and state transfer between antipodal vertices
- Exact form of stationary state for $d \neq n$ + numerical results

Stefanak and Skoupy, Quantum walk state transfer on a hypercube, arXiv:2302.07581

Physica Scripta 98, 104003 (2023)

Thank you for your attention