A general theorem in spectral extremal graph theory

Algebraic Graph Theory Seminar

John Byrne (joint work with Dheer Noal Desai and Michael Tait)

University of Delaware

June 10, 2024

Definitions

Definitions

Let $\mathcal{F}$ be a family of graphs.

Definitions

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if G has no subgraph isomorphic to some $F \in \mathcal{F}$.

Definitions

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if G has no subgraph isomorphic to some $F \in \mathcal{F}$. The *spectral radius* of G is the largest eigenvalue of its adjacency matrix.

Definitions

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if G has no subgraph isomorphic to some $F \in \mathcal{F}$. The *spectral radius* of G is the largest eigenvalue of its adjacency matrix.

- $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices.

Definitions

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if G has no subgraph isomorphic to some $F \in \mathcal{F}$. The *spectral radius* of G is the largest eigenvalue of its adjacency matrix.

- $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices.
- $\mathrm{spex}(n, \mathcal{F})$ is the maximum spectral radius of an $\mathcal{F}$-free graph on n vertices.

Definitions

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if G has no subgraph isomorphic to some $F \in \mathcal{F}$. The *spectral radius* of G is the largest eigenvalue of its adjacency matrix.

- $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices.
- $\mathrm{spex}(n, \mathcal{F})$ is the maximum spectral radius of an $\mathcal{F}$-free graph on n vertices.
- $\mathrm{EX}(n, \mathcal{F})$ and $\mathrm{SPEX}(n, \mathcal{F})$ are the sets of graphs obtaining the maximum in each case.

Definitions

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if G has no subgraph isomorphic to some $F \in \mathcal{F}$. The *spectral radius* of G is the largest eigenvalue of its adjacency matrix.

- $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices.
- $\mathrm{spex}(n, \mathcal{F})$ is the maximum spectral radius of an $\mathcal{F}$-free graph on n vertices.
- $\mathrm{EX}(n, \mathcal{F})$ and $\mathrm{SPEX}(n, \mathcal{F})$ are the sets of graphs obtaining the maximum in each case.

Our goal is to find conditions on $\mathcal{F}$ under which we can relate $\mathrm{EX}(n, \mathcal{F})$ and $\mathrm{SPEX}(n, \mathcal{F})$ as $n \to \infty$.

Non-bipartite graphs I

Non-bipartite graphs I

For non-bipartite forbidden graphs, then asymptotics of the edge extremal function are known:

Non-bipartite graphs I

For non-bipartite forbidden graphs, then asymptotics of the edge extremal function are known:

Theorem (Erdős-Stone-Simonovits)

For any graph F, we have

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right)\binom{n}{2} + o(n^2).$$

Non-bipartite graphs I

For non-bipartite forbidden graphs, then asymptotics of the edge extremal function are known:

Theorem (Erdős-Stone-Simonovits)

For any graph F, we have

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right)\binom{n}{2} + o(n^2).$$

Moreover, we know something about the structure of the extremal graphs:

Non-bipartite graphs I

For non-bipartite forbidden graphs, then asymptotics of the edge extremal function are known:

Theorem (Erdős-Stone-Simonovits)

For any graph F, we have

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right)\binom{n}{2} + o(n^2).$$

Moreover, we know something about the structure of the extremal graphs:

Theorem (Erdős-Simonovits)

Let F be a graph with $r = \chi(F) - 1$. Then any graph in $\mathrm{EX}(n, \mathcal{F})$ can be obtained from $T_{n,r}$ by adding and deleting $o(n^2)$ many edges.

Non-bipartite graphs I

For non-bipartite forbidden graphs, then asymptotics of the edge extremal function are known:

Theorem (Erdős-Stone-Simonovits)

For any graph F, we have

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right)\binom{n}{2} + o(n^2).$$

Moreover, we know something about the structure of the extremal graphs:

Theorem (Erdős-Simonovits)

Let F be a graph with $r = \chi(F) - 1$. Then any graph in $\mathrm{EX}(n, \mathcal{F})$ can be obtained from $T_{n,r}$ by adding and deleting $o(n^2)$ many edges.

Here $T_{n,r}$ denotes the *Turán graph*, the complete r-partite graph on n vertices with partite sets as equal as possible.

Non-bipartite graphs II

Non-bipartite graphs II

Similar results hold for the spectral extrema.

Non-bipartite graphs II

Similar results hold for the spectral extrema.

Theorem (Nikiforov)

Let F be a graph with $r = \chi(F) - 1$. Then any graph in $\mathrm{SPEX}(n, \mathcal{F})$ can be obtained from $T_{n,r}$ by adding and deleting $o(n^2)$ many edges.

Non-bipartite graphs II

Similar results hold for the spectral extrema.

Theorem (Nikiforov)

Let F be a graph with $r = \chi(F) - 1$. Then any graph in $\mathrm{SPEX}(n, \mathcal{F})$ can be obtained from $T_{n,r}$ by adding and deleting $o(n^2)$ many edges.

What can be said about the relation between $\mathrm{SPEX}(n, F)$ and $\mathrm{EX}(n, F)$?

Non-bipartite graphs II

Similar results hold for the spectral extrema.

Theorem (Nikiforov)

Let F be a graph with $r = \chi(F) - 1$. Then any graph in $\mathrm{SPEX}(n, \mathcal{F})$ can be obtained from $T_{n,r}$ by adding and deleting $o(n^2)$ many edges.

What can be said about the relation between $\mathrm{SPEX}(n, F)$ and $\mathrm{EX}(n, F)$?

Theorem (Wang-Kang-Xue)

If $\mathrm{ex}(n, F) = e(T_{n,r}) + O(1)$, then $\mathrm{SPEX}(n, F) \subseteq \mathrm{EX}(n, F)$ for n large enough.

Bipartite graphs I

Bipartite graphs I

The Erdős-Stone-Simonovits Theorem gives that if F is bipartite, then $\mathrm{ex}(n, F) = o(n^2)$. The correct exponent is only known in some special cases.

Bipartite graphs I

The Erdős-Stone-Simonovits Theorem gives that if F is bipartite, then $\mathrm{ex}(n, F) = o(n^2)$. The correct exponent is only known in some special cases.

Proposition

- $\mathrm{ex}(n, F) < n^{2-\varepsilon}$ if and only if F is bipartite;
- $\mathrm{ex}(n, F) > n^{1+\varepsilon}$ if and only if F has a cycle;
- if F is a forest, then $\mathrm{ex}(n, F) = \Theta(n)$.

Bipartite graphs I

The Erdős-Stone-Simonovits Theorem gives that if F is bipartite, then $\mathrm{ex}(n, F) = o(n^2)$. The correct exponent is only known in some special cases.

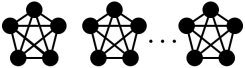
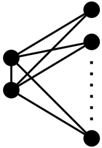
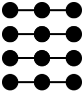
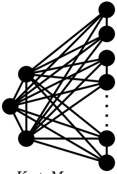
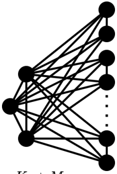
Proposition

- $\mathrm{ex}(n, F) < n^{2-\varepsilon}$ if and only if F is bipartite;
- $\mathrm{ex}(n, F) > n^{1+\varepsilon}$ if and only if F has a cycle;
- if F is a forest, then $\mathrm{ex}(n, F) = \Theta(n)$.

A linear extremal number can also be obtained by forbidding certain infinite families of graphs, e.g. if $\mathcal{F} = \{\text{all cycles}\}$.

Motivating examples

Motivating examples

F	$\mathrm{EX}(n, F)$	$\mathrm{SPEX}(n, F)$
 P_6	 $\frac{n}{5} \cdot K_5$	 $K_2 + \overline{K_{n-2}}$
 $4 \cdot P_3$	 $K_3 + M_{n-3}$	 $K_3 + M_{n-3}$

Previous work for forests

Previous work for forests

Let $G + H$ denote the *join* of graphs G and H, obtained by taking the disjoint union of G and H and letting every vertex in G be adjacent to every vertex in H.

Previous work for forests

Let $G + H$ denote the *join* of graphs G and H, obtained by taking the disjoint union of G and H and letting every vertex in G be adjacent to every vertex in H.

Theorem (Zhai, Yuan, You)

If $\mathrm{EX}(n, F) = K_k + \overline{K_{n-k}}$ (or $K_k + (K_2 \cup \overline{K_{n-k-2}})$) then $\mathrm{SPEX}(n, F) = K_k + \overline{K_{n-k}}$ (or $K_k + (K_2 \cup \overline{K_{n-k-2}})$) for n large enough.

Main result I

Main result I

We need some definitions before stating our main result:

Main result I

We need some definitions before stating our main result:

Definition

Let F, G be graphs. Then let $\mathrm{ex}^G(n, F)$ denote the maximum number of edges in an n-vertex F-free graph which contains a copy of G.

Main result I

We need some definitions before stating our main result:

Definition

Let F, G be graphs. Then let $\mathrm{ex}^G(n, F)$ denote the maximum number of edges in an n-vertex F-free graph which contains a copy of G.

Definition

We denote by $K_{k,\infty}$ the complete bipartite graph with one partite set of size k and the other countably infinite.

Main result I

We need some definitions before stating our main result:

Definition

Let F, G be graphs. Then let $\mathrm{ex}^G(n, F)$ denote the maximum number of edges in an n-vertex F-free graph which contains a copy of G.

Definition

We denote by $K_{k,\infty}$ the complete bipartite graph with one partite set of size k and the other countably infinite.

Note that

$$F \subseteq K_{k,\infty} \iff F \subseteq K_{k,n-k} \text{ for } n \text{ large enough.}$$

Main result II

Main result II

Theorem (B., Desai, Tait 2024)

Suppose $\mathcal{F}$ satisfies:

Main result II

Theorem (B., Desai, Tait 2024)

Suppose $\mathcal{F}$ satisfies:

- $\mathrm{ex}(n, \mathcal{F}) = O(n)$

Main result II

Theorem (B., Desai, Tait 2024)

Suppose $\mathcal{F}$ satisfies:

- $\mathrm{ex}(n, \mathcal{F}) = O(n)$
- $K_{k+1,\infty} \supseteq F \in \mathcal{F}$

Main result II

Theorem (B., Desai, Tait 2024)

Suppose $\mathcal{F}$ satisfies:

- $\mathrm{ex}(n, \mathcal{F}) = O(n)$
- $K_{k+1,\infty} \supseteq F \in \mathcal{F}$
- $\mathrm{EX}^{K_{k,n-k}}(n, \mathcal{F}) \ni H$ for large enough n, where either:

Main result II

Theorem (B., Desai, Tait 2024)

Suppose $\mathcal{F}$ satisfies:

- $\mathrm{ex}(n, \mathcal{F}) = O(n)$
- $K_{k+1,\infty} \supseteq F \in \mathcal{F}$
- $\mathrm{EX}^{K_{k,n-k}}(n, \mathcal{F}) \ni H$ for large enough n, where either:
  (a) $H = K_{k,n-k}$

Main result II

Theorem (B., Desai, Tait 2024)

Suppose $\mathcal{F}$ satisfies:

- $\mathrm{ex}(n, \mathcal{F}) = O(n)$
- $K_{k+1,\infty} \supseteq F \in \mathcal{F}$
- $\mathrm{EX}^{K_{k,n-k}}(n, \mathcal{F}) \ni H$ for large enough n, where either:
  (a) $H = K_{k,n-k}$
  (b) $H = K_k + \overline{K_{n-k}}$

Main result II

Theorem (B., Desai, Tait 2024)

Suppose $\mathcal{F}$ satisfies:

- $\mathrm{ex}(n, \mathcal{F}) = O(n)$
- $K_{k+1,\infty} \supseteq F \in \mathcal{F}$
- $\mathrm{EX}^{K_{k,n-k}}(n, \mathcal{F}) \ni H$ for large enough n, where either:
  - (a) $H = K_{k,n-k}$
  - (b) $H = K_k + \overline{K_{n-k}}$
  - (c) $H = K_k + (K_2 \cup \overline{K_{n-k-2}})$

Main result II

Theorem (B., Desai, Tait 2024)

Suppose $\mathcal{F}$ satisfies:

- $\mathrm{ex}(n, \mathcal{F}) = O(n)$
- $K_{k+1,\infty} \supseteq F \in \mathcal{F}$
- $\mathrm{EX}^{K_{k,n-k}}(n, \mathcal{F}) \ni H$ for large enough n, where either:
  (a) $H = K_{k,n-k}$
  (b) $H = K_k + \overline{K_{n-k}}$
  (c) $H = K_k + (K_2 \cup \overline{K_{n-k-2}})$
  (d) $H = K_k + X$ where $e(X) \leq Qn + O(1)$ for some $Q \in [0, 3/4)$ and $\mathcal{F}$ is finite

Main result II

Theorem (B., Desai, Tait 2024)

Suppose $\mathcal{F}$ satisfies:

- $\mathrm{ex}(n, \mathcal{F}) = O(n)$
- $K_{k+1,\infty} \supseteq F \in \mathcal{F}$
- $\mathrm{EX}^{K_{k,n-k}}(n, \mathcal{F}) \ni H$ for large enough n, where either:
  - (a) $H = K_{k,n-k}$
  - (b) $H = K_k + \overline{K_{n-k}}$
  - (c) $H = K_k + (K_2 \cup \overline{K_{n-k-2}})$
  - (d) $H = K_k + X$ where $e(X) \leq Qn + O(1)$ for some $Q \in [0, 3/4)$ and $\mathcal{F}$ is finite
  - (f) $H = K_k + X$ where $e(X) \leq Qn + O(1)$ for some $Q \geq 3/4$, all but a bounded number of vertices of X have constant degree d, and $K_k + (\infty \cdot K_{1,d+1})$ is not $\mathcal{F}$-free.

Main result II

Theorem (B., Desai, Tait 2024)

Suppose $\mathcal{F}$ satisfies:

- $\mathrm{ex}(n, \mathcal{F}) = O(n)$
- $K_{k+1,\infty} \supseteq F \in \mathcal{F}$
- $\mathrm{EX}^{K_{k,n-k}}(n, \mathcal{F}) \ni H$ for large enough n, where either:
  - (a) $H = K_{k,n-k}$
  - (b) $H = K_k + \overline{K_{n-k}}$
  - (c) $H = K_k + (K_2 \cup \overline{K_{n-k-2}})$
  - (d) $H = K_k + X$ where $e(X) \leq Qn + O(1)$ for some $Q \in [0, 3/4)$ and $\mathcal{F}$ is finite
  - (f) $H = K_k + X$ where $e(X) \leq Qn + O(1)$ for some $Q \geq 3/4$, all but a bounded number of vertices of X have constant degree d, and $K_k + (\infty \cdot K_{1,d+1})$ is not $\mathcal{F}$-free.

Then if (a), (b) or (c) holds $\mathrm{SPEX}(n, \mathcal{F}) = \{H\}$

Main result II

Theorem (B., Desai, Tait 2024)

Suppose $\mathcal{F}$ satisfies:

- $\mathrm{ex}(n, \mathcal{F}) = O(n)$
- $K_{k+1,\infty} \supseteq F \in \mathcal{F}$
- $\mathrm{EX}^{K_{k,n-k}}(n, \mathcal{F}) \ni H$ for large enough n, where either:
  - (a) $H = K_{k,n-k}$
  - (b) $H = K_k + \overline{K_{n-k}}$
  - (c) $H = K_k + (K_2 \cup \overline{K_{n-k-2}})$
  - (d) $H = K_k + X$ where $e(X) \leq Qn + O(1)$ for some $Q \in [0, 3/4)$ and $\mathcal{F}$ is finite
  - (f) $H = K_k + X$ where $e(X) \leq Qn + O(1)$ for some $Q \geq 3/4$, all but a bounded number of vertices of X have constant degree d, and $K_k + (\infty \cdot K_{1,d+1})$ is not $\mathcal{F}$-free.

Then if (a), (b) or (c) holds $\mathrm{SPEX}(n, \mathcal{F}) = \{H\}$ and if (d) or (f) holds $\mathrm{SPEX}(n, \mathcal{F}) \subseteq \mathrm{EX}^{K_k + \overline{K_{n-k}}}(n, \mathcal{F})$.

Example application

Example application

Let F be the star forest $K_{1,d_1} \cup \cdots \cup K_{1,d_j}$ where $d_1 \geq \cdots \geq d_j$.

Example application

Let F be the star forest $K_{1,d_1} \cup \cdots \cup K_{1,d_j}$ where $d_1 \geq \cdots \geq d_j$.

- Any forest satisfies $\mathrm{ex}(n, \mathcal{F}) = O(n)$.

Example application

Let F be the star forest $K_{1,d_1} \cup \cdots \cup K_{1,d_j}$ where $d_1 \geq \cdots \geq d_j$.

- Any forest satisfies $\mathrm{ex}(n, \mathcal{F}) = O(n)$.
- We have $F \subseteq K_{j,\infty}$, so let $k = j - 1$.

Example application

Let F be the star forest $K_{1,d_1} \cup \cdots \cup K_{1,d_j}$ where $d_1 \geq \cdots \geq d_j$.

- Any forest satisfies $\mathrm{ex}(n, \mathcal{F}) = O(n)$.
- We have $F \subseteq K_{j,\infty}$, so let $k = j - 1$.
- Any graph of the form $Y + X$ where $|V(Y)| = k$ is F-free if and only if $\Delta(X) < d_j$, so $\mathrm{EX}^{K_{k,n-k}}(n, F) \ni K_k + X$ where X is $(d_j - 1)$-regular.

Example application

Let F be the star forest $K_{1,d_1} \cup \cdots \cup K_{1,d_j}$ where $d_1 \geq \cdots \geq d_j$.

- Any forest satisfies $\mathrm{ex}(n, \mathcal{F}) = O(n)$.
- We have $F \subseteq K_{j,\infty}$, so let $k = j - 1$.
- Any graph of the form $Y + X$ where $|V(Y)| = k$ is F-free if and only if $\Delta(X) < d_j$, so $\mathrm{EX}^{K_{k,n-k}}(n, F) \ni K_k + X$ where X is $(d_j - 1)$-regular.
- (f) is satisfied, so we conclude that every graph in $\mathrm{SPEX}(n, F)$ is of this form.

Example application

Let F be the star forest $K_{1,d_1} \cup \cdots \cup K_{1,d_j}$ where $d_1 \geq \cdots \geq d_j$.

- Any forest satisfies $\mathrm{ex}(n, \mathcal{F}) = O(n)$.
- We have $F \subseteq K_{j,\infty}$, so let $k = j - 1$.
- Any graph of the form $Y + X$ where $|V(Y)| = k$ is F-free if and only if $\Delta(X) < d_j$, so $\mathrm{EX}^{K_{k,n-k}}(n, F) \ni K_k + X$ where X is $(d_j - 1)$-regular.
- (f) is satisfied, so we conclude that every graph in $\mathrm{SPEX}(n, F)$ is of this form.
- This recovers a result of Chen, Liu, and Zhang.

Applications

Applications

We can characterize $\mathrm{SPEX}(n, \mathcal{F})$ when $\mathcal{F}$ is:

Applications

We can characterize $\mathrm{SPEX}(n, \mathcal{F})$ when $\mathcal{F}$ is:

Finite families

- Path forest (Chen-Liu-Zhang)
- Star forest (Chen-Liu-Zhang)
- Certain small trees
- Almost any tree (Fang-Lin-Shu-Zhang)
- Spectral Erdős-Sós theorem (Cioaba-Desai-Tait)

Infinite families

- Arithmetic progression of cycles
- Consecutive even cycle lengths
- k disjoint long cycles
- Disjoint equicardinal cycles
- Graphs with K_k-minors (Tait)
- Graph with F_k-minors (He-Li-Feng)
- k disjoint chorded cycles
- 2 disjoint cycles, one with a chord
- Cycles with $k(k-2)+1$ chords
- Cycles with k incident chords

Proof idea I

Proof idea I

Our main tools are:

- **Eigenvector equations:** if λ is an eigenvalue of G with eigenvector x, then for any $u \in V(G)$,

$$\lambda x_u = \sum_{v \sim u} x_v, \quad \lambda^2 x_u = \sum_{v \sim u} \sum_{w \sim v} x_w.$$

Proof idea I

Our main tools are:

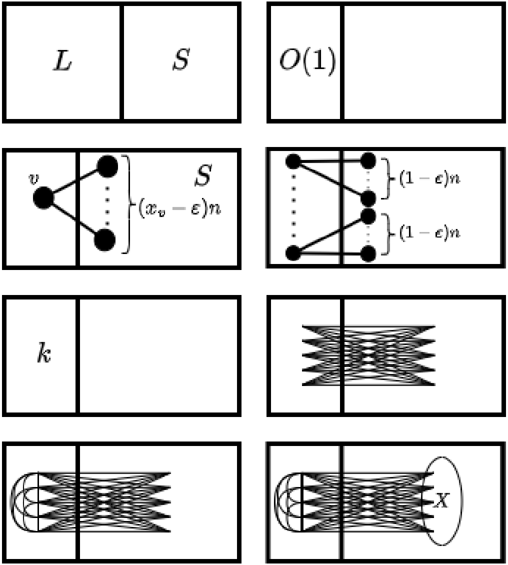
- **Eigenvector equations:** if λ is an eigenvalue of G with eigenvector x, then for any $u \in V(G)$,

$$\lambda x_u = \sum_{v \sim u} x_v, \quad \lambda^2 x_u = \sum_{v \sim u} \sum_{w \sim v} x_w.$$

- **Rayleigh quotient:** We have

$$\lambda = \max_{\|x\|=1} 2 \sum_{uv \in E(G)} x_u x_v.$$

Proof idea II



$Q = 3/4$ threshold I

$Q = 3/4$ threshold I

Theorem (B., Desai, Tait 2024)

Suppose $\mathcal{F}$ satisfies:

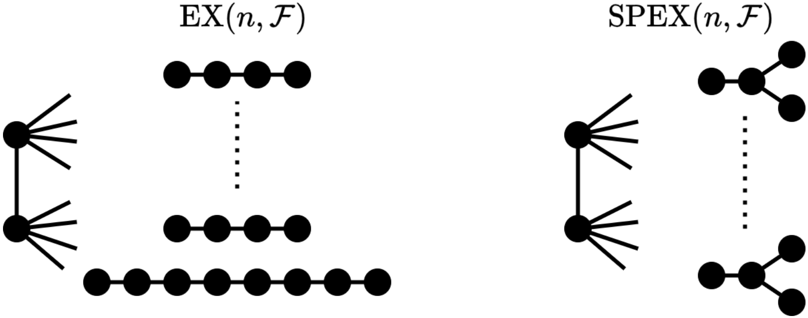
- $\mathrm{ex}(n, \mathcal{F}) = O(n)$
- $K_{k+1,\infty} \supseteq F \in \mathcal{F}$
- $\mathrm{EX}^{K_{k,n-k}}(n, \mathcal{F}) \ni H$ for large enough n, where either:
  - (a) $H = K_{k,n-k}$
  - (b) $H = K_k + \overline{K_{n-k}}$
  - (c) $H = K_k + (K_2 \cup \overline{K_{n-k-2}})$
  - (d) $H = K_k + X$ where $e(X) \leq Qn + O(1)$ for some $Q \in [0, 3/4)$ and $\mathcal{F}$ is finite
  - (f) $H = K_k + X$ where $e(X) \leq Qn + O(1)$ for some $Q \geq 3/4$, all but a bounded number of vertices of X have constant degree d, and $K_k + (\infty \cdot K_{1,d+1})$ is not $\mathcal{F}$-free.

Then if (a), (b) or (c) holds $\mathrm{SPEX}(n, \mathcal{F}) = \{H\}$ and if (d) or (f) holds $\mathrm{SPEX}(n, \mathcal{F}) \subseteq \mathrm{EX}^{K_k + \overline{K_{n-k}}}(n, \mathcal{F})$.

$Q = 3/4$ threshold II

$Q = 3/4$ threshold II

We will find a finite graph family $\mathcal{F}$ for which:



$Q = 3/4$ threshold III

$Q = 3/4$ threshold III

Let $\mathcal{F}$ consist of:

- $3 \cdot K_{1,4}$
- $K_{2,6} \cup X_5 \cup Y_5$, for all connected graphs X_5, Y_5 on 5 vertices
- $K_{2,6} \cup K_{1,3} \cup X_5$, for all connected graphs X_5 on 5 vertices
- $K_{2,6} \cup X_9$, for all connected graphs X_9 on 9 vertices
- $K_{2,6} \cup X_8$, for all connected graphs X_8 on 8 vertices except for P_8
- $X_5 \cup Y_5 \cup Z_5 \cup W_5$, for all connected graphs X_5, Y_5, Z_5, W_5 on 5 vertices
- $C_i \cup C_j \cup C_k$ for $3 \leq i, j, k \leq 7$

A_α version

A_α version

- Let $\mathrm{spex}_\alpha(n, \mathcal{F})$ be the maximum spectral radius of the matrix $A_\alpha(G) := \alpha D(G) + (1-\alpha)A(G)$ over $\mathcal{F}$-free graphs G on n vertices. In particular, the signless Laplacian $Q(G) = 2A_{1/2}(G)$.

A_α version

- Let $\mathrm{spex}_\alpha(n, \mathcal{F})$ be the maximum spectral radius of the matrix $A_\alpha(G) := \alpha D(G) + (1-\alpha)A(G)$ over $\mathcal{F}$-free graphs G on n vertices. In particular, the signless Laplacian $Q(G) = 2A_{1/2}(G)$.
- Our second goal is to relate $\mathrm{SPEX}(n, \mathcal{F})$ and $\mathrm{SPEX}_\alpha(n, \mathcal{F})$.

A_α version

- Let $\mathrm{spex}_\alpha(n, \mathcal{F})$ be the maximum spectral radius of the matrix $A_\alpha(G) := \alpha D(G) + (1-\alpha)A(G)$ over $\mathcal{F}$-free graphs G on n vertices. In particular, the signless Laplacian $Q(G) = 2A_{1/2}(G)$.
- Our second goal is to relate $\mathrm{SPEX}(n, \mathcal{F})$ and $\mathrm{SPEX}_\alpha(n, \mathcal{F})$.

Theorem (B., Desai, Tait 2024)

Let $\mathcal{F}$ be a family of graphs containing some bipartite graph F for which $F - v$ is a forest, or such that $\mathrm{ex}(n, \mathcal{F}) = O(n)$. Suppose that for n large enough, $\mathrm{SPEX}(n, \mathcal{F}) \ni H$, where either:

(a) $H = K_k + \overline{K_{n-k}}$

(b) $H = K_k + (K_2 \cup \overline{K_{n-k-2}})$.

Then for any $\alpha \in (0,1)$, for any n large enough, $\mathrm{SPEX}_\alpha(n, \mathcal{F}) = H$.

Applications

Applications

We can characterize $\mathrm{SPEX}_\alpha(n, \mathcal{F})$ when $\mathcal{F}$ is:

Applications

We can characterize $\mathrm{SPEX}_\alpha(n, \mathcal{F})$ when $\mathcal{F}$ is:

Finite families

- Linear forests (Chen-Liu-Zhang)
- Certain small trees
- Spectral Erdős-Sós Theorem (Chen-Li-Li-Yu-Zhang)

Infinite families

- Even cycles (Li-Yu)
- Intersecting even cycles
- All k consecutive even cycles
- All k disjoint cycles (Li-Yu-Zhang)
- All long cycles
- All disjoint equicardinal cycles
- Graphs with K_k-minors (Chen-Liu-Zhang)
- Graphs with F_k-minors (Wang-Zhang)

Open questions

Open questions

- Many more applications can be proven using only a slight modification of our methods. Can one prove a stronger result which captures all of these?

Open questions

- Many more applications can be proven using only a slight modification of our methods. Can one prove a stronger result which captures all of these?
  - Can we completely drop the assumption that $\mathcal{F}$ is finite?

Open questions

- Many more applications can be proven using only a slight modification of our methods. Can one prove a stronger result which captures all of these?
    - Can we completely drop the assumption that $\mathcal{F}$ is finite?
    - What if the extremal graphs are of the form $W + X$ where $W \notin \{\overline{K_K}, K_k\}$?

Open questions

- Many more applications can be proven using only a slight modification of our methods. Can one prove a stronger result which captures all of these?
  - Can we completely drop the assumption that $\mathcal{F}$ is finite?
  - What if the extremal graphs are of the form $W + X$ where $W \notin \{\overline{K_K}, K_k\}$?
  - If $K_{k,n-k} \in \mathrm{SPEX}(n, \mathcal{F})$ then what is the range of α for which we can show $K_{k,n-k} \in \mathrm{SPEX}_\alpha(n, \mathcal{F})$?

Open questions

- Many more applications can be proven using only a slight modification of our methods. Can one prove a stronger result which captures all of these?
  - Can we completely drop the assumption that $\mathcal{F}$ is finite?
  - What if the extremal graphs are of the form $W + X$ where $W \notin \{\overline{K_K}, K_k\}$?
  - If $K_{k,n-k} \in \mathrm{SPEX}(n, \mathcal{F})$ then what is the range of α for which we can show $K_{k,n-k} \in \mathrm{SPEX}_\alpha(n, \mathcal{F})$?
- Do the spectral extremal results still hold when $\mathrm{ex}(n, \mathcal{F}) = n^r$ for $1 < r < 3/2$?

Open questions

- Many more applications can be proven using only a slight modification of our methods. Can one prove a stronger result which captures all of these?
  - Can we completely drop the assumption that $\mathcal{F}$ is finite?
  - What if the extremal graphs are of the form $W + X$ where $W \notin \{\overline{K_K}, K_k\}$?
  - If $K_{k,n-k} \in \mathrm{SPEX}(n, \mathcal{F})$ then what is the range of α for which we can show $K_{k,n-k} \in \mathrm{SPEX}_\alpha(n, \mathcal{F})$?
- Do the spectral extremal results still hold when $\mathrm{ex}(n, \mathcal{F}) = n^r$ for $1 < r < 3/2$?
- How much can the counterexample for $Q = 3/4$ be simplified?